2008

TRIAL

HIGHER SCHOOL CERTIFICATE

GIRRAWEEN HIGH SCHOOL

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board – approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1 – 10
- All questions are of equal value

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Attempt Questions 1 – 10
All questions are of equal value

Question 1 (12 marks). **Marks**

(a) Evaluate $\dfrac{\pi+2}{\pi-2}$ correct to one decimal place. **2**

(b) Solve $2x+1\le 7$ and graph the solution on a number line. **2**

(c) Rationalise the denominator of $\dfrac{1}{\sqrt{6}-2}$. **2**

(d) Find the limiting sum of the geometric series **2**

$$9-3+1-\ldots$$

(e) Factorise $6x^2-x-2$. **2**

(f) At Octopus Fones annual sale, all mobile phones are discounted by 40%. Mia paid \$630 for a mobile phone at the sale. What was the original price of the phone? **2**

Marks

Question 2 (12 marks). Start on a SEPARATE page.

(a) Differentiate with respect to x:

(i) $\frac{1}{x}$. **1**

(ii) $\frac{3x}{e^x - 1}$. **2**

(iii) $(1 + \cos x)^8$. **2**

(b) (i) Find $\int 2\sec^2 3x\,dx$. **2**

(ii) Evaluate $\int_0^1 \frac{2x}{x^2+1}dx$. **2**

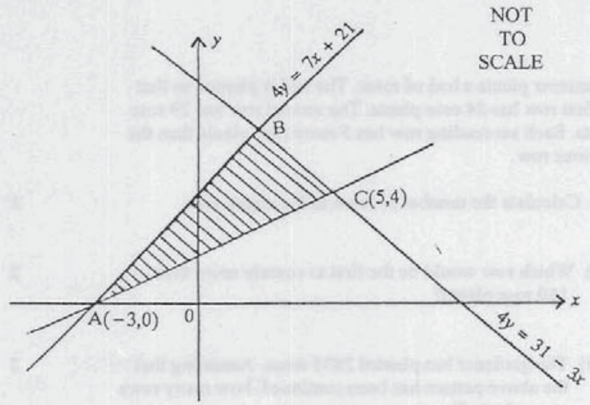
(Leave answer in exact form).

(c) Find the equation of the normal to the curve $y = e^{4x} - 1$ at the point on the curve where $x = 0$. **3**

Marks

Question 3 (12 marks). Start on a SEPARATE page.

(a)



In the diagram, the lines $4y = 7x + 21$ and $4y = 31 - 3x$ intersect at the point B. A and C are the points $(-3, 0)$ and $(5, 4)$ respectively.

(i) Calculate the gradient of AC. **1**

(ii) Show that the line AC has equation $x - 2y + 3 = 0$. **1**

(iii) Show that B has coordinates (1, 7). **1**

(iv) Show that the perpendicular distance from B to the line AC is $2\sqrt{5}$ units. **2**

(v) Find the exact length of the interval AC. (Express answer as a simplified surd). **1**

(vi) Find the area of triangle ABC. **1**

Question 3 continues on page 5

Question 3 (continued) **Marks**

(b) A gardener plants a bed of roses. The bed is planted so that the first row has 24 rose plants. The second row has 29 rose plants. Each succeeding row has 5 more rose plants than the previous row.

(i) Calculate the number of roses in the eighth row. **1**

(ii) Which row would be the first to contain more than 150 rose plants? **2**

(iii) The gardener has planted 2895 roses. Assuming that the above pattern has been continued, how many rows were planted? **2**

End of Question 3

Please turn over

Marks

Question 4 (12 marks). Start on a SEPARATE page.

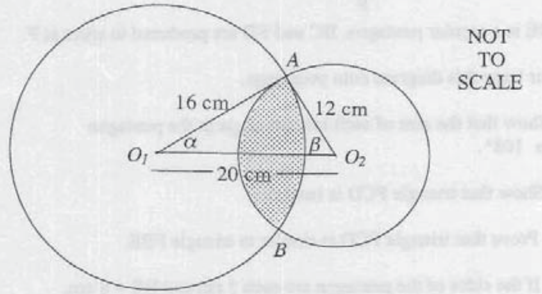
(a) The sides of a triangle are 7 cm, 5 cm and 4 cm. Find the size of the angle opposite the largest side. (Give answer correct to the nearest minute). **2**

(b) A fair die is rolled twice. Find the probability that:

(i) the second score is greater than the first score. **2**

(ii) the total of the two scores is 7 or 11. **1**

(c)



A circle with centre at O_1, and radius 16 cm intersects with another circle with centre at O_2 and radius 12 cm. Their points of intersection are A and B and the distance between their centres, O_1O_2, is 20 cm. The $\angle AO_1O_2 = \alpha$ and $\angle AO_2O_1 = \beta$.

