GIRRAWEEN HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

2012

General Instructions:

- Reading time - 5 minutes
- Working time – 3 hours
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working for questions 11-16

Total marks – 100

Section I **10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II **90 Marks**

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Section 1 Circle the letter corresponding to the correct answer.

1. A normal die is tossed 3 times. The probability of getting no twos is:

   A. $\frac{1}{2}$ B. 0 C. $\frac{5}{6}$ D. $\frac{125}{216}$

2. Solve $3(2x-5)+3\leq-2(x+2)$

   A. $x\leq1$ B. $x\geq1$ C. $x\leq0$ D. x ⍰ 5

3. Solve $12-\sqrt[3]{5n-2}=15$

   A. $n=0$ B. $n=-5$ C. $n=5$ D. $n=\frac{9}{5}$

4. Simplify $\dfrac{\frac{1}{xy}+\frac{1}{x^2}}{\frac{1}{xy^2}+\frac{1}{x^2y}}$

   A. $\frac{y}{x+y}$ B. 1 C. y D. x

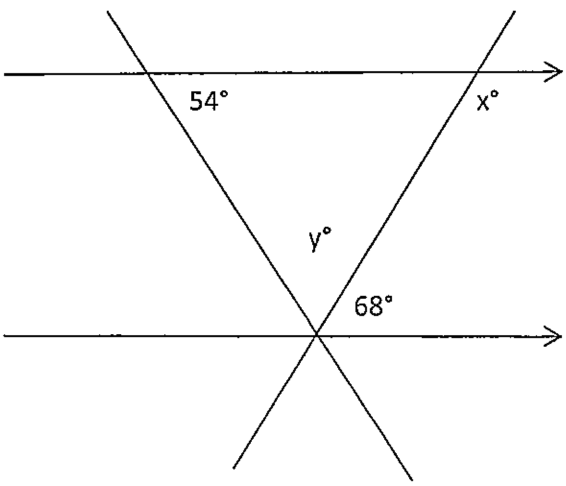
5. Solve $|4-8x|\geq20$

   A. $-2\leq x\leq3$ B. x ⍰ -2 C. $x\leq-2, x\geq3$ D. $x\leq-2, x\geq2$

6. The vertex of the parabola $f(x)=2x^2-16x+24$ is

   A. (4, -24) B. (4, -8) C. (-4, 8) D. (-4, -12)

7. Find the values of x and y in the following diagram.



   A. X = 112, y = 58 B. x = 112, y = 48 C. x = 126, y = 58 D. x = 126, y = 48

8. The pilot of an aeroplane notices a pond at an angle of depression of 36°.
If the distance from a point on the ground directly below the airplane to the pond is 500m, what is the height of the aeroplane (to the nearest metre)?

A. 688m B. 363m C. 850m D. 618m

9. A right cylinder has a surface area of 1600π cm^2, and the radius and the height are equal.
The radius of the cylinder is:

A. $20\sqrt{2}$ B. 40 C. $10\sqrt{2}$ D. 20

10. Mrs Venkataya's lolly jar has 12 red and 8 green mints. Duncan picks a mint and eats it, then picks another mint. Find the probability that he picks a red mint first and then the green one.

A. $\frac{6}{25}$ B. $\frac{21}{95}$ C. $\frac{24}{95}$ D. $\frac{21}{100}$

Section II (90 Marks) Write your answers on the paper provided.

Question 11 (15 marks)

a. Factorise $49x^2 - 16$ **1**

b. Find the exact value of $\tan\frac{2\pi}{3}$ **1**

c. The line $6x - ky = 4$ passes through (3, 2). Find the value of k. **2**

d. $A(-1,-2)$, $B(0,3)$ and $C(5,4)$ are the vertices of ΔABC.

(i) Plot these points on a number plane and draw ΔABC.

(ii) E is the midpoint of AC. Find the coordinates of E. **1**

(iii) Find the gradient of AC. **1**

(iv) A line l is drawn through B, perpendicular to AC.
Find the equation of the line l. **2**

(v) Show that E lies on the line l. **1**

(vi) On your diagram, draw line l and plot E. Prove that $\Delta BEC \equiv \Delta BEA$. **3**

(vii) AC is a diameter of a circle.

