Name: ______________________________

Teacher: ______________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2012

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics

Time allowed: 3 hours
(plus 5 minutes reading time)

Outcomes Assessed	Questions
Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-10
Manipulates algebraic expressions to solve problems from topic areas such as geometry, co-ordinate geometry, quadratics, trigonometry, probability and logarithms	11,13
Demonstrates skills in the processes of differential and integral calculus and applies appropriate techniques to solve problems	12, 14
Synthesises mathematical solutions to harder problems and communicates them in appropriate form	15, 16

Total Marks 70

Section I 10 marks
Multiple Choice, attempt all questions,
Allow about 15 minutes for this section
Section II 90 Marks
Attempt Questions 11-16,
Allow about 2 hours 45 minutes for this section

General Instructions:

- Questions 11-16 are to be started in a new booklet
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used

Section I	Total 10	Marks
Q1-Q10		
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

SECTION I Multiple Choice (10 marks)

CIRCLE CORRECT RESPONSE

1. What is the value of $\dfrac{(1.49)^2 - 1.98}{\sqrt{11.62 + 8.34 \times 2.72}}$ correct to three significant figures?

   A. 0.040 B. 0.0409

   C. 0.041 D. 0.0410

2. Find the values that satisfy $x^2 - x < 6$

   A. $x < -2, x > 3$ B. $x < -3, x > 2$

   C. $-3 < x < 2$ D. $-2 < x < 3$

3. If $\log_a 5 = 1.03$ and $\log_a 2 = 0.64$ then the value of $\log_a 10$ is:

   A. 2.06 B. 1.67

   C. 0.6592 D. 3.2

4. $\sum_{n=5}^{14} 17 - 2n =$

   A. -11 B. -20

   C. -18 D. -28

5. Expressed in radian measure, 235^0 is:

A. $\dfrac{\pi}{235}$ B. $\dfrac{235}{\pi}$

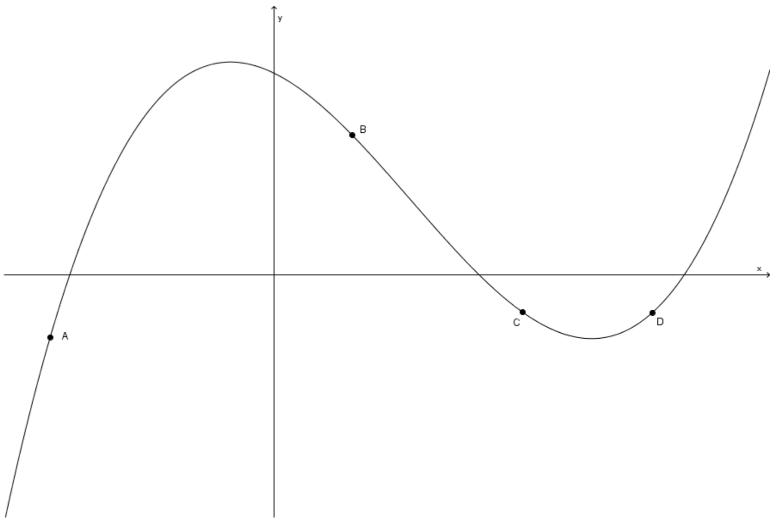
C. $\dfrac{47\pi}{36}$ D. $\dfrac{36\pi}{47}$

6. The third and seventh terms of a geometric series are 1.25 and 20 respectively. What is the first term?

A. ± 2 B. $\dfrac{5}{16}$

C. ± 4 D. $\dfrac{5}{1024}$

7. State which point on the sketch fits the description $y < 0, \dfrac{dy}{dx} > 0, \dfrac{d^2y}{dx^2} < 0$



A. A B. B

C. C D. D

8. What is the greatest value of the function $y = 4 - 2\cos x$?

A. 2

B. 4

C. 6

D. 8

9. A game is played in which two coloured dice are thrown once. The six faces of the red die are numbered 3, 5, 7, 8, 9 and 11. The six faces of the white die are numbered 1, 2, 4, 6, 10 and 12. The player wins if the number on the white die is larger than the number on the red die. What is the probability that the player wins once in two successive games?

A. $\frac{7}{18}$

B. $\frac{11}{18}$

C. $\frac{77}{162}$

D. $\frac{77}{324}$

10. What is the centre and radius of the circle with the equation $x^2 + y^2 + 6x - 8y - 11 = 0$?

A. Centre $(-3, -4)$ and radius 36

B. Centre $(-3, 4)$ and radius 36

C. Centre $(-3, -4)$ and radius 6

D. Centre $(-3, 4)$ and radius 6

SECTION II All necessary working must be shown

Question 11 (15 marks) **Begin a NEW booklet** **Marks**

(a) Evaluate $\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$ **2**

(b) If $\sqrt{50} - 3\sqrt{75} + \sqrt{18} = a\sqrt{2} - b\sqrt{3}$, find the value for a and b. **2**

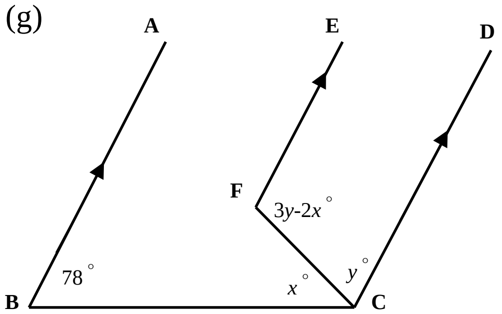
(c) State the domain and range of the function: $f(x) = 2\sqrt{x-1} + 3$ **2**

(d) Evaluate a, in simplest exact form, if $\int_1^3 \frac{a}{1+x}\,dx = \log_e 16$. **2**

(e) Solve for x: $|2x - 1| = 3x - 4$ **2**

(f) For what values of k will $x^2 - (k-3)x + k = 0$ have equal roots? **2**

(g)