(i) Show that triangle O_1AO_2 is a right-angled triangle. **1**

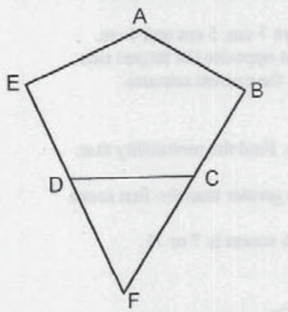
(ii) Find the area of the quadrilateral AO_2BO_1. **1**

(iii) Find the size of the angles α and β. **2**

(iv) Find the shaded area enclosed by these circles. (Give your answer correct to the nearest cm^2). **3**

Marks

Question 5 (12 marks). Start on a SEPARATE page.



ABCDE is a regular pentagon. BC and ED are produced to meet at F.

Copy or trace this diagram onto your page.

(i) Show that the size of each interior angle in the pentagon is $108°$. **1**

(ii) Show that triangle FCD is isosceles. **1**

(iii) Prove that triangle FCD is similar to triangle FBE. **2**

(iv) If the sides of the pentagon are each 5 cm and BE = 8 cm, determine the length of CF. **2**

(b) If α and β are the roots of the quadratic equation $3x^2 + 4x + 7 = 0$, find the values of :

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\alpha^2 + \beta^2$ **1**

(iv) $\frac{1}{\alpha} + \frac{1}{\beta}$ **1**

Question 5 continues on page 8

Question 5 (continued) **Marks**

(c) A particle moves in a straight line so that its acceleration, a m/s^2, at time t seconds is given by $a = 3(4+t)^2$. Initially the particle is moving with a velocity of 64 m/s. **2**

Find the velocity of the particle as a function of time.

Question 6 (12 marks). Start on a SEPARATE page.

(a) Solve the following equation for x: **2**

$$e^{2x} - e^x - 6 = 0.$$

(b) Let $f(x) = \frac{4x^3 - x^4}{9}$.

(i) Find the coordinates of the points where the curve crosses the axes. **2**

(ii) Find any stationary points, and determine their nature. **3**

(iii) Find any points of inflexion. **3**

(iv) Sketch the graph of $y = f(x)$, indicating clearly the intercepts, stationary points and points of inflexion. **2**

End of Question 6

Please turn over

Marks

Question 7 (12 marks). Start on a SEPARATE page.

(a) A parabola whose equation is $y = kx^2$, where k is a constant, has the line $y = -6x + 3$ as a tangent.

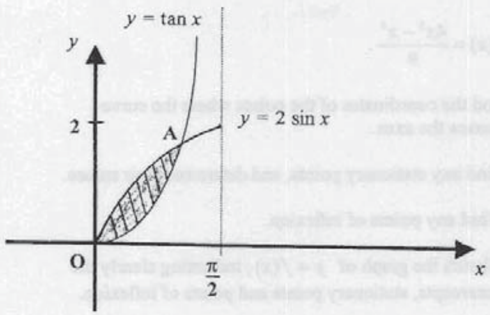
(i) By equating the two given equations, find a quadratic equation in terms of x and k. **1**

(ii) By using the discriminant of the quadratic equation found, find the value of k. **2**

(iii) Find the coordinates of the point of contact between the tangent and the parabola. **2**

(iv) Sketch the parabola and the tangent line, showing the coordinates of the point of contact and where the tangent line cuts the x – and y – axes. **2**

(b)



The diagram shows the graphs of $y = 2\sin x$ and $y = \tan x$ for $0 \le x \le \frac{\pi}{2}$. A is the point of intersection of the two graphs.

(i) Find the coordinates of point A. **2**

(ii) Show that $\frac{d}{dx}[(\ln \cos x)] = -\tan x$ **1**

(ii) Find the area of the shaded region in the diagram. **2**

Marks

Question 8 (12 marks). Start on a SEPARATE page.

(a) A population of bacteria in a medium are growing at a rate proportional to the current population. The population obeys the model

$$P = P_0 e^{kt}$$

where P_0 is the population of bacteria at noon on 1 August and t is measured in hours. When $t = 6$ the population has grown from 900 000 to 1.4 million.

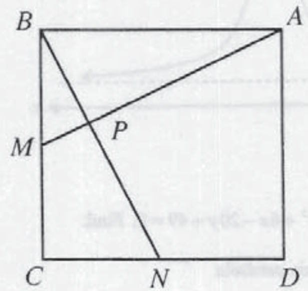
(i) Show that $\frac{dP}{dt} = kP$ **1**

(ii) What is the value of k? **2**

(iii) What will the population be when $t = 10$? **1**

(iv) When will the population reach 3 million? **1**

(b)



ABCD is a square. M and N are the midpoints of BC and CD respectively.