(I) Calculate the radius of the circle. **2**

(II) Find the equation of the circle. **1**

Question 12 (15 marks)

a. On a number plane shade in the region given by the two conditions $x^2+y^2 \geq 2$ and $2x+y \geq 2$. **2**

b. Differentiate

(i) $4x^3-2x+\dfrac{2}{x^2}+3$ **2**

(ii) $x\log_e x$ **2**

c. Find

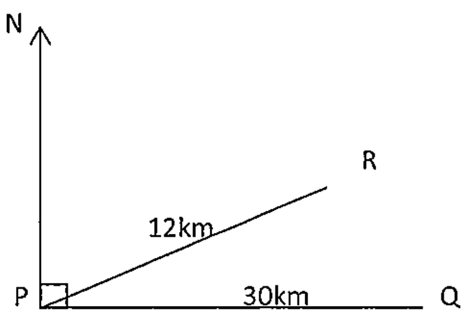
(i) $\int(e^{-x}+1)dx$ **1**

(ii) $\int_0^{\frac{\pi}{2}}\cos 2x\,dx$ **2**

d. (i) Differentiate e^{x^2} **1**

(ii) Hence, evaluate $\int_0^1 xe^{x^2}dx$ **1**

e.



The diagram shows a point P which is 30km due west of point Q.

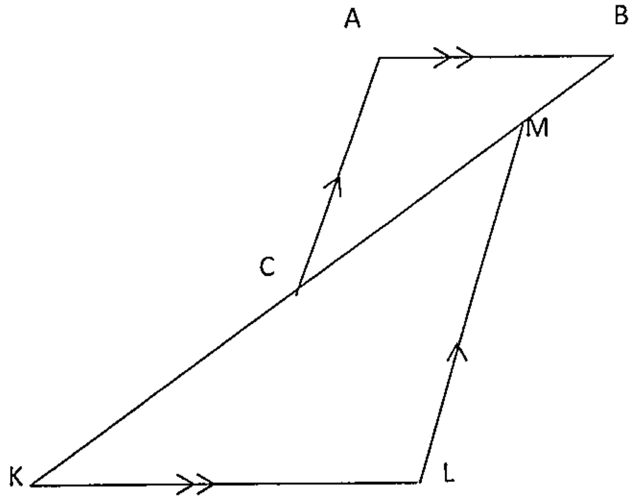
The point R is 12km from P and has a bearing of $070°$ from P.

(i) Find the distance of R from Q (to 1 decimal place). **2**

(ii) Find the bearing of R from Q (to the nearest degree). **2**

Question 13 (15 marks)

a.



In the diagram, AB // KL and AC // ML. AB = 4cm, MB = 2cm, KL = 8cm and KC = 7cm.

(i) Show that ΔABC /// ΔKLM. **3**

(ii) Find the length of CM. **2**

b. Rajnikant received 30 tonnes of topsoil for his yard. He uses a wheel barrow which can hold 150kg to move the soil.

(i) How many loads in the wheelbarrow will he need? **1**

(ii) He begins at the pile of topsoil and deposits the first load 3m away from the pile. Each successive load is dumped half a metre further from the pile.

(I) How far from the pile will he leave the final load? **2**

(II) What is the total distance that he will have travelled if he finished back at the starting point? **2**

c. For the parabola

$x^2 - 6x - 2y + 7 = 0$, find

(i) the focal length. **2**

(ii) the coordinates of the vertex. **1**

(iii) the coordinates of the focus. **1**

(iv) the equation of the directrix. **1**

Question 14 (15 marks)

a. Monique and Kaavya are playing in a tennis tournament and will play 2 sets. Each has an equal chance of winnng the first set. If Monique wins the first set, the probability of her winning the second set is 0.7. If Monique loses the first set, the probability of her winning the second set is 0.4.

(i) Draw a probability tree to represent this information. **2**

(ii) Find the probability that Monique wins exactly one set. **2**

b. Consider the curve given by $y = 1 - 12x + x^3$ for $-4 \le x \le 4$.

