Girraween High School

2015 Year 12 Trial Higher School Certificate

Mathematics (2 unit)

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

- For **Section II: Questions 11 – 16** MUST be returned in clearly marked ***separate*** **sections.**
- On each page of your answers, clearly write:
  - the **QUESTION** being answered
  - **YOUR NAME**
  - your **Mathematics TEACHER'S NAME.**
- Start each new question on a **NEW PAGE.**
- You may ask for extra pieces of paper if you need them.

SECTION I

10 Marks

Attempt all of Questions 1 – 10

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1 – 10.

1. What is the value of $\dfrac{2\pi^2}{19}$, correct to 3 significant figures?

(A) 1.03

(B) 1.04

(C) 1.038

(D) 1.039

2. Which of the following is equal to $\dfrac{1}{3\sqrt{5} + \sqrt{2}}$?

(A) $\dfrac{3\sqrt{5} - \sqrt{2}}{13}$

(B) $\dfrac{3\sqrt{5} + \sqrt{2}}{13}$

(C) $\dfrac{3\sqrt{5} - \sqrt{2}}{43}$

(D) $\dfrac{3\sqrt{5} + \sqrt{2}}{43}$

3. Which of the following is equivalent to $\dfrac{\log_a 8}{\log_a 4}$?

(A) $1\dfrac{1}{2}$

(B) 2

(C) $\log_a 2$

(D) $\log_a 4$

4. The quadratic equation $2x^2 - 4x - 3 = 0$ has roots α and β.

What is the value of $(\alpha + \beta) - (\alpha\beta)$?

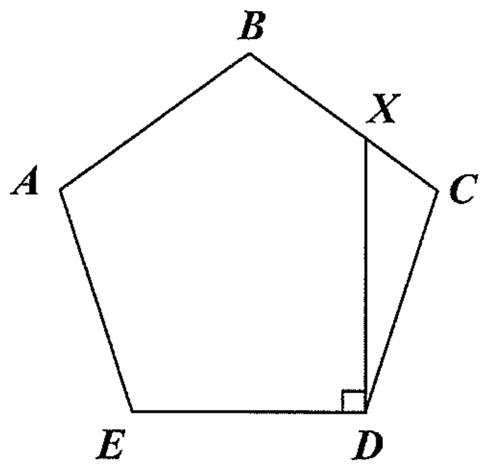
(A) $-\frac{1}{2}$

(B) $-3\frac{1}{2}$

(C) $\frac{1}{2}$

(D) $3\frac{1}{2}$

5.



NOT TO SCALE

$ABCDE$ is a regular pentagon and $DX \perp DE$.

The size of $\angle CXD$ is :

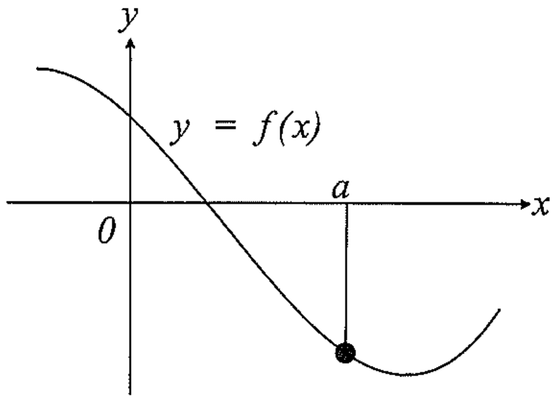
(A) 36°

(B) 54°

(C) 64°

(D) 72°

6. The diagram shows the graph of $y = f(x)$.



Which of the following statements is true?

(A) $f'(a) < 0$ and $f''(a) < 0$

(B) $f'(a) < 0$ and $f''(a) > 0$

(C) $f'(a) > 0$ and $f''(a) < 0$

(D) $f'(a) > 0$ and $f''(a) > 0$

7. A parabola has focus $(0, -4)$ and directrix $y = 2$.

What is the equation of the parabola?