(i) Prove triangles ABM and BCN are congruent. **4**

(ii) Prove that AM and BN are perpendicular. **3**

Marks

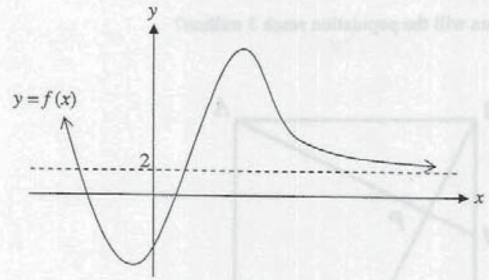
Question 9 (12 marks). Start on a SEPARATE page.

(a) The table shows the values of a function $f(x)$ for four values of x.

x	2	3	4	5
$f(x)$	0.693	1.099	1.386	1.609

Use the trapezoidal rule with these four values to find an approximation to $\int_2^5 f(x)dx$. **2**

(b) The diagram below shows a sketch of the curve $y = f(x)$. Copy or trace the diagram on your page and use it to draw a sketch of the gradient function $y = f'(x)$. **2**



(c) A parabola has equation $x^2 + 6x - 20y + 49 = 0$. Find:

(i) the focal length of this parabola **2**

(ii) the coordinates of the vertex **1**

(iii) the coordinates of the focus **1**

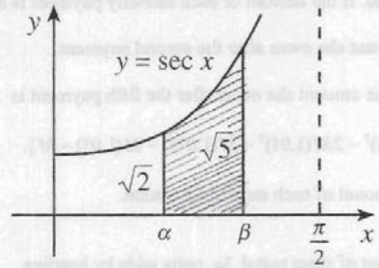
(iv) the equation of the directrix **1**

Question 9 continues on page 12

Marks

Question 9 (continued)

(d)



The shaded region in the diagram is bounded by the curve $y = \sec x$, the x – axis and the lines $x = \alpha$ and $x = \beta$.

Find the volume of the solid of revolution formed when the shaded region is rotated about the x – axis. **3**

End of Question 9

Please turn over

Question 10 (12 marks). Start on a SEPARATE page.

(a) Arina borrows \$20000 from City Credit at 12% p.a. interest. She pays it back at regular monthly intervals over four years. However, because she is a good customer, she is given two months interest free. If the amount of each monthly payment is M dollars

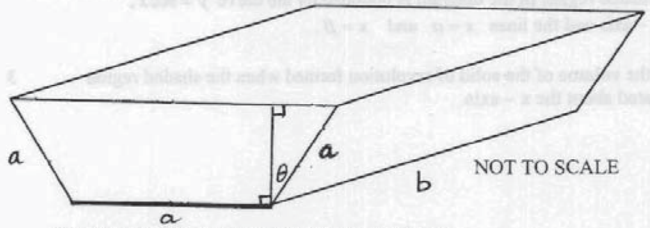
(i) Find the amount she owes after the second payment. **1**

(ii) Show that the amount she owes after the fifth payment is **2**

$$\$[20000(1.01)^3 - 2M(1.01)^3 - M(1.01)^2 - M(1.01) - M].$$

(iii) Find the amount of each monthly payment. **3**

(b) A gutter is made out of sheet metal $3a$ units wide by bending it as shown in the diagram. The length of the gutter is b units.



(i) Show that the volume of the gutter is given by **2**

$$V = a^2 b\cos\theta(1+\sin\theta).$$

(ii) Show that $\dfrac{dV}{d\theta} = a^2 b(1 - 2\sin^2\theta - \sin\theta).$ **2**

(iii) Determine the value of θ, in degrees, so that the gutter has maximum volume. **2**

End of paper

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

Mathematics Trial HSC 2008, Solutions

Question 1

(a) $\frac{\pi+2}{\pi-2} = 4.503876788$

$= 4.5$ (1 dec. pl.) (2)

(b) $2x+1 \le 7$

$2x \le 6$

$\therefore x \le 3$

(2)

(c) $\frac{1}{\sqrt{6}-2} = \frac{1}{\sqrt{6}-2} \times \frac{\sqrt{6}+2}{\sqrt{6}+2}$

$= \frac{\sqrt{6}+2}{6-4}$

$= \frac{\sqrt{6}+2}{2}$ (2)

(d) $9-3+1-\ldots$

$a=9,\ r=-\frac{3}{9}=-\frac{1}{3}$

$S_\infty = \frac{a}{1-r}$

$= \frac{9}{1+\frac{1}{3}}$

$= 6\frac{3}{4}$ (2)

(e) $6x^2-x-2$

$= (2x+1)(3x-2)$ (2)