(i) Find the stationary points and determine their nature. **4**

(ii) Find the point of inflexion. **2**

(iii) Sketch the curve for $-4 \le x \le 4$. **3**

(iv) What is the minimum value of y for $-4 \le x \le 4$ and where does it occur? **2**

Question 15 (15 marks)

a. If α and β are the roots of the equation $3x^2 + 4x + 7 = 0$, find the value of $\alpha^2 + \beta^2$. **2**

b. The table shows the values of a function $f(x)$

x	0	2	4	6	8
$f(x)$	0.9	1.4	1.8	2.1	1.7

Use Simpson's Rule to find an approximation of $\int_0^8 f(x)dx$, using five function values. **2**

c. A ball is dropped into a long vertical tube filled with honey. The rate at which the ball decelerates is proportional to its velocity i.e. $\frac{dV}{dt} = -kV$,

where V is the velocity in cm/s, t is the time in seconds and k is a constant.
Initially, V = 100cm/s and when t = 0.25, V = 85cm/s.

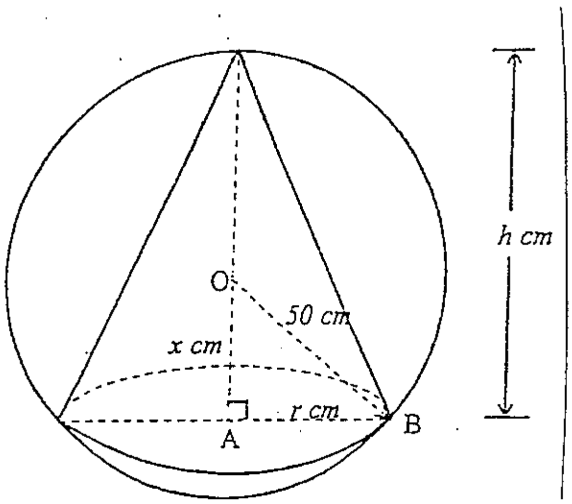
(i) Show that $V = Ce^{-kt}$ is a solution to the differential equation. 1

(ii) Find the value of C. 1

(iii) Find the value of k. 2

(iv) Find the velocity when t = 2 seconds. 1

d.



The diagram shows a cone of base radius r cm and height h cm. inscribed in a sphere of radius 50cm.

The centre of the sphere is O and $\angle OAB = 90°$ and $OA = xcm$.

(i) Show that $r = \sqrt{2500 - x^2}$. 2

(ii) Hence, show that the volume of the cone is given by $V = \frac{\pi}{3}(2500 - x^2)(50 + x)$. 1

(iii) Find the radius of the largest cone which can be inscribed in the circle. 3

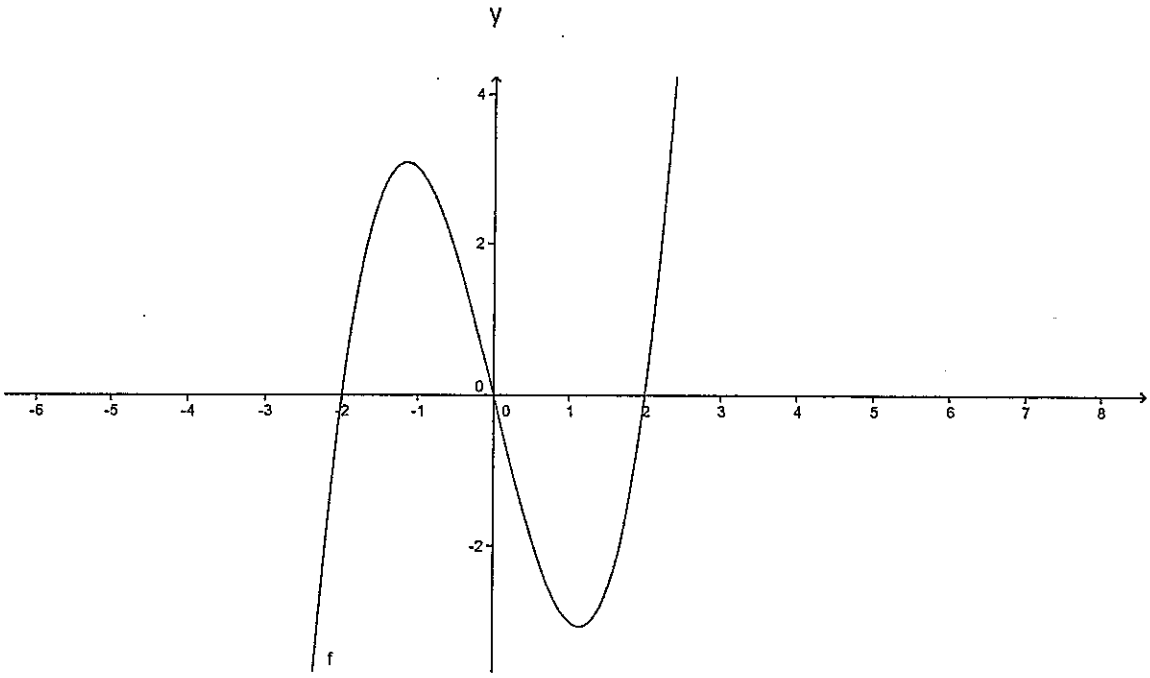
Question 16 (15 marks)