AB∥EF∥DC. Find the value of x and y, giving reasons. **3**

Question12 (15 marks) **Begin a NEW booklet** **Marks**

(a) Differentiate with respect to x:

i) $y = \sqrt{1 - x^2}$ **1**

ii) $y = \frac{cosx}{x}$ **1**

(b) Find:

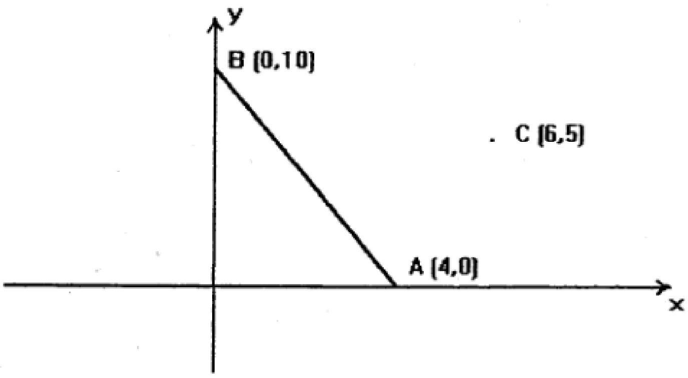
i) $\int \frac{1}{e^{3x}} dx$ **1**

ii) $\int (1 + 3sec^2 \frac{x}{2}) dx$ **1**

(c) The gradient of a curve is given by $f'(x) = 6x^2 - 6x + 5$. If the curve passes through the point $(2, 13)$, find its equation. **2**

(d) Find the equation of the normal to the curve $y = \ln\left(\frac{2x-1}{x+1}\right)$, at the point where $x = 2$. **3**

(e) **6**



The points A, B and C have coordinates (4, 0) , (0, 10) and (6, 5) respectively as shown on the diagram. Copy the diagram into your booklets.

(i) Find the gradient of the line AB.

(ii) Show that the equation of the line through point C perpendicular to AB is given by

$2x - 5y + 13 = 0$. Label this line q.

(iii) Find the perpendicular distance of B from the line CD, giving your answer in surd form with rational denominator.

(iv) If the line q meets the x-axis at D, calculate the size of angle ADC, correct to the nearest degree.

Question 13 (15 marks) **Begin a NEW booklet** **marks**

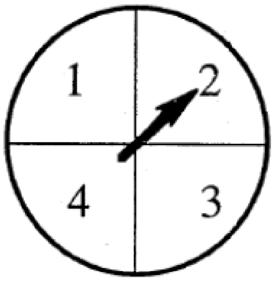
(a) A party of walkers leave their base camp A at noon and walk in the direction $150°T$ until they reach a point B on the bank of the river. Unable to cross the river, they follow it on a bearing of $015°T$ and retire at 2 pm when they reach point C. The bearing and distance of C from A are $120°T$ and $2.5\ km$ respectively. **4**

i) In your booklet, draw a diagram and mark on it the given information, in the correct positions.

ii) Find the distances AB and BC.

iii) Hence, calculate the average speed, correct to one decimal place, of these walkers from A to C.

(b) **3**



Dani and Chris used the spinner shown above to play a game. Dani spun the spinner twice and added the results of the two spins to get his score. Chris then took his turn. The player with the highest score won the game.

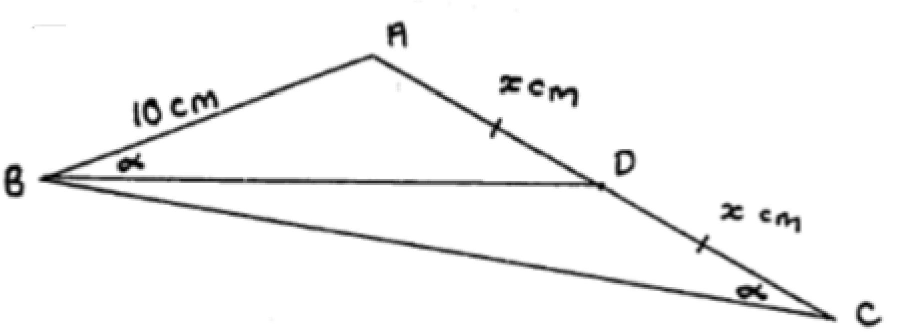
i) Use a tree diagram or a sample space to show all the possible scores Dani could have achieved when he played the game.

ii) What is the probability that Dani scored 6 in the game?

iii) Dani's score was 6. What is the probability that Chris won the game?

Question 13 is continued over

(c)



In the diagram above, $AB = 10\ cm$, $AD = DC = x\ cm$ and $\angle ABD = \angle BCD$. **4**

i) Prove $\Delta ABD\ |||\ \Delta ACB$

ii) Hence find the exact value of x.

iv) If α and β are the roots of the quadratic equation $5x^2 - 3x - 2 = 0$, find the value of: **4**

i) $\alpha + \beta$

ii) $\alpha\beta$

iii) $\alpha^2 + \beta^2$

iv) $(3 - \alpha)(3 - \beta)$

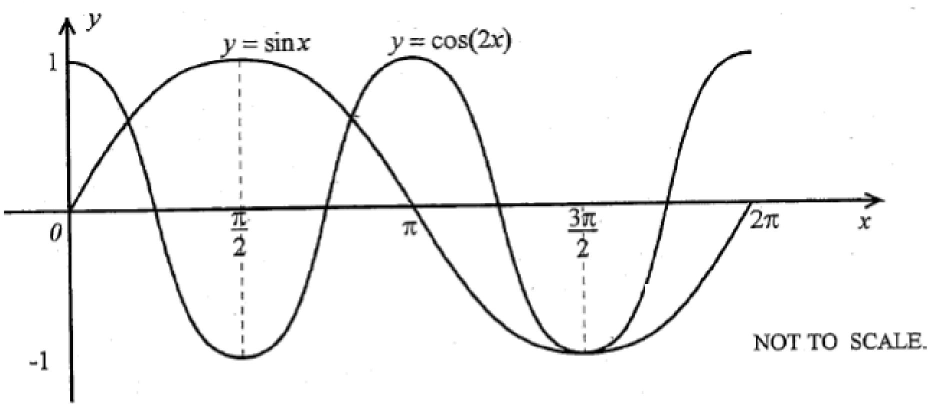
Question14 (15 marks) **Begin a NEW booklet** **marks**