(f) $60\% = \$630$

$1\% = \$\frac{630}{60}$

$\therefore 100\% = \$\frac{630}{60} \times 100$

$= \$1050$ (2)

Question 2

(a) (i) $\frac{d}{dx}\left(\frac{1}{x}\right)$

$= -1x^{-2}$

$= \frac{-1}{x^2}$ (1)

(ii) $\frac{d}{dx}\left(\frac{3x}{e^x-1}\right)$

$= \frac{(e^x-1).3 - 3x(e^x)}{(e^x-1)^2}$

$= \frac{3e^x-3-3xe^x}{(e^x-1)^2}$

$= \frac{3(e^x-1-xe^x)}{(e^x-1)^2}$ (2)

(iii) $\frac{d}{dx}\left[(1+\cos x)^8\right]$

$= 8(1+\cos x)^7 \times \frac{d}{dx}(\cos x)$

$= -8\sin x(1+\cos x)^7$ (2)

(b) (i) $\int 2\sec^2 3x\, dx$

$= \frac{2\tan 3x}{3} + c$ (2)

(ii) $\int_0^1 \frac{2x}{x^2+1}\,dx$

$= \left[\ln(x^2+1)\right]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$ (2)

(c) $y = e^{4x}-1$

$\therefore \frac{dy}{dx} = 4e^{4x}$

when $x=0$, $y=e^0-1=0$

" $x=0$, $\frac{dy}{dx} = 4e^0 = 4$

$\therefore$ gradient of normal is $-\frac{1}{4}$

$\therefore$ equation of normal is

$y-0 = -\frac{1}{4}(x-0)$

$\therefore y = -\frac{1}{4}x$

or $x+4y=0$ (3)

Question 3

(a) (i) $m_{AC} = \frac{4-0}{5+3}$

$= \frac{4}{8}$

$= \frac{1}{2}$ (1)

(ii) using $(-3,0)$ and $m=\frac{1}{2}$ equation of AC is

$y-0 = \frac{1}{2}(x+3)$

$2y = x+3$

$\therefore x-2y+3=0$ (1)

(iii) $4y = 7x+21 \ldots (1)$

$4y = 31-3x \ldots (2)$

$\therefore 7x+21 = 31-3x$

$7x+3x = 31-21$

$10x = 10$

$\therefore x=1$

using (1), $4y = 31-3\times1$

$\therefore 4y = 28$

$\therefore y=7$

$\therefore B(1,7)$ (1)

(iv) $d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

using $B(1,7)$ & $x-2y+3=0$

$\therefore d = \frac{|(1\times1)+(-2\times7)+3|}{\sqrt{1^2+(-2)^2}}$

$= \frac{|-10|}{\sqrt{5}}$

$= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{10\sqrt{5}}{5}$

$= 2\sqrt{5}$ units (2)

(v) $AC = \sqrt{(5+3)^2+(4-0)^2}$

$= \sqrt{64+16}$

$= \sqrt{80}$

$= \sqrt{16}\times\sqrt{5}$

$= 4\sqrt{5}$ units (1)

(vi) Area ΔABC

$= \frac{1}{2}\times AC \times d$

$= \frac{1}{2}\times 4\sqrt{5}\times 2\sqrt{5}$

$= 20$ unit^2 (1)

(b) 24, 29, 34, ...

A.P. with $a = 24$, $d = 5$

(i) $T_n = a + (n-1)d$

$\therefore T_8 = 24 + 7 \times 5$

$= 59.$ (1)

(ii) $24 + (n-1)5 > 150$

$24 + 5n - 5 > 150$

$19 + 5n > 150$

$5n > 131$

$\therefore n > 26\frac{1}{5}$

$\therefore n = 27.$ (2)

(iii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore 2895 = \frac{n}{2}[2 \times 24 + (n-1)5]$

$5790 = n[48 + (n-1)5]$

$5790 = n(43 + 5n)$

$\therefore 5n^2 + 43n - 5790 = 0$

$\therefore n = \frac{-43 \pm \sqrt{43^2 + 4 \times 5 \times 5790}}{2 \times 5}$

$= \frac{-43 \pm \sqrt{117649}}{10}$

$= \frac{-43 \pm 343}{10}$

$= 30$ or -38.6

$\therefore n = 30$ since $n > 0$ (2)

Question 4

(a)

Using cosine rule,

$\cos\theta = \frac{4^2 + 5^2 - 7^2}{2 \times 4 \times 5}$

$= -\frac{8}{40}$

$\therefore \theta = 101^\circ\ 32'$ (2)

(b) (i)

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,
5,1	5,2	5,3	5,4	5,5	5,
6,1	6,2	6,3	6,4	6,5	6,