a. (i) Show that 2^x can be written as $e^{x\ln 2}$. **1**

(ii) Hence, find $\int_0^1 2^x \, dx$. **2**

b. Find the volume of the solid of revolution formed by rotating the curve $y = x - \frac{1}{x}$ about the x-axis between $x = 1$ and $x = 4$. **3**

c. The sketch below shows the graph of $y = f'(x)$ which is the derivative of a function $y = f(x)$.



(i) Find the x value(s) of all the turning points on the curve $y = f(x)$. **1**

(ii) Find the x value(s) of any point of inflection on the curve $y = f(x)$. **1**

(iii) Draw a possible sketch of $y = f''(x)$, the second derivative of $y = f(x)$. **1**

d. Joshua wins $100 000 in a lottery. He places it in an investment account which pays Interest 9% p.a. compounded monthly. He plans to withdraw $M at the end of each year immediately after the twelfth interest payment and he wants the investment to last 25 years.

(i) Write an expression for A_1, the amount of money in the account immediately after his first withdrawal of $M. **1**

(ii) Write a similar expression for A_2. **2**

(iii) Find the value of M to the nearest dollar. **3**

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

GHS TRIAL HSC MATHEMATICS

SOLUTIONS 2012

Section 1 (10 marks)

1. D 2. A 3. B 4. C
5. C 6. B 7. A 8. B
9. D 10. C

Section 2 (90 marks)

Q11 (15 marks)

a) $49x^2-16$

$= (7x)^2 - 4^2$

$= (7x-4)(7x+4)$ ①

b) $\tan \frac{2\pi}{3} = -\sqrt{3}$ ①

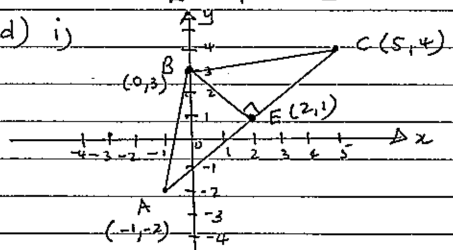
c) $6x - ky = 4$ $(3,2)$

$6(3) - 2k = 4$

$2k = 14$

$k = 7$ ②

d) i)



ii) $E = \left(\frac{-1+5}{2}, \frac{-2+4}{2}\right)$

$E = (2,1)$ ①

iii) $m_{AC} = \frac{y_2-y_1}{x_2-x_1}$

$= \frac{4+2}{5+1}$ ①

$m_{AC} = 1$

iv) $m_L = -1$, pt $B(0,3)$

Equation of L

$y - y_1 = m(x-x_1)$

$y - 3 = -1(x-0)$

$y - 3 = -x$

$x + y - 3 = 0$ ②

v) $l: x+y-3=0$ $E(2,1)$

LHS $= 2+1-3$

$= 0 =$ RHS

$\therefore$ E lies on l ①

vi) In $\triangle BEA$ and $\triangle BCE$,

$AE = EC$ (E midpoint of AC)

$\angle BEA = \angle BEC = 90^\circ$ ($BE \perp AC$)