(a) Rapunzel has discovered something interesting about the way her hair grows. **6**
When she was imprisoned in her tower (on the 1st of January) her hair was
9 metres long. She measured her hair every year on the 1st of January, and made a note of its length. She recorded the following lengths on her cell wall for the first 4 years of her imprisonment:

9 m, 10 m, 11.05 m, 12.15 m, 13.3 m

i) How much did Rapunzel's hair grow in the second year?
ii) Write a series showing the amount her hair had grown in each of the first 4 years.
iii) Using the series in part (ii), find how much Rapunzel's hair grew in the eighteenth year.
iv) Prince Charming arrived to rescue Rapunzel after she had been in the tower for 21 years. If the tower is 40 metres high, and assuming that her hair continues to grow according to the series in (ii), will Rapunzel's hair be long enough to reach the ground ? Justify your answer by finding the length of Rapunzel's hair at this time.

(b)



The diagram shows parts of the curves $y = sinx$ and $y = cos2x$. **6**

i) Show that the curves intersect at $x = \frac{5\pi}{6}$ and $x = \frac{3\pi}{2}$.

ii) Calculate the area between the curves from $x = \frac{5\pi}{6}$ to $x = \frac{3\pi}{2}$. Leave the answer in exact form.

c) The region enclosed by the y axis, the line $y = x - 1$ and the line $y = 3$ is rotated about the y axis.

i) Sketch the region **1**

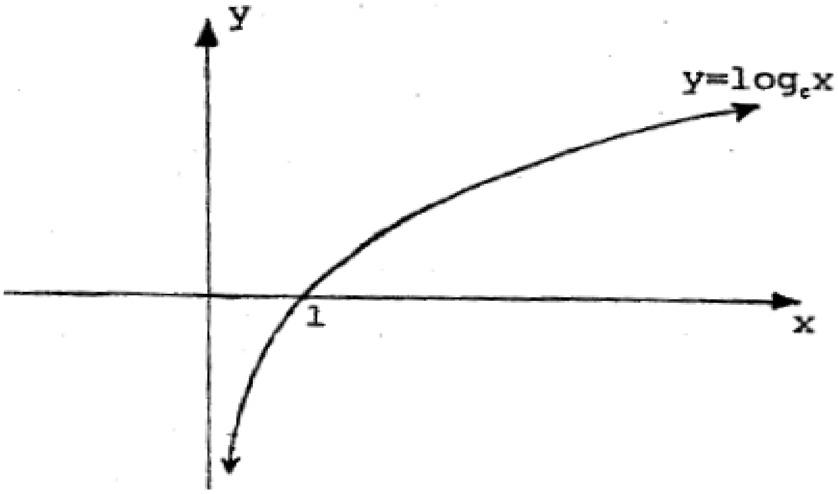
ii) Show that the volume of the resulting solid of revolution is given by **1**

$$V = \pi \int_{-1}^{3} (y^2 + 2y + 1)dy$$

iii) Hence, calculate this volume correct to two decimal places. **1**

Question 15 (15 marks) **Begin a NEW booklet** **marks**

(a) **6**

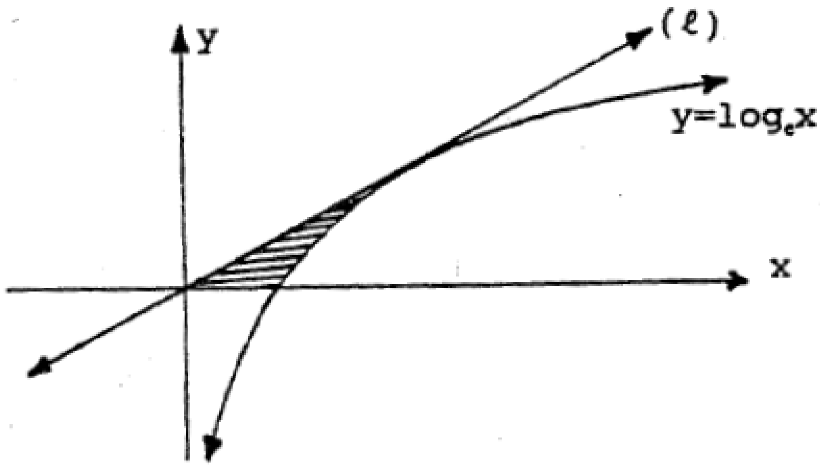


The diagram shows the graph of the curve $y = \log_e x$, not drawn to scale.

(i) Find the equation of the tangent (l) at $x = e$ to the curve and show that it passes through the origin.

(ii) Show that the area bounded by the curve, the y-axis and the lines $y = 0$ and $y = 1$ is $(e - 1)$ square units.

(iii) Using the results of part (ii) or otherwise, find the area of the shaded region between $y = \log_e x$, the tangent (l) and the x-axis as shown in the diagram below.



Question 15 is continued over

b) A river 60m wide is measured for depth every 10m across its width. **2**

The measurements from bank to bank are given in the table. By using Simpson's Rule, estimate the cross-sectional area of the river at this point.

Measurement number	1	2	3	4	5	6	7
Depth in metres	0	5.5	11.0	13.2	8.5	4.5	0

(c) (i) Expand $e^{-x}(1-e^{-x})$ **1**

(ii) For the curve $y = e^{-x} - e^{-2x}$: **6**

(α) find where it cuts the axes.