(A) $x^2 = -12(y + 1)$

(B) $x^2 = -24(y + 4)$

(C) $x^2 = 12(y + 1)$

(D) $x^2 = 24(y + 4)$

8. What is the derivative of $\dfrac{x}{\sin x}$?

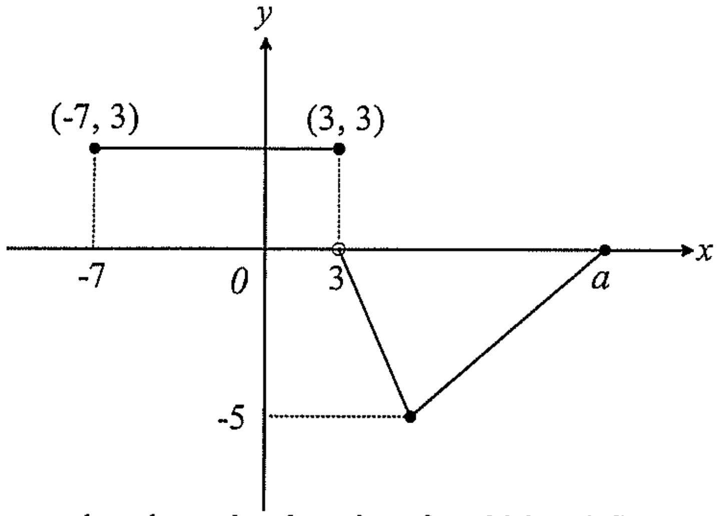
(A) $\dfrac{-x\cos x - \sin x}{\sin^2 x}$

(B) $\dfrac{x\cos x - \sin x}{\sin^2 x}$

(C) $\dfrac{\sin x + x\cos x}{\sin^2 x}$

(D) $\dfrac{\sin x - x\cos x}{\sin^2 x}$

9. The diagram shows the graph of $y = f(x)$.



NOT TO SCALE

Use the graph to determine the value of a which satisfies the condition $\int_{-7}^{a} f(x)\,dx = 0$.

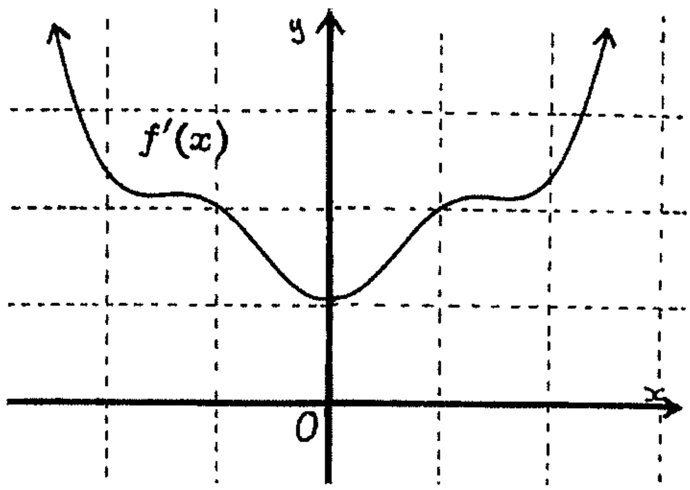
(A) 9

(B) 12

(C) 13

(D) 15

10.



It is known that $y = f(x)$ passes through the origin.

By examining the graph of $y = f'(x)$ shown above, state which of the following statements is true?

(A) $f(x)$ is an even function with a point of inflexion at $f(0)$.

(B) $f(x)$ is an even function with a local maximum at $f(0)$.

(C) $f(x)$ is an odd function with a point of inflexion at $f(0)$.

(D) $f(x)$ is an odd function with a local maximum at $f(0)$.

The examination continues on the next page.

SECTION II

90 Marks

Attempt all of Questions 11 – 16

Allow about 2 hours and 45 minutes for this section.

Answer each question on the paper provided.

Start each new question on a **NEW PAGE**.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) **Marks**

Start a NEW PAGE.

(a) Factorise $2x^2 + 11x - 21$. **2**

(b) Solve $|3x - 5| < 4$. **2**

(c) Find the equation of the tangent to the curve $y = x^3$ at the point where $x = 2$. **2**

(d) Differentiate $\left(e^{3x} - 5\right)^4$. **2**

(e) Evaluate $\displaystyle\int_1^2 \frac{1}{(3x - 1)^2}\, dx$ **3**

(f) (i) Find $\dfrac{d}{dx}(\sin x^3)$. **2**

(ii) Hence find $\displaystyle\int x^2 \cos x^3\, dx$. **2**

The examination continues on the next page.

Question 12 (15 Marks) **Marks**

Start a NEW PAGE.