BE is common

$\therefore \triangle BEA \equiv \triangle BEC$ (SAS) ③

vii) I) AC is diameter

$\therefore$ AE is the radius (E midpt of AC)

$AE = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \sqrt{3^2+3^2}$

$= \sqrt{18}$

radius $= 3\sqrt{2}$ units ②

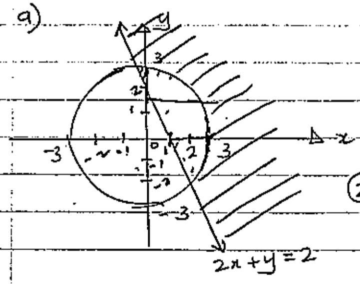
II) Equation of circle

centre $(2,1)$ $r = 3\sqrt{2}$

$(x-2)^2 + (y-1)^2 = 18$ ①

Q12 (15 marks)

a)



②

b) i) $y = 4x^3 - 2x + 2x^{-2} + 3$

$\frac{dy}{dx} = 12x^2 - 2 - 4x^{-3}$

$= 12x^2 - 2 - \frac{4}{x^3}$ ②

ii) $y = x\log_e x$

$u = x$, $u' = 1$, $v = \log_e x$, $v' = \frac{1}{x}$

$\frac{dy}{dx} = vu' + uv'$

$= \log_e x + 1$ ②

c) i) $\int (e^{-x}+1)\,dx$

$= -e^{-x} + x + c$ ①

ii) $\int_0^{\pi/2} \cos 2x \, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\sin \pi - \sin 0\right]$

$= 0$ ②

d) i) $y = e^{x^2}$

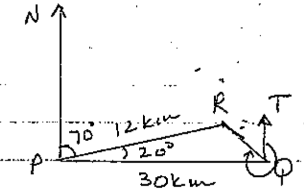
$\frac{dy}{dx} = 2xe^{x^2}$ ①

ii) $\int_0^1 xe^{x^2}\,dx$

$= \left[\frac{1}{2}e^{x^2}\right]_0^1$

$= \frac{1}{2}(e-1)$ ②

e)



i) In $\triangle PRQ$

$RQ^2 = 12^2 + 30^2 - 2\times12\times30\cos 20^\circ$

$RQ = 19.2$ km ②

ii) In $\triangle PRQ$

$\frac{\sin \angle RQP}{12} = \frac{\sin 20^\circ}{19.2}$

$\sin \angle RQP = \frac{12\sin 20^\circ}{19.2}$

$\angle RQP = 12^\circ 34'$

Bearing of R from Q

$= 12^\circ 34' + 270^\circ$

$= 282^\circ$ (nearest degree) ②

Q13 (15 marks)

a) i) In ΔACB and ΔLMK

$\angle ACB = \angle KML$ (alternate $\angle$s $AB \parallel ML$)

$\angle ABC = \angle MKL$ (alternate $\angle$s, $AB \parallel KL$)

$\therefore \Delta ACB \parallel\parallel \Delta KLM$ (equiangular) (3)

ii) $\frac{BC}{KM} = \frac{AB}{KL}$ (ratio of matching sides of similar Δs)

$\frac{2+x}{x+7} = \frac{1}{2}$

$4 + 2x = x + 7$

$x = 3$ cm (2)

b) i) Loads $= \frac{30\,000 \text{ kg}}{150 \text{ kg}}$

$= 200$ loads (1)

ii) I) 3, 3.5, 4, ...

$T_n = a + (n-1)d$

$= 3 + 199(0.5)$ (2)

$= 102.5$ m

II) 6, 7, 8, ...