(1

P(2nd score > 1st score)

$= \frac{15}{36}$

$= \frac{5}{12}.$ (1)

(ii) P(total of 2 scores is 7 or 11,

$= \frac{8}{36}$

$= \frac{2}{9}.$ (1)

(c) (i) $16^2 = 256$

$12^2 = 144$

$20^2 = 400.$

Since $20^2 = 16^2 + 12^2$

Δ is right-angled at A (1)

(ii) Area of AO_2BO_1

$= 2 \times \left(\frac{1}{2} \times 16 \times 12\right)$

$= 192\ cm^2$ (1)

(iii) $\sin\alpha = \frac{12}{20}$

$\therefore \alpha = 36^\circ 52'$

$\therefore \beta = 90^\circ - 36^\circ 52'$

$= 53^\circ\ 8'$ (2)

(iv) Shaded area

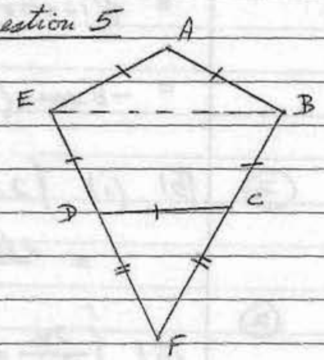
$= \frac{1}{2} \times 16^2 (2 \times 0.644 - \sin 2 \times 0.644)$

$+ \frac{1}{2} \times 12^2 (2 \times 0.927 - \sin 2 \times 0.927)$

$= 106\ cm^2$ (nearest cm^2). (3)

Question 5

(a)



(i)

$\angle$ sum of polygon $= (n-2) \times 180^\circ$

$\therefore \angle$ sum of pentagon $= (5-2) \times 180^\circ$

$= 540^\circ$

$\therefore$ each interior $\angle = \frac{540^\circ}{5} = 108^\circ$ (1)

(ii) $\angle EDC + \angle CDF = 180^\circ$ (str. $\angle$

$\therefore 108^\circ + \angle CDF = 180^\circ$

$\therefore \angle CDF = 72^\circ$

Similarly, $\angle DCF = 72^\circ$

$\therefore \Delta FCD$ is isosceles (1

(iii) $FC = FD$ (isos. Δ CDF

$CB = DE$ (sides of regul
pentagon)

$\therefore FB = FE$

$\therefore \frac{FC}{FB} = \frac{FD}{FE}$

$\angle F$ is common

$\therefore \Delta FCD\ |||\ \Delta FBE$ (2)

(iv) $\frac{FC}{FB} = \frac{DC}{EB}$ (matching sid
of sim. Δs
proportiona

Let $FC = x$,

$\therefore \frac{x}{x+5} = \frac{5}{8}$

$\therefore 8x = 5(x+5)$

$8x = 5x + 25$

$\therefore 3x = 25$

$\therefore x = 8\frac{1}{3}$ cm (2)

(b) (i) $3x^2 + 4x + 7 = 0$

$\alpha + \beta = \frac{-4}{3}$ (1)

(ii) $\alpha\beta = \frac{7}{3}$ (1)

(iii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{4}{3}\right)^2 - 2\times\frac{7}{3}$

$= \frac{16}{9} - \frac{14}{3}$

$= -2\frac{8}{9}$ ①

(iv) $\frac{1}{\alpha}+\frac{1}{\beta}$

$= \frac{\alpha+\beta}{\alpha\beta}$

$= -\frac{4}{3} \div \frac{7}{3}$

$= -\frac{4}{7}$ ①

(c) $a = 3(4+t)^2$

$\therefore v = \int 3(4+t)^2\,dt$

$= \frac{3(4+t)^3}{3} + c$

$= (4+t)^3 + c$

when $t=0$, $v=64$

$\therefore 64 = c + 4^3$

$\therefore c = 0$

$\therefore V = (4+t)^3$ ②

Question 6

(a) $e^{2x} - e^{x} - 6 = 0$

Let $t = e^x$

$\therefore t^2 - t - 6 = 0$

$(t-3)(t+2) = 0$

$\therefore t = 3$ or $t = -2$

$\therefore e^x = 3$ or $e^x = -2$ (no solution)

$\therefore x \ln e = \ln 3$

$\therefore x = \ln 3$ ②

(b) $f(x) = \frac{4x^3 - x^4}{9}$

(i) Curve cuts x-axis when $f(x) = 0$

$\therefore \frac{4x^3 - x^4}{9} = 0$

$\therefore 4x^3 - x^4 = 0$ since $9 \neq 0$

$\therefore x^3(4-x) = 0$ ②

$\therefore x = 0$ or $x = 4$

$\therefore$ coords. are $(0,0)$ & $(4,0)$

(ii) Stat. pts when $f'(x) = 0$.