(a) What is the radius of a circle in which an arc of length 15 cm subtends an angle of 60° at the centre? **2**

Give your answer correct to the nearest mm.

(b) Solve $2 \log_e x = \log_e (3x + 10)$. **2**

(c) Differentiate $y = \dfrac{\sin x}{1 + \cos x}$ **3**

and hence show that $\dfrac{dy}{dx} = \dfrac{1}{1 + \cos x}$.

(d) Find $\displaystyle\int \frac{10x}{x^2 - 8}\, dx$. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \cos (2x)\, dx$ **2**

(f) Mayomi and Christina each throw a die.

(i) Find the probability that they throw the same number. **2**

(ii) Find the probability that the number thrown by Christina is greater than the number thrown by Mayomi. **2**

The examination continues on the next page.

Question 13 (15 Marks) **Marks**

Start a NEW PAGE.

(a) The first and last terms of an arithmetic series are 10 and 60 respectively, and the sum of the series is 3535.

Find:

(i) the number of terms in the series. **2**

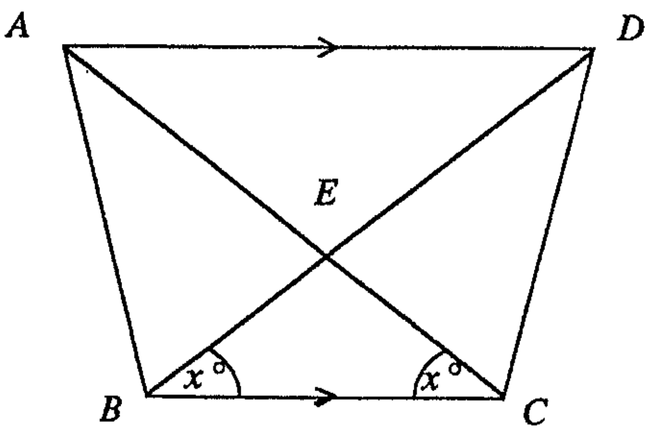
(ii) the common difference. **2**

(b) "Although the number of unemployed is increasing, the Government's policies to reduce unemployment seem to be taking effect." **2**

Given E is the number of unemployed, what does the above statement imply about $\frac{dE}{dt}$ and $\frac{d^2E}{dt^2}$?

(c) What is the volume of the solid of revolution formed by rotating the curve $y = \sec x$ about the x-axis for $0 \leq x \leq \frac{\pi}{3}$? **2**

(d)



In the diagram AD is parallel to BC and $\angle DBC = \angle ACB = x°$.

(i) Show that $AE = DE$. **2**

(ii) Prove that the triangles ΔABC and ΔDCB are congruent. **3**

(iii) Deduce that $\angle ABD = \angle DCA$. **2**

The examination continues on the next page.

Question 14 (15 Marks) **Marks**

Start a NEW PAGE.

(a) Consider the geometric series $1 + (\sqrt{5} - 2) + (\sqrt{5} - 2)^2 + \ldots$

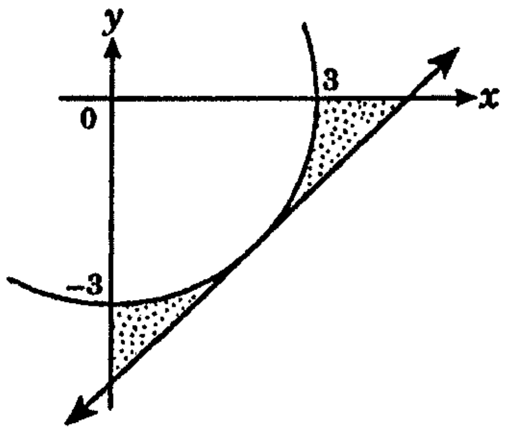
(i) Explain why the geometric series has a limiting sum. **1**

(ii) Find the exact value of the limiting sum. **2**

Write your answer with a rational denominator.