$a = 6$, $d = 1$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$= 100[12 + 199(1)]$

$= 21\,100$ m

$= 21.1$ km (2)

c) $x^2 - 6x = 2y - 7$

$x^2 - 6x + 9 = 2y + 2$

$(x-3)^2 = 2(y+1)$

i) $4a = 2$

$a = \frac{1}{2}$ (2)

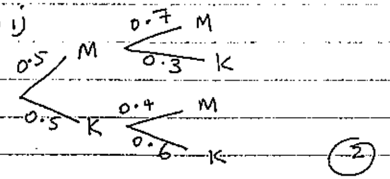
(ii) Vertex : $(3, -1)$ (1)

iii) Focus $(3, -\frac{1}{2})$ (1)

iv) Directrix : $y = -\frac{3}{2}$ (1)

Q14 (15 marks)

a) i)



(2)

ii) P(MK or KM)

$= (0.5 \times 0.3) + (0.5 \times 0.4)$

$= 0.35$ (2)

b) i) $y = 1 - 12x + x^3$

$-4 \le x \le 4$

$\frac{dy}{dx} = -12 + 3x^2$

SP $\Rightarrow \frac{dy}{dx} = 0$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

when $x = 2$, $y = -15$ pt $(2, -15)$

when $x = -2$, $y = 17$ pt $(-2, 17)$

$\frac{d^2y}{dx^2} = 6x$

At $x = 2$, $\frac{d^2y}{dx^2} = 12 > 0$

$\therefore$ minimum at $(2, -15)$

At $x = -2$, $\frac{d^2y}{dx^2} = -12 < 0$

$\therefore$ maximum at $(-2, 17)$ (4)

ii) Point of inflexion $\Rightarrow \frac{d^2y}{dx^2} = 0$

ie. $6x = 0$

$x = 0$

$\therefore$ Possible point of inflexion

Test for point of inflexion

x	-1	0	1
$\frac{d^2y}{dx^2}$	-6	0	6

there is a change in concavity

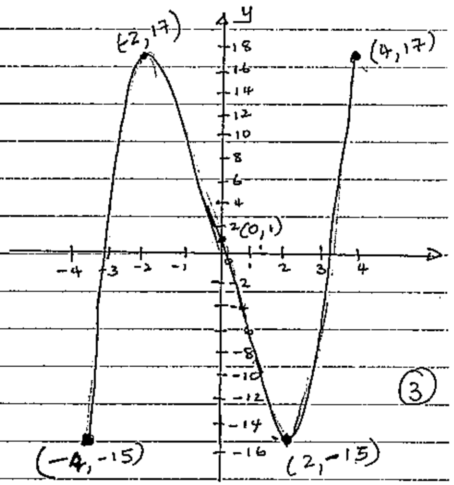
$\therefore$ there is a point of inflexion at $(0, 1)$ (2)

(iii) At $x = -4$, $y = 1 - 12(-4) + (-4)^3$

$= -15$ pt $(-4, -15)$

At $x = 4$, $y = 1 - 12(4) + (4)^3$

$= 17$ pt $(4, 17)$



(3)

iv) Minimum value of $y = -15$ occurs at $x = -4$ and $x =$ (2)

Q15 (15 marks)

a) $3x^2 + 4x + 7 = 0$

$\alpha + \beta = \frac{-b}{a} = \frac{-4}{3}$ $\alpha\beta = \frac{c}{a} = \frac{7}{3}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{-4}{3}\right)^2 - 2\left(\frac{7}{3}\right)$

$= -2\frac{8}{9}$ (2)

b) $h = 2$

$\int_0^8 f(x)\,dx$

$\doteqdot \frac{2}{3}\{0.9 + 4(1.4 + 2.1) + 2(1.8) + 1.7\}$

$\doteqdot 13.47$ (to 2dp) (2)

c) $\frac{dV}{dt} = -kV$

when $t = 0$, $V = 100\text{ cm}^3/\text{s}$

when $t = 0.25$, $V = 85\text{ cm}^3/\text{s}$

i) $V = Ce^{-kt}$

$\frac{dV}{dt} = -kCe^{-kt}$

$= -kV$

$\therefore V = Ce^{-kt}$ is a solution. (1)

ii) when $t = 0$, $V = 100\text{ cm}^3/\text{s}$

$100 = Ce^0$

$\therefore C = 100$ (1)

iii) when $t = 0.25$, $V = 85$

$100e^{-0.25k} = 85$

$-0.25k = \ln 0.85$

$k = -\frac{\ln 0.85}{0.25}$

$\doteqdot 0.650076$ (2)