$f'(x) = \frac{1}{9}(12x^2 - 4x^3)$

$= 0$

$\therefore 12x^2 - 4x^3 = 0$

$4x^2(3-x) = 0$

$\therefore x = 0$ or $x = 3$

When $x=0$, $y=0$

" $x=3$, $y = \frac{4\times3^3 - 3^4}{9} = 3$

$\therefore$ stat. pts at $(0,0)$ & $(3,3)$

Test $(0,0)$:

When $x = -0.1$, $f'(x) = 0.014 > 0$

" $x = 0.1$, $f'(x) = 0.013 > 0$

$\therefore (0,0)$ is a horizontal P.O.I

Test $(3,3)$:

When $x = 2.9$, $f'(x) = 0.37 > 0$

" $x = 3.1$, $f'(x) = -0.43 < 0$ ③

$\therefore$ max. t. pt. at $(3,3)$.

(iii) Possible pts. of inflexion when $f''(x) = 0$.

$f''(x) = \frac{1}{9}(24x - 12x^2)$

$= 0$

$\therefore 24x - 12x^2 = 0$

$12x(2-x) = 0$

$\therefore x = 0$ or $x = 2$.

when $x=0$, $y=0$

" $x=2$, $y = \frac{4\times2^3 - 2^4}{9}$

$= 16/9$.

Now $(0,0)$ is a horizontal pt. of inflexion from (ii) above.

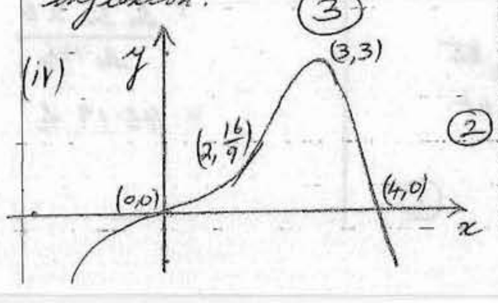
Test $\left(2, \frac{16}{9}\right)$:

When $x = 1.9$, $f''(x) = 0.25 > 0$

" $x = 2.1$, $f''(x) = -0.28 < 0$

$\therefore$ since change in concavity occurs $\left(2, \frac{16}{9}\right)$ is a point of inflexion. ③

(iv) ②



Question 7

(a) (i) $kx^2 = -6x + 3$

$\therefore kx^2 + 6x - 3 = 0$ (1

(ii) $\Delta = 0$,

$\therefore 6^2 - 4k(-3) = 0$

$36 + 12k = 0$

$\therefore k = -3$ ②

(iii) $-3x^2 + 6x - 3 = 0$

$\therefore 3x^2 - 6x + 3 = 0$

$3(x^2 - 2x + 1) = 0$

$3(x-1)^2 = 0$

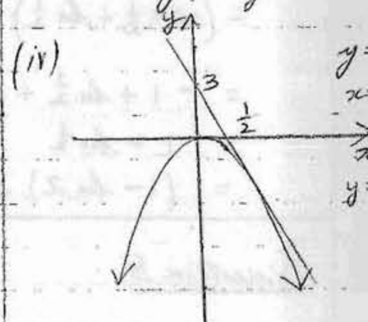
$\therefore x = 1$. ②

using $y = -3x^2$

$= -3\times1^2 = -3$

$\therefore$ coords. of pt. of contact (1;

(iv)



$y = -6x + 3$

$x=0, y=3$

$y=0, x=$

②

(b) (i) $2\sin x = \tan x$

$= \frac{\sin x}{\cos x}$ (cos x ≠

$\therefore 2\sin x\cos x = \sin x$

$\therefore 2\sin x\cos x - \sin x = 0$

$\therefore \sin x(2\cos x - 1) = 0$

$\therefore \sin x = 0$ or $2\cos x - 1 = 0$

$\therefore x = 0$ or $\cos x = \frac{1}{2}$

$\therefore x = \frac{\pi}{3}$

when $x = \frac{\pi}{3}$,

$y = \tan\frac{\pi}{3} = \sqrt{3}$

$\therefore A\left(\frac{\pi}{3}, \sqrt{3}\right)$ (2)

(ii) $\frac{d}{dx}[\ln\cos x]$

$= \frac{-\sin x}{\cos x}$

$= -\tan x$ (1)

(iii) $A = \int_0^{\pi/3} (2\sin x - \tan x)\,dx$

$= \left[-2\cos x + \ln(\cos x)\right]_0^{\pi/3}$

$= \left(-2\times\frac{1}{2} + \ln\frac{1}{2}\right) - (-2\times 1 + \ln 1)$

$= -1 + \ln\frac{1}{2} + 2$

$= 1 + \ln\frac{1}{2}$

$= (1 - \ln 2)$ units^2 (2)