(b) Find the values of k for which the quadratic equation $x^2 + (k - 2)x + 4 = 0$ has no real roots. **2**

(c) (i) Prove that the line $3x - 4y = 15$ is a tangent to the circle $x^2 + y^2 = 9$. **2**



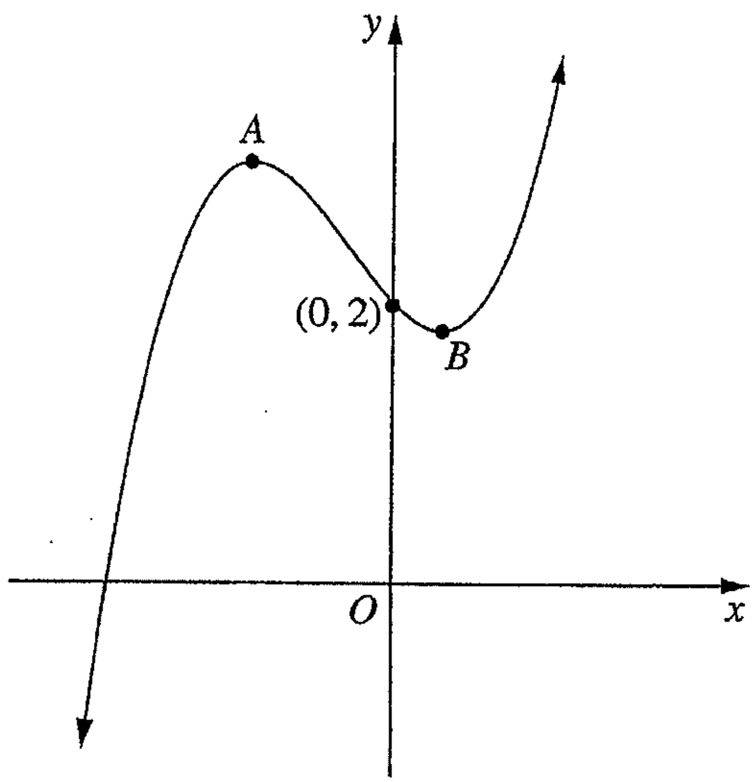
(ii) Calculate the exact area in the fourth quadrant between the tangent $3x - 4y = 15$ and the circle $x^2 + y^2 = 9$ (as shown by the shaded area in the diagram above). **2**

Question 14 continues on the next page.

Question 14 (continued)

(d) The graph of $y = x^3 + x^2 - x + 2$ is sketched below.

The points A and B are the turning points.



NOT TO SCALE

(i) Find the coordinates of A and B. **3**

(ii) For what values of x is the curve concave up? **2**

Give reasons for your answer.

(iii) For what values of k has the equation $x^3 + x^2 - x + 2 = k$ three real solutions? **1**

End of Question 14.

The examination continues on the next page.

Question 15 (15 Marks) **Marks**

Start a NEW PAGE.

(a) A standard pack of 52 cards consists of four suits (Diamonds, Hearts, Clubs and Spades) with 13 cards in each suit.

(i) One card is drawn from the pack and kept on the table. A second card is drawn and placed beside it on the table. **1**

What is the probability that the second card is from a different suit to the first?

(ii) The two cards are replaced and the pack shuffled. Four cards are chosen from the pack and placed side by side on the table. **2**

What is the probability that these four cards are all from different suits?

(b) A particle is initially at rest at the origin.

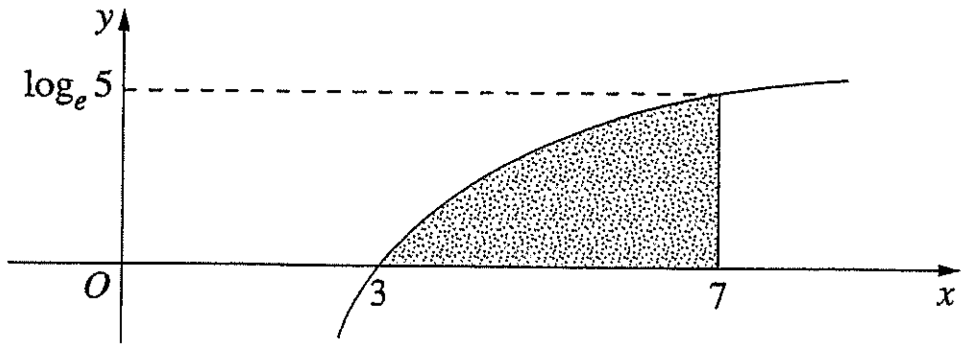
Its acceleration as a function of time t is given by $\ddot{x} = 4\sin 2t$.