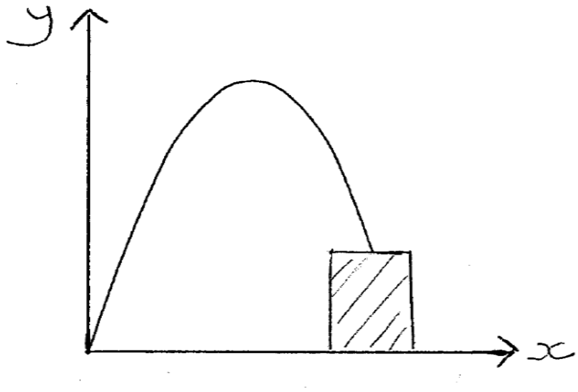
(β) find the co-ordinates of the stationary points and determine their nature.

(γ) determine the values of x for which the curve is monotonic decreasing and hence, or otherwise, discuss the behaviour of the curve for large values of x.

(δ) sketch the curve.

Question 16 (15 marks) **Begin a NEW booklet** **marks**

(a) A water spout reaches a maximum height of 40m at a point 20m away from its source on the ground. It lands on the roof of a building 25 m away. How high is the building? (A water spout traces out a parabola.) **3**



(b) Fabio and Su Lin worked out that they would save \$50000 in 5 years by depositing all their combined monthly salary of $\$S$ at the beginning of each month into a savings account and withdrawing \$1600 at the end of each month for living expenses. The savings account paid interest at the rate of 3% p.a. compounding monthly. **5**

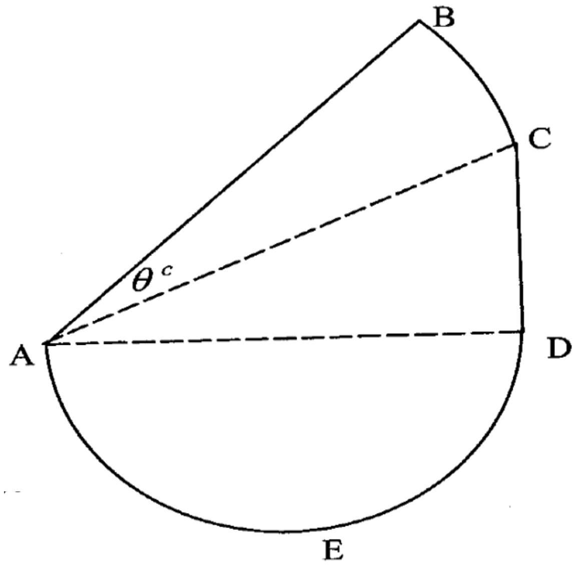
(i) Show that at the end of the second month, the balance in their savings account, immediately after making their withdrawal of \$1600, would be given by $\$[(1.0025^2+1.0025)S-1600(1.0025+1)]$.

(ii) Hence write down an expression for the balance in their account at the end of the 60^{th} month.

(iii) Calculate their combined monthly salary.

Question 16 continued

(c) **7**



ABC is a sector of centre A and radius AB.

ACD is a right-angled triangle with $\angle ADC = 90^0$.

AED is a semi-circle of radius 2r. CD = 3r. The perimeter of ABCDE is 24m.

(i) Show that $\theta = \dfrac{24 - 2\pi r - 8r}{5r}$

(ii) Show that the total Area $= 60\text{r} - 14\text{r}^2 - 3\pi \text{r}^2$

(iii) Show that the total area is a maximum when $\text{r} = \dfrac{30}{3\pi + 14}$

(iv) Find the maximum area to the nearest m^2.

END OF EXAMINATION

Q1. D Q2. D Q3. B Q4. B

Q5. C Q6. B Q7. A Q8. C

Q9. C Q10. D

(10 marks)

Q11

a) $\lim_{x\to 3} \frac{x^3-27}{x-3}$

$= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3}$ ✓

$= 9+9+9$

$= 27$ ✓

. students factorised this incorrectly $x^3-27 = (x+3)(x-3)(x+3)$?

. many simple maths errors.

b) $\sqrt{50} - 3\sqrt{75} + \sqrt{18} = a\sqrt{2} - b\sqrt{3}$

$5\sqrt{2} - 15\sqrt{3} + 3\sqrt{2} = a\sqrt{2} - b\sqrt{3}$ ✓

$8\sqrt{2} - 15\sqrt{3} = a\sqrt{2} - b\sqrt{3}$

$\therefore a = 8$ and $b = 15$ ✓

well done

c) $f(x) = 2\sqrt{x-1} + 3$

D: $x \geq 1$ ✓

R: $y \geq 3$ ✓

students used $>$ instead of $\geq$

d) $\int_1^3 \frac{a}{1+x}\,dx = \log_e 16$

$a[\ln(1+x)]_1^3 = \log_e 16$

$a(\ln 4 - \ln 2) = \ln 16$ ✓

$a \ln\left(\frac{4}{2}\right) = \ln 16$

$a = \frac{\ln 16}{\ln 2}$

$= \frac{4\ln 2}{\ln 2}$

$\underline{a = 4}$ ✓

. students took $\frac{1}{a}$ out of the integral instead of a

. students left the answer as $\frac{\ln 16}{\ln 2}$

e) ... $= 3x-4$

$2x-1 = 3x-4$, $-2x+1 = 3x-4$

$3 = x$; $5 = 5x$, $x = 1$

Check: LHS = 5, RHS = 5

Check: LHS = 1, RHS = −1 ✓

$\therefore$ not a soln ✓

$\therefore$ the only soln is $x = 3$

students did not check their answers.

must check both solns

f) $x^2 - (k-3)x + k = 0$

for equal roots $\Delta = 0$

ie $b^2 - 4ac = 0$

$[-(k-3)]^2 - 4\times1\times k = 0$ ✓

$k^2 - 6k + 9 - 4k = 0$

$k^2 - 10k + 9 = 0$

$(k-9)(k-1) = 0$

$\therefore k = 1$ or 9 ✓

mostly well done

g) $3y - 2x + y = 180$ (co-int. ∠'s, EF ∥ DC)

$4y - 2x = 180$

$2y - x = 90$ ①

$78 + x + y = 180$ (co-int. ∠'s, AB ∥ CD)