Question 8

(a) (i) $P = P_0 e^{kt}$

$\frac{dP}{dt} = P_0 \times k e^{kt}$

$= k \times P_0 e^{kt}$

$= kP$ (1)

(ii) $1400000 = 900000 e^{6k}$

$\therefore \frac{14}{9} = e^{6k}$

$\therefore \ln\frac{14}{9} = 6k$

$\therefore k = \frac{1}{6}\ln\frac{14}{9}$

$\therefore k = 0.0736$ (4 d.p.) (2)

(iii) When $t = 10$,

$P = 900000 e^{10k}$

where $k = \frac{1}{6}\ln\frac{14}{9}$

$= 1879540.637$

$= 1879540$ (nearest whole no.) (1)

(iv) $3000000 = 900000 e^{kt}$

where $k = \frac{1}{6}\ln\frac{14}{9}$

$\therefore \frac{30}{9} = e^{kt}$

$\therefore \ln\frac{30}{9} = kt$

$\therefore t = \frac{1}{k}\ln\frac{30}{9}$

$= \ln\frac{30}{9} \div \frac{1}{6}\ln\frac{14}{9}$

$= \frac{\ln 30 \times 6}{\ln 14/9}$

$= 46.19$ h (1)

(i) In $\triangle$s ABM and BCN

$AB = BC$ (sides of a square)

$BM = CN$ (M, N are mid-point of equal sides)

$\angle ABM = \angle BCN = 90^\circ$ ($\angle$s of a square)

$\therefore \triangle ABM \equiv \triangle BCN$ (SAS) (4)

(ii) $\angle CBN = \angle BAM$ (matching $\angle$s of congruent $\triangle$s).

Let $\angle ABP = x^\circ$, $\angle BAM = y^\circ$

$\therefore \angle CBN = y^\circ$

$\therefore x^\circ + y^\circ = 90^\circ$ ($\angle ABC$)

$\therefore \angle ABP = 90^\circ$ ($\angle$ sum of $\triangle ABP$)

$\therefore AM \perp BN$. (3)

Question 9

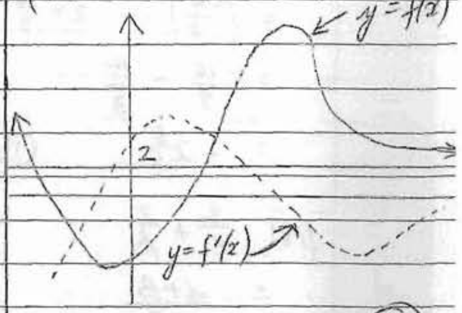
(a) $\int_2^5 f(x)\,dx$

$\doteqdot \frac{1}{2}\left[0.693 + 2(1.099 + 1.386) + 1.609\right]$

$= \frac{1}{2} \times 7.272$

$= 3.636$. (2)

(b)



(2)

(c) (i) $x^2 + 6x = 20y - 49$

$x^2 + 6x + 3^2 = 20y - 49 + 3^2$

$(x+3)^2 = 20(y-2)$

of form $(x-h)^2 = 4a(y-k)$

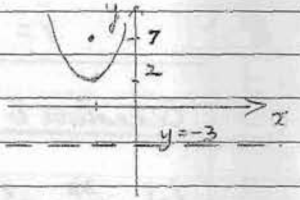
$4a = 20$

$\therefore a = 5$ (2)

(ii) Vertex (h, k)

$\therefore V(-3, 2)$ (1)

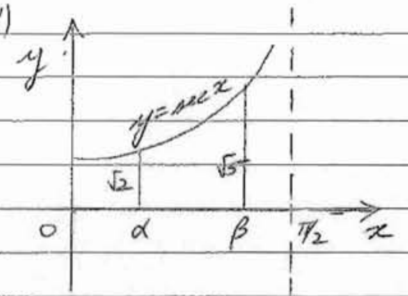
(iii) Focus $(-3, 7)$ (1)



(iv) Equation of directrix is

$y = -3$. (1)

(d)



$$V_x = \pi \int_\alpha^\beta \sec^2 x \, dx$$

$$= \pi \left[\tan x \right]_\alpha^\beta$$

$$= \pi(\tan\beta - \tan\alpha)$$

Now $\sec\alpha = \sqrt{2}$ & $\sec\beta = \sqrt{5}$

$\therefore \sec^2\alpha = 2 \qquad \sec^2\beta = 5$

$\therefore 2 - \tan^2\alpha = 1$ & $5 - \tan^2\beta = 1$

$\therefore \tan^2\alpha = 1 \qquad \tan^2\beta = 4$

$\therefore \tan\alpha = \pm 1 \qquad \tan\beta = \pm 2$

Now $\tan\alpha$ & $\tan\beta$ are positive since α and β lie between 0 and $\pi/2$

$\therefore \tan\alpha = 1$ and $\tan\beta = 2$

$$\therefore V_x = \pi(2-1)$$

$$= \pi \text{ unit}^3$$ (3)

Question 10

(a) (i) $n = 48$, $r = 1\%$ p. month, $A_{48} = 0$, $R = 1.01$.