(i) Show that the velocity of the particle is given by $\dot{x} = 2 - 2\cos 2t$. **2**

(ii) Sketch the graph of the velocity for $0 \leq t \leq 2\pi$ AND determine the time at which the particle first comes to rest after $t = 0$. **3**

(iii) Find the distance travelled by the particle between $t = 0$ and the time at which the particle first comes to rest after $t = 0$. **2**

(c)



In the diagram, the shaded region is bounded by $y = \log_e(x - 2)$, the x-axis and the line $x = 7$. **5**

Find the exact value of the area of the shaded region.

The examination continues on the next page.

Question 16 (15 Marks) **Marks**

Start a NEW PAGE.

(a) A drug is used to control a medical condition. It is known that the quantity Q of drug remaining in the body after t hours satisfies an equation of the form

$$Q = Q_0 e^{-kt}$$

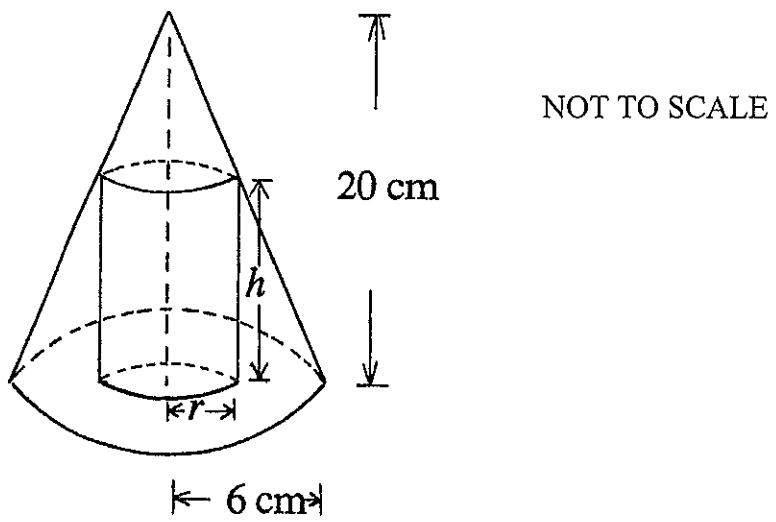
where Q_0 and k are constants.

The initial dose is 6 milligrams and after 15 hours the amount remaining in the body is half the initial dose.

(i) Find the values of Q_0 and k, correct to 3 decimal places where necessary. **2**

(ii) When will one-eighth of the initial dose remain? **2**

(b)



A cylinder of radius r cm and height h cm is inscribed in a cone with base radius 6 cm and height 20 cm as in the diagram.

(i) Show that the volume V of the cylinder is given by **2**

$$V = \frac{10\pi r^2 (6 - r)}{3}.$$

(ii) Hence find the values of r and h for the cylinder which has maximum volume. **2**

Question 16 continues on the next page.

Question 16 (continued)

(c) A farmer borrows \$80 000 to purchase new machinery. The interest is calculated monthly at the rate of 2% per month, and is compounded each month.

The farmer intends to repay the loan with interest in two equal annual instalments of $\$M$ at the end of the first and second years.

(i) How much does the farmer owe at the end of the first month? **1**

(ii) Write an expression involving M for the total amount owed by the farmer after 12 months, just after the first instalment of $\$M$ has been paid. **2**

(iii) Find an expression for the amount owed at the end of the second year and deduce that **2**

$$M = \frac{80\,000 \times (1.02)^{24}}{(1.02)^{12} + 1}$$

(iv) What is the total interest over the two year period? **2**

End of examination.

MULTIPLE-CHOICE

1) B 2) C 3) A 4) D 5) B
6) B 7) A 8) D 9) D 10) C

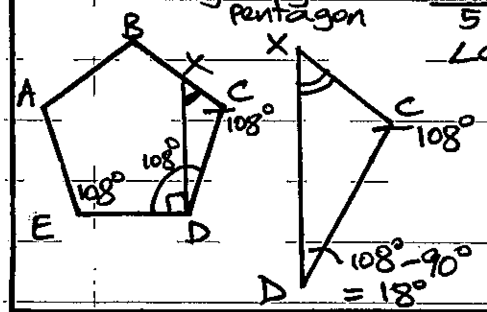
1) $\frac{2\pi^2}{19} = 1.038905726$
$= 1.04$ (3 sig figs) (B)