(iv) $V = 100e^{+\frac{\ln 0.85}{0.25}t}$

when $t = 2$

$V = 100e^{+\frac{\ln 0.85}{0.25}(2)}$

$= 27.2\text{ cm}^3/\text{s}$ (1)

d) i) In ΔOAB,

$r^2 + x^2 = 50^2$

$r^2 = 2500 - x^2$

$r = \sqrt{2500 - x^2}$ (2)

(ii) $V = \frac{1}{3}\pi r^2 h$

h = radius of sphere $+ x$

$= 50 + x$

$V = \frac{\pi}{3}(2500 - x^2)(50 + x)$ (1)

(iii) $V = \frac{\pi}{3}(125\,000 + 2500x - 50x^2 - x^3)$

Max when $\frac{dV}{dx} = 0$

$\frac{dV}{dx} = \frac{\pi}{3}(2500 - 100x - 3x^2)$

ie. $3x^2 + 100x - 2500 = 0$

$(3x - 50)(x + 50) = 0$

$x = \frac{50}{3}$ or -50

$\therefore x = \frac{50}{3}$ $(x > 0)$

when $x = \frac{50}{3}$, $\frac{d^2V}{dx^2} = 6\left(\frac{50}{3}\right) + 100 > 0$ $\therefore$ max

when $x = \frac{50}{3}$

$r = \sqrt{2500 - \left(\frac{50}{3}\right)^2}$

Radius of largest cone $= 47.14$ cm (to 2dp)

(3)

Q16 (15 marks)

a) i) $e^{\ln 2} = 2$

$\therefore 2^x = e^{x\ln 2}$ (1)

ii) $\int_0^1 e^{x\ln 2}\,dx$

$= \frac{1}{\ln 2}\left[e^{x\ln 2}\right]_0^1$

$= \frac{1}{\ln 2}\left[e^{\ln 2} - e^0\right]$

$= \frac{1}{\ln 2}\left[2 - 1\right]$

$= \frac{1}{\ln 2}$ (2)

b) $y = x - \frac{1}{x}$

$y^2 = x^2 - 2 + \frac{1}{x^2}$

$V = \pi\int_1^4 y^2\,dx$

$= \pi\int_1^4 \left(x^2 - 2 + \frac{1}{x^2}\right)dx$

$= \pi\int_1^4 (x^2 - 2 + x^{-2})\,dx$

$= \pi\left[\frac{x^3}{3} - 2x - \frac{1}{x}\right]_1^4$

$= \pi\left[\left(\frac{64}{3} - 8 - \frac{1}{4}\right) - \left(\frac{1}{3} - 2 - 1\right)\right]$

$= \pi\left(13\frac{1}{12} - -2\frac{2}{3}\right)$

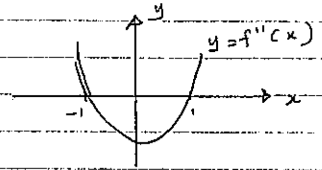
$= \frac{63\pi}{4}\ u^3$

(3)

c) i) $x = -2, 0, 2$ (1)

ii) $x = -1, 1$ (1)

iii)



(1)

d) i) $A_0 = 100\,000$

$r = \frac{9\%}{12} = 0.75\%$

$= 0.0075$

$A_1 = 100\,000(1.0075)^{12} - M$ (1)

ii) $A_2 = A_1(1.0075)^{12} - M$

$= (100\,000(1.0075)^{12} - M)(1.0075)^{12} - M$

$= 100\,000(1.0075)^{24} - (1.0075)^{12}M - M$

$= 100\,000(1.0075)^{24} - M(1.0075^{12} + 1)$ (2)

iii) After 25 years $A_n = 0$

$A_{25} = 100\,000(1.0075)^{300} - M\left[1.0075^{288} + 1.0075^{276} + \cdots + 1.0075^{12} + 1\right]$

GP with $a = 1$, $r = (1.0075)^{12}$, $n = 25$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1((1.0075)^{12 \times 25} - 1)}{(1.0075)^{12} - 1}$

$= 89.63$

$\therefore A_{25} = 0$

$\therefore 100\,000(1.0075)^{300} = M(89.63)$

$M = \frac{100\,000(1.0075)^{300}}{89.63}$

$\doteqdot 10\,496$ (3)