$x + y = 102$ ②

1 for setting up eqns

①+②: $3y = 192$

$\underline{y = 64}$ ✓

Sub in ②: $x + 64 = 102$

$\underline{x = 38.}$ ✓

mostly well done

1) i) $y=\sqrt{1-x^2}$

$y=(1-x^2)^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\times -2x$

$$\boxed{\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}}$$

✓ accept $\frac{dy}{dx}=-x(1-x^2)^{-1/2}$

Watch Chain rule

ii) $y=\frac{\cos x}{x}$ (u / v)

$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\cdot\frac{dv}{dx}}{v^2}$

$=\frac{x\cdot -\sin x-\cos x\cdot 1}{x^2}$

$$\boxed{\frac{dy}{dx}=\frac{-x\sin x-\cos x}{x^2}}$$ ✓

Well done

2) i) $\int\frac{1}{e^{3x}}\,dx=\int e^{-3x}\,dx$

$$\boxed{=-\frac{1}{3}e^{-3x}+c}$$ ✓

Some confusion $\neq -3e^{-3x}$

ii) $\int\left(1+3\sec^2\frac{x}{2}\right)dx$

$=x+\frac{3\tan\frac{x}{2}}{\frac{1}{2}}+c$

$$\boxed{=x+6\tan\frac{x}{2}+c}$$ ✓

Well done

3) $f'(x)=6x^2-6x+5$

$f(x)=\frac{6x^3}{3}-\frac{6x^2}{2}+5x+c$

$f(x)=2x^3-3x^2+5x+c$ ✓

Wide range of constants.

through $(2,13)$: $13=16-12+10+c$

$c=-1$

$\therefore\ \boxed{f(x)=2x^3-3x^2+5x-1}$ ✓

4) $y=\ln\left(\frac{2x-1}{x+1}\right)$.

$y=\ln(2x-1)-\ln(x+1)$

$y'=\frac{2}{2x-1}-\frac{1}{x+1}$

at $x=2$: $y'=\frac{2}{3}-\frac{1}{3}$

$=\frac{1}{3}\implies m=\frac{1}{3}$ ✓

$\therefore \perp m=-3$

when $x=2$: $y=\ln 1=0$ ✓ $\implies(2,0)$

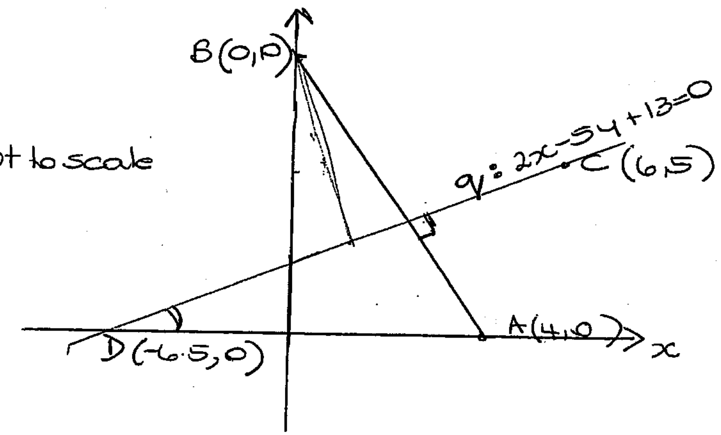
$y-y_1=m(x-x_1)$

$y-0=-3(x-2)$ ✓

$\boxed{y=-3x+6}$ or $\boxed{3x+y-6=0}$

Some log differen problems.

5)



i) $m_{AB}=\frac{10-0}{0-4}$

$m_{AB}=\frac{-5}{2}$ ✓

ii) $\perp m=\frac{2}{5}$ ✓ $C(6,5)$

$y-y_1=m(x-x_1)$

$y-5=\frac{2}{5}(x-6)$

$5y-25=2x-12$

$2x-5y+13=0$

(as req'd) ✓ for working

Mostly Well done

$a=2 \quad b=-5 \quad c=13$

perp. dist. $= \left|\dfrac{ax+by+c}{\sqrt{a^2+b^2}}\right|$

$= \left|\dfrac{0-50+13}{\sqrt{4+25}}\right|$ ✓

$= \left|\dfrac{-37}{\sqrt{29}}\right|$

$= \dfrac{37}{\sqrt{29}} \times \dfrac{\sqrt{29}}{\sqrt{29}}$

$= \dfrac{37\sqrt{29}}{29}$ units. ✓

Some people didn't rationalised as asked

v) CD(q) cuts x axis $\Rightarrow y=0$
$2x=-13$
$x=-6.5 \Rightarrow D(-6.5, 0)$ } not needed.

now $m=\frac{2}{5}$ from ii)

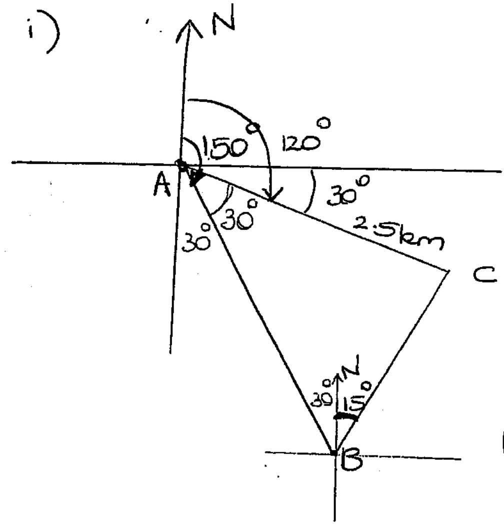
$\therefore \tan\theta = m$

$\tan\angle ADC = \frac{2}{5}$

$\angle ADC = 22°$ (nearest deg.) ✓

21.8° OK

a) i)



ii) $\dfrac{AB}{\sin 105°} = \dfrac{2.5}{\sin 45}$

$AB = \dfrac{2.5 \sin 105}{\sin 45}$

$= 3.415063509$ ✓

$AB = 3.42$ km (2dp)

$\dfrac{BC}{\sin 30} = \dfrac{2.5}{\sin 45}$

$BC = \dfrac{2.5 \times \frac{1}{2}}{\frac{1}{\sqrt{2}}}$

$= 1.767766953$ ✓

$BC = 1.77$ km (2dp)

iii) av. speed $= \dfrac{D}{T}$

$= \dfrac{AB+BC}{2}$ ✓

$= 2.591415231$

$= 2.6$ km/h (correct to 1dp).