2) $\frac{1}{3\sqrt{5}+\sqrt{2}} \times \frac{3\sqrt{5}-\sqrt{2}}{3\sqrt{5}-\sqrt{2}} = \frac{3\sqrt{5}-\sqrt{2}}{9\times5-2}$
$= \frac{3\sqrt{5}-\sqrt{2}}{43}$ (C)

3) $\frac{\log_a 8}{\log_a 4} = \frac{\log_a 2^3}{\log_a 2^2}$
$= \frac{3\log_a 2}{2\log_a 2}$
$= \frac{3}{2}$ or $1\frac{1}{2}$ (A)

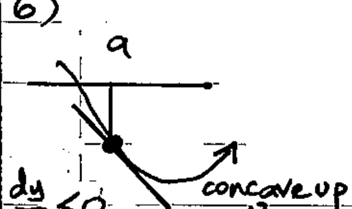
4) $2x^2-4x-3=0$ $\alpha+\beta = \frac{-b}{a}$ $\alpha\beta = \frac{c}{a}$
$= \frac{-(-4)}{2}$ $= \frac{-3}{2}$
$= 2$
$(\alpha+\beta) - (\alpha\beta) = 2 - (-\frac{3}{2})$
$= 3\frac{1}{2}$ (D)

5) Angle sum pentagon (5 sided) $= (n-2)\times180°$
$= 3\times180°$
$= 540°$
$\therefore$ each angle regular pentagon $= \frac{540°}{5} = 108°$



$\angle CXD$
$= 180° - (108°+18°)$
$= 180° - 126°$
$= 54°$ (B)

6)

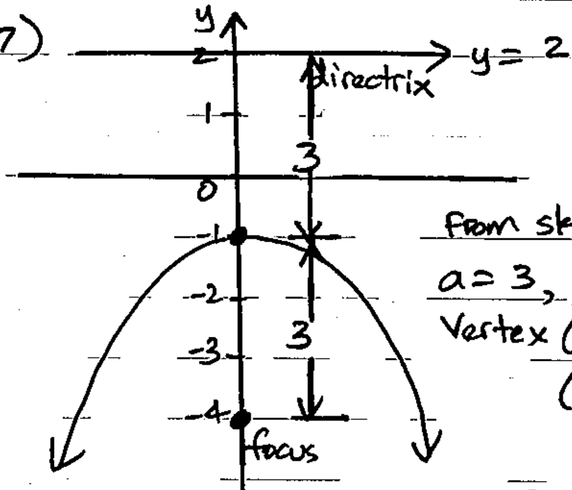


$\frac{dy}{dx} < 0$ $\therefore \frac{d^2y}{dx^2} > 0$
At $x=a$:

At $x=a$:
* gradient negative
$\therefore f'(a) < 0$
* concave UP
$\therefore f''(a) > 0$

$\therefore f'(a) < 0$ and $f''(a) > 0$ (B)

7)



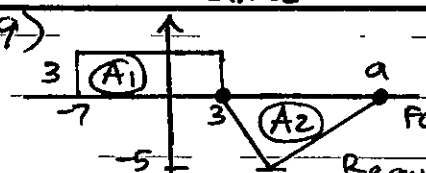
From sketch:
$a = 3$,
Vertex $(0,-1)$
(h,k)

Parabola of form $(x-h)^2 = -4a(y-k)$
$(x-0)^2 = -4\times3(y-(-1))$
$x^2 = -12(y+1)$ (A)

8) $y = \frac{x}{\sin x}$ $u = x$ $v = \sin x$
$\frac{du}{dx} = 1$ $\frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$= \frac{(\sin x).(1) - (x).\cos x}{\sin^2 x}$
$= \frac{\sin x - x\cos x}{\sin^2 x}$ (D)

9)



$A_1 = 3\times10$
$= 30$ units^2
For $\int_{-7}^{a} f(x)\,dx = 0$
Require $A_1 = A_2$
$A_2 = \frac{1}{2}bh = \frac{1}{2}\times b\times5 = 30$
$5b = 60$
$b = 12$
$\therefore a = 12+3 = 15$ (D)