$A_1 = 20000 - M$

$A_2 = (20000 - M) - M$

$= 20000 - 2M.$ (1)

(ii) $A_3 = (20000 - 2M)1.01 - M$

$= 20000(1.01) - 2M(1.01) - M$

$A_4 = A_3(1.01) - M$

$= [20000(1.01) - 2M(1.01) - M]1.01 - M$

$= 20000(1.01)^2 - 2M(1.01)^2 - M(1.01) - M$

$A_5 = A_4(1.01) - M$

$= [20000(1.01)^2 - 2M(1.01)^2 - M(1.01) - M] \times 1.01 - M$

$= 20000(1.01)^3 - 2M(1.01)^3 - M(1.01)^2 - M(1.01) - M$

as required.

(2)

(iii) $A_{48} = 20000(1.01)^{46} - 2M(1.01)^{46} - M(1.01)^{45} - M(1.01)^{44} - \cdots - M(1.01) - M$

$= 20000(1.01)^{46} - 2M(1.01)^{46} - M[1.01^{45} + 1.01^{44} + \cdots + 1.01$

$= 20000(1.01)^{46} - 2M(1.01)^{46} - M[1 + 1.01 + \cdots + 1.01^{44} + 1.01^{45}]$

a geometric series with $a = 1$, $r = 1.01$, $n = 46$

$= 20000(1.01)^{46} - 2M(1.01)^{46} - M\left(\frac{1.01^{46} - 1}{0.01}\right)$

Since $A_{48} = 0$,

$\therefore 2M(1.01)^{46} + M\left(\frac{1.01^{46} - 1}{0.01}\right) = 20000(1.01)^{46}$

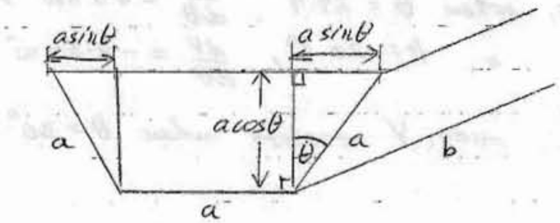
$\therefore M\left[2(1.01)^{46} + \frac{1.01^{46} - 1}{0.01}\right] = 20000(1.01)^{46}$

$61.20680318\, M = 31609.17709$

$\therefore M = 516.4324151$

$\therefore M = \$516.43$ (3)

(b)



(i) $A = \frac{1}{2}(2a + 2a\sin\theta) \times a\cos\theta$

$= (a + a\sin\theta)a\cos\theta$

$= a^2\cos\theta + a^2\sin\theta\cos\theta$

$\therefore V = b(a^2\cos\theta + a^2\sin\theta\cos\theta)$

$= a^2 b\cos\theta + a^2 b\sin\theta\cos\theta$

$= a^2 b\cos\theta(1 + \sin\theta)$ as required (2)

(ii) $\frac{dV}{d\theta} = a^2b \times \frac{d}{d\theta}(\cos\theta) + a^2b \times \frac{d}{d\theta}(\sin\theta\cos\theta)$

$= -a^2b\sin\theta + a^2b(-\sin^2\theta + \cos^2\theta)$

$= a^2b(-\sin\theta - \sin^2\theta + \cos^2\theta)$

$= a^2b(-\sin\theta - \sin^2\theta + 1 - \sin^2\theta)$

$= a^2b(1 - 2\sin^2\theta - \sin\theta)$ (2)

(iii) For maximum V, $\frac{dV}{d\theta} = 0$.

$\therefore 1 - 2\sin^2\theta - \sin\theta = 0$ since $a^2b > 0$.

$\therefore 2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\therefore 2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2}$ $\qquad$ $\sin\theta = -1$

$\therefore \theta = \pi/6$ or $\theta = -\pi/2, 3\pi/2, \ldots$

$\therefore \theta = \frac{\pi}{6}$ since θ is acute

$\therefore \theta = 30°$.

Now, when $\theta = 29.9°$, $\frac{dV}{d\theta} = 4.5 \times 10^{-3} > 0$

" $\theta = 30.1°$, $\frac{dV}{d\theta} = -4.5 \times 10^{-3} < 0$ (2)

$\therefore$ max. V occurs when $\theta = 30°$.