* OR 2.6 km/h if using full display on calculator

Comments

* Diagrams were very poorly done. They must be $\frac{1}{3}$ A4 page in pencil using a ruler!

* many students did not mark on the given info as asked. They marked 'deduced' info instead.

Errors in ii) were mainly due to incorrect angle sizes.

* some tried to use right Δ ratios ??

* many students included AC!

1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

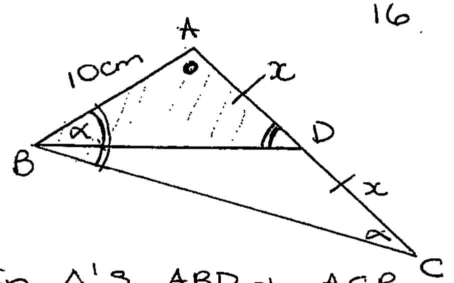
or can use a tree diagram ✓

*poorly done – many students recorded the outcomes, not the scores as asked in the question.

i) $P(\text{Dani}, 6) = \frac{3}{16}$ ✓

ii) $P(\text{Chris wins}) = P(\text{score} > 6)$
$= P(\text{score } 7) + P(\text{score } 8)$
$= \frac{3}{16}$ ✓

generally well done!



) In Δ's ABD & ACB
∠A is common
∠ABD = ∠ACB (= α) given ✓
∴ Δ ABD ||| ΔABC equiangular ✓

* MOST STUDENTS NEED TO REVISE SETTING OUT SIMILARITY PROOFS and the CORRECT TESTS!!!
(Many are confusing congruence tests with similarity tests)

∴ $\frac{AD}{AB} = \frac{AB}{AC}$ (corres. sides of sim. Δ's are in same ratio)

→ reason should be stated for setting up the the proportion statement

$\frac{x}{10} = \frac{10}{2x}$ ✓
$2x^2 = 100$
$x^2 = 50$
$x = 5\sqrt{2}$ ✓ $(x > 0)$

) $5x^2 - 3x - 2 = 0$
$a = 5 \quad b = -3 \quad c = -2$

i) $\alpha + \beta = \frac{-b}{a}$
$\boxed{\alpha + \beta = \frac{3}{5}}$ ✓

ii) $\alpha\beta = \frac{c}{a}$
$\boxed{\alpha\beta = \frac{-2}{5}}$ ✓

iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= \frac{9}{25} + \frac{4}{5}$
$= \boxed{\frac{29}{25}}$ ✓

iv) $(3-\alpha)(3-\beta) = 9 - 3\alpha - 3\beta + \alpha\beta$
$= 9 - 3(\alpha+\beta) + \alpha\beta$

*generally very well done but some students must learn the main expressions $\alpha+\beta$, $\alpha\beta$, $(\alpha+\beta)^2$

34

QUESTION 14

(a) 9m, 10m, 11.05m, 12.15m, 13.3m

i) 1.05m ✓

ii) $1 + 1.05 + 1.10 + 1.15 \ldots$ ✓

iii) $T_n = a + (n-1)d$ AP: $a = 1, d = 0.05$
$T_{18} = 1 + 17 \times 0.05$
$= 1.85$ ✓
∴ grew 1.85m in the 18th year

iv) $S_n = \frac{n}{2}[2a + (n-1)d]$
$S_{21} = \frac{21}{2}[2 + 20 \times .05]$ ✓
$S_{21} = 31.5$

Length of hair after 21 yrs
$= 9 + S_{21}$
$= 40.5$m ✓

Since the tower is 40m high, her hair will be long enough. ✓

* Most students are not reading the requirements of the question
i.e. (i) 2nd year, not 1st!
(ii) "Write a series ..."
(iii) "... in the 18th year..."
• many did not know the definition of a series.

• Several students found T_{21} instead of S_{21}.

• Many forgot to add the 31.5m growth to the initial 9m length, thus forever trapping Rapunzel unjustly in her tower forever.

(b) $y = \sin x$

$x = \frac{5\pi}{6}$: $y = \sin\frac{5\pi}{6}$
$y = \frac{1}{2}$

$y = \cos 2x$
$= \cos 2\left(\frac{5\pi}{6}\right)$
$= \cos\frac{5\pi}{3}$
$= \frac{1}{2}$

∴ curves intersect at $x = \frac{5\pi}{6}$ ✓

$x = \frac{3\pi}{2}$: $y = \sin\frac{3\pi}{2}$
$= -1$

$y = \cos 2\left(\frac{3\pi}{2}\right)$
$= -1$

∴ curves intersect at $x = \frac{3\pi}{2}$ ✓

• while many made and solved the equation $\cos 2x = \sin x$, too many used an equation to show. Solve or show, don't mix the methods!
show substitutions

ii) $A = \int_{\frac{5\pi}{6}}^{\frac{3\pi}{2}} (\cos 2x - \sin x)\, dx$

$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{\frac{5\pi}{6}}^{\frac{3\pi}{2}}$ ✓

$= \left(\frac{1}{2}\sin 3\pi + \cos\frac{3\pi}{2}\right) - \left(\frac{1}{2}\sin\frac{5\pi}{3} + \cos\frac{5\pi}{6}\right)$ ✓

$= (0 + 0) - \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$ ✓

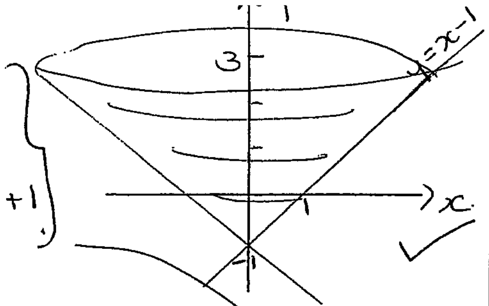
• despite being clearly asked to evaluate between $\frac{5\pi}{6}$ and $\frac{3\pi}{2}$, many tried to add together every possible area/integral (wasting time).
• Many also seem unaware that $\frac{5\pi}{6} < \frac{3\pi}{2}$.
• exact value – this doesn't mean leave in trig form when the angle give exact trig values!!