10) From graph of $y = f'(x)$:
$f'(x) > 0$ for all x
So $f(x)$ ALWAYS increasing
$\therefore$ No stationary points / No Max/Min turning points.
at $x=0$: $f'(x)$ has stationary point
$\therefore$ At $x=0$, $f''(x)=0$ i.e. $f''(0)=0$
So $x=0$ may be point of inflexion.
Need to check for change of concavity "around" $x=0$:
• to immediate left of $x=0$: $f''(x)<0$ concave DOWN
• to immediate right of $x=0$: $f''(x)>0$ concave UP
$\therefore$ at $x=0$, $f''(0)=0$ & concavity changes $\rightarrow$ point of inflexion.
ODD function: $f(x)$ will be symmetrical about the origin
$\therefore$ ODD function, pt inflexion $x=0$ (C)

Question 11 (15 Marks)

(a) $2x^2+11x-21$ (-42: 14, -3)

$= 2x^2+14x-3x-21$

$= 2x(x+7)-3(x+7)$

$= (2x-3)(x+7)$ (2m)

(b) $|3x-5| < 4$

$\pm(3x-5) < 4$

$+(3x-5) < 4$ or $-(3x-5) < 4$

$3x < 9$ $3x-5 > -4$

$x < 3$ $3x > 1$

$x > \frac{1}{3}$

$\therefore \frac{1}{3} < x < 3$ (2m)

(c) $y = x^3$

$\frac{dy}{dx} = 3x^2$

when $x=2$: $\frac{dy}{dx} = 3\times 4 = 12$

when $x=2$: $y = 2^3$, $y=8$

Require tangent $m=12$, $(2,8)$:

$y-y_1 = m(x-x_1)$

$y-8 = 12(x-2)$

$y-8 = 12x-24$

$y = 12x-16$

or $12x-y-16=0$ (2m)

(d) $y=(e^{3x}-5)^4$

$\frac{dy}{dx} = 4(e^{3x}-5)^3 . (3e^{3x})$

$= 12e^{3x}(e^{3x}-5)^4$ (2m)

(e) $\int_1^2 \frac{1}{(3x-1)^2}dx = \int_1^2 (3x-1)^{-2}dx$

$= \left[\frac{(3x-1)^{-1}}{3\times(-1)}\right]_1^2$

$= \left[\frac{-1}{3(3x-1)}\right]_1^2$

$= \frac{-1}{15} - \left(\frac{-1}{6}\right)$

$= \frac{-1}{15} + \frac{1}{6}$

$= \frac{1}{10}$ (3m)

Question 11 continued:

11)(f)(i) $y=\sin x^3$

$y=\sin u$ $u=x^3$

$\frac{dy}{du}=\cos u$ $\frac{du}{dx}=3x^2$

$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$

$= \cos u \times 3x^2$

$= 3x^2\cos x^3$ (2m)

f)(ii) If $\frac{d}{dx}(\sin x^3) = 3x^2\cos x^3$

then $\int 3x^2\cos x^3 dx = \sin x^3 + c_1$

$\therefore \int x^2\cos x^3 dx = \frac{1}{3}\sin x^3 + C$ (2m)

Question 12 (15 Marks)

(a) $l=15$ $\theta = 60° = \frac{\pi}{3}$

$l=r\theta$

$15 = r.\frac{\pi}{3}$

$r = \frac{45}{\pi}$

$r = 14.32394488$

$\therefore r = 14.3$ cm, or $r = 143$ mm (correct to nearest mm.) (2m)

(b) $2\log_e x = \log_e(3x+10)$

$\log_e x^2 = \log_e(3x+10)$

$\therefore x^2 = 3x+10$

$x^2-3x-10=0$

$(x+2)(x-5)=0$

$x=-2$, or $x=5$

$x\neq -2$ as for $y=\log_e x$, $x>0$.

$\therefore$ only solution is $x=5$. (2m)

Question 12 continued:

12)(c) $y = \dfrac{\sin x}{1+\cos x}$

$u = \sin x \qquad v = 1+\cos x$

$\dfrac{du}{dx} = \cos x \qquad \dfrac{dv}{dx} = -\sin x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+\cos x).(\cos x) - (\sin x).(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} \quad \text{since } \sin^2 x + \cos^2 x = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+\cos x}$$

(3m)

12)(d) $\displaystyle\int \frac{10x}{x^2-8}\,dx = 5\int \frac{2x}{x^2-8}\,dx$

$= 5\log_e(x^2-8)+C$

since $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log_e f(x) + c$

(2m)

12)(e) $\displaystyle\int_0^{\frac{\pi}{4}} \cos(2x)\,dx$

$$= \frac{1}{2}\left[\sin(2x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \frac{1}{2}(1-0)$$

$$= \frac{1}{2}$$

12)(f)(i) P(throw same number)

= P(1,1) or P(2,2) or P(3,3) or P(4,4) or P(5,5) or P(6,6)

$= 6 \times \frac{1}{36}$

$= \frac{6}{36}$

$= \frac{1}{6}$

(2m)

Question 12(f) continued:

12)f)(ii) Let Mayomi be first number, then Christina throw second number.