$y = x - 1$

$x = y + 1$

$x^2 = (y+1)^2$

$x^2 = y^2 + 2y + 1$



$V = \pi \int x^2 \, dy$

$= \pi \int (y^2 + 2y + 1) \, dy$ ← show expansion for mark

$= \pi \left[\frac{y^3}{3} + \frac{2y^2}{2} + y \right]_{-1}^{3}$

$= \pi \left[(9 + 9 + 3) - \left(-\frac{1}{3} + 1 - 1 \right) \right]$

$= \pi \left[21\frac{1}{3} \right]$

$V = \frac{64}{3}\pi \; u^3$ ✓

$= 67.02 \; u^3$ (2dp)

- Sketch the region – you must indicate the region somehow: shading is the standard way to do this. Simply drawing all the lines does not fulfil the requirements of the question!
- As the integral is already given, you must make the link between (i) as the starting point and thus integral! Too many wrote
  $V = \pi \int_{-1}^{3} x^2 \, dy$
  $= \pi \int_{-1}^{3} y^2 + 2y + 1 \, dy$
  (this scored no marks)
- too many stopped at $V = \frac{64}{3}\pi$ (those that could at least get the $\frac{64}{3}$) and did not fulfil the requirements of the question
- π also had a habit of mysteriously disappearing – leaving $21\frac{1}{3}$ as a common answer!

Question 15 (15 marks)

Solution	comments
a) i) $y = \ln x$	
$y' = \frac{1}{x}$	
at $x = e$, $y' = \frac{1}{e}$, $y = \ln e = 1$	
Eqn. of tangent:	well answered
$y - 1 = \frac{1}{e}(x - e)$ ← ① mark for eqn. of tangent	
$ey - e = x - e$	
$x - ey = 0$.	
sub $(0,0)$ into	many forgot to read the q carefully and did not attempt this part.
$x - ey = 0$	
$0 - e(0) = 0$	
$0 = 0$ ① mark for showing origin is on tangent.	
LHS = RHS	
∴ tangent passes through the origin.	
ii) $A = \int_0^1 e^y \, dy$ ← ① mark for integral ($y = \ln x$ ∴ $x = e^y$)	A common mistake was to try and $\int \ln x \, dx$
$= \left[e^y \right]_0^1$ ① mark for working.	
$= e^1 - e^0$	
$= (e - 1)$ units²	
iii) $A = (e - 1) - A$ of Δ	More CARE needed.
$= (e - 1) - \frac{e \times 1}{2}$ ← ① mark	
$= e - 1 - \frac{e}{2}$	
$= \left(\frac{e}{2} - 1\right)$ units² ← ① mark.	
b) $h = 10$	Many students did not recognise $h = 10$ from q.

x	0	10	20	30	40	50	60
	0	5.5	11	13.2	8.5	4.5	0
	y_1	y_2	y_3	y_4	y_5	y_6	y_7

$A \doteqdot \frac{h}{3}\left\{ y_1 + y_7 + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5) \right\}$

$= \frac{10}{3}\left\{ 0 + 0 + 4(5.5 + 13.2 + 4.5) + 2(11 + 8.5) \right\}$ ← ① mark

$= 439.\dot{3}$ units² ← ① mark

15 continued. | comments

c) i) $e^{-x}(1-e^{-x})$

$= e^{-x} - e^{-2x}$ ① for expanding correctly

ii) $y = e^{-x} - e^{-2x}$

x) sub $x=0$: $y = e^0 - e^0 = 1-1$, $y=0$

sub $y=0$: $0 = e^{-x}(1-e^{-x})$, $e^{-x} \neq 0$, $e^{-x}=1$ when $x=0$

∴ cuts x & y axes at origin. ① for origin.

Comment: some errors in not recognising $e^{-x} \neq 0$ and when $e^{-x}=1$ $x=0$

3) $\frac{dy}{dx} = -e^{-x} + 2e^{-2x}$

at stationary pts., $\frac{dy}{dx}=0$

∴ $-e^{-x} + 2e^{-2x} = 0$

$e^{-x}(2e^{-x} - 1) = 0$

$e^{-x} \neq 0$, $e^{-x} = \frac{1}{2}$

$\ln(\frac{1}{2}) = -x$

$x = -\ln(\frac{1}{2})$

$x = -(\ln 1 - \ln 2)$

$x = \ln 2$.

If $x = \ln 2$, $y = e^{-\ln 2} - e^{-2\ln 2}$

$= \frac{1}{2} - \frac{1}{4}$

$= \frac{1}{4}$

∴ stationary pt. is at $(\ln 2, \frac{1}{4})$ $(\doteqdot 0.69, \frac{1}{4})$ ① for stat. pt.

Comment: many students had trouble solving $2e^{-x} - 1 = 0$ ⇓ $x = \ln 2$

$\frac{d^2y}{dx^2} = e^{-x} - 4e^{-2x}$

at $x = \ln 2$, $\frac{d^2y}{dx^2} < 0$

∴ $(\ln 2, \frac{1}{4})$ is a maximum turning pt. ① for nature of stat. pt.

Question 15 continued. | comments

c) ii) y) Since there is only one turning point and it is a maximum turning point, then curve is monotonically decreasing for $x > \ln 2$. ① for $x > \ln 2$

OR

$\frac{dy}{dx} = -e^{-x} + 2e^{-2x} < 0$

i.e. $e^{-x}(2e^{-x} - 1) < 0$

when $2e^{-x} - 1 < 0$ as $e^{-x} > 0$ for all real x

∴ $e^{-x} < \frac{1}{2}$

i.e. $\ln e^{-x} < \ln \frac{1}{2}$

$-x < \ln \frac{1}{2}$

$x > -\ln \frac{1}{2}$

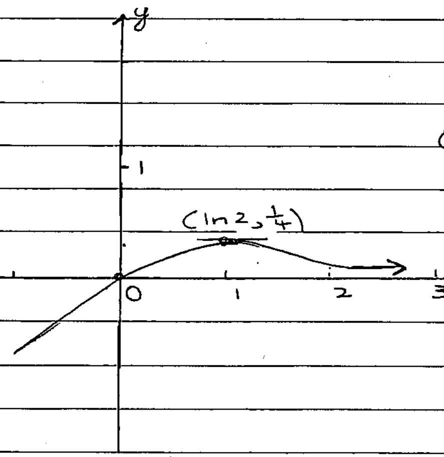
i.e. $x > \ln 2$.

Comment: poorly answered. many students found their answer to be $x < \ln 2$ instead of $x > \ln 2$

as $x \to +\infty$, $y = e^{-x}(1 - e^{-x}) \to 0$ (because $e^{-x} \to 0$) ① for $y \to 0$

Comment: well answered

δ)



① for sketch

Comment: poorly answered and often contradicted previous working.

Question 16 (15 marks) — comments

i) eqn. of parabola is in form: → ① mark.

$(x-h)^2 = -4a(y-k)$

i.e. $(x-20)^2 = -4a(y-40)$ — * vertex $(h,k) = (20,40)$

Find 'a':

$(0-20)^2 = -4a(0-40)$

$400 = 160a$

$a = 2.5$ — ① for 'a'

∴ eqn. of parabola:

$(x-20)^2 = -10(y-40)$

when $x = 25$:

$(25-20)^2 = -10(y-40)$

$25 = -10y + 400$

$-375 = -10y$

∴ $y = 37.5$ — ① for height of building.

∴ height of building is 37.5 metres.

Comments:

- extension I students tried to use projectile motion formulae.
- students used $y = ax^2 + bx + c$ which turns out to be much more difficult.
- calculation error was common

2) $r = 3\%$ p.a $= 0.25\%$/month.

A_n = balance in savings account after n months, after withdrawal

i) $A_1 = S(1.0025) - 1600$ ← ① mark

$A_2 = (A_1 + S)1.0025 - 1600$

$= A_1(1.0025) + S(1.0025) - 1600$

$= [S(1.0025 - 1600)]1.0025 + S(1.0025) - 1600$

$= S(1.0025)^2 - 1600(1.0025) + S(1.0025) - 1600$

$= S[1.0025 + 1.0025^2] - 1600[1 + 1.0025]$

① mark for deriving A_2

ii) $A_{60} = S[1.0025 + 1.0025^2 + \dots + 1.0025^{60}] - 1600[1 + 1.0025 + \dots + 1.0025^{59}]$

⇒ ① mark for A_{60}

Question 16 ct'd — comments

b) iii) $A_{60} = 50000$

$50000 = S(1.0025 + \dots + 1.0025^{60}) - 1600(1 + 1.0025 + \dots + 1.0025^{59})$

$50000 = S\left[\frac{1.0025(1.0025^{60} - 1)}{0.0025}\right] - 1600\left[\frac{1(1.0025^{60} - 1)}{0.0025}\right]$ ← ① mark

$50000 = S[401(1.0025^{60} - 1)] - 1600[400(1.0025^{60} - 1)]$

$S = \frac{50000 + 640000[1.0025^{60} - 1]}{401(1.0025^{60} - 1)}$

$S \doteqdot 2367.56$ ← ① mark

Comments:

- students made the second geometric sum with 59 terms.
- students made $S_{60} = 0$

Question 16 ct'd | comments

c) i) In right $\triangle ADC$:

$AC = 5r$ (Pythagoras' Thm)

$\therefore AB = 5r$

$AB + BC + CD + DEA = 24$ (given)

$\therefore 5r + (5r)\theta + 3r + \frac{1}{2}\times\pi\times 4r = 24$ ← ① mark

$8r + 5r\theta + 2\pi r = 24$

$5r\theta = 24 - 2\pi r - 8r$

$\theta = \dfrac{24 - 2\pi r - 8r}{5r}$

① mark for working

Comments: · students did not show working. · many wrote a "vague" equation and then just wrote the answer.

ii) A = A of sector ABC + A of $\triangle ADC$ + A of semicircle

$= \frac{1}{2}(5r)^2\theta + \frac{4r\times 3r}{2} + \frac{1}{2}\times\pi\times(2r)^2$ ← ① mark

$= \frac{25r^2}{2}\left(\frac{24-2\pi r-8r}{5r}\right) + 6r^2 + 2\pi r^2$

$= \frac{5r}{2}(24 - 2\pi r - 8r) + 6r^2 + 2\pi r^2$

$= 60r - 5\pi r^2 - 20r^2 + 6r^2 + 2\pi r^2$

$A = 60r - 14r^2 - 3\pi r^2$

① mark for working

Comments: As above.

iii) A is maximum when $A' = 0$ and $A'' < 0$

$A' = 60 - 28r - 6\pi r$

$A' = 0$ when

$60 - 28r - 6\pi r = 0$

$60 = 6\pi r + 28r$

$60 = r(6\pi + 28)$

$r = \dfrac{60}{6\pi + 28}$ $\div 2$ / $\div 2$

$r = \dfrac{30}{3\pi + 14}$

① mark for finding r

$A'' = -28 - 6\pi < 0$

$\therefore$ A is maximum when $r = \dfrac{30}{3\pi + 14}$.

① mark for testing for maximum

Comments: Mostly well done. · students did not test their answers.

Q16 continued | Comments

iv) $A \doteq 60(1{\cdot}28) - 14(1{\cdot}28)^2 - 3\times\pi\times(1{\cdot}28)^2$ $r = \dfrac{30}{3\pi+14} \doteq 1{\cdot}28$

$= 38\ m^2$ (nearest m^2).

① mark for answer.

Comments: · Calculator error occurred regularly.