Possible outcomes:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

15 possible outcomes where Christina throws number greater than Mayomi.

∴ P(Christina's number > Mayomi's number)

$= \frac{15}{36}$

$= \frac{5}{12}$

(2m)

12)f(ii) ALTERNATIVE METHOD

from f(i), P(throw same number) $= \frac{1}{6}$

∴ P(do NOT throw same number) $= \frac{5}{6}$

Of this, $\frac{1}{2} \times \frac{5}{6}$ = P(Christina > Mayomi) = P(Mayomi > Christina)

∴ P(Christina's number > Mayomi's number)

$= \frac{1}{2} \times \frac{5}{6}$

$= \frac{5}{12}$

(2m)

Question 13 (15 Marks)

(a)(i) Arithmetic series: $a = 10$, $l = 60$, $S_n = 3535$

$S_n = \frac{n}{2}(a+l)$

$3535 = \frac{n}{2}(10+60)$

$35n = 3535$

$n = \frac{3535}{35}$

$\therefore n = 101$

(2m)

13)(a)(ii) $l = a + (n-1)d$

$60 = 10 + 100 \times d$

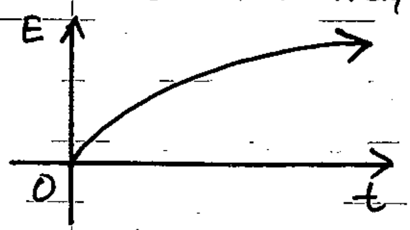
$100d = 50$

$\therefore d = \frac{1}{2}$

(2m)

Question 13 continued:

13)(b) Possible graph of unemployment (E) over time may be:



Unemployment is increasing: so $\frac{dE}{dt} > 0$. Since policy to reduce unemployment is taking effect, then even though the number of unemployed is still increasing, rate of increase is slowing down, i.e. curve is concave down.

i.e. $\frac{d^2E}{dt^2} < 0$

so in this case: $\frac{dE}{dt} > 0$, $\frac{d^2E}{dt^2} < 0$. (2m)

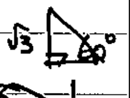
13)(c) Volume about x-axis $= \pi \int y^2\,dx$

$= \pi \int_0^{\frac{\pi}{3}} \sec^2 x\,dx$

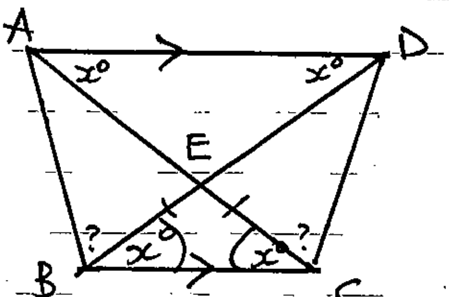
$= \pi \left[\tan x\right]_0^{\frac{\pi}{3}}$

$= \pi \left[\tan\frac{\pi}{3} - \tan 0\right]_0^{\frac{\pi}{3}}$

$= \pi\sqrt{3}$ cubic units (2m)



13)(d)



(i) $\angle ADE = \angle CBE$ (alternate angles equal AD // BC)

and $\angle DAE = \angle ACB$ (alternate angles equal AD // BC)

$\therefore \triangle ADE$ is isosceles triangle

$\therefore AE = DE$ (sides opposite equal angles are equal) (2m)

Question 13(d) continued:

13)(d)(ii) $\triangle BEC$ is isosceles

(since given $\angle DBC = x° = \angle ACB$)

$\therefore EB = EC$ (sides opposite equal angles are equal)

But $AC = AE + EC$

and $DB = DE + EB$

but $AE = DE$, and $EB = EC$

$\therefore AC = DB$.

Now in $\triangle ABC$, $\triangle DCB$:

$AC = DB$ (as shown) S

$\angle ACB = \angle DBC$ (given) A

BC is common S

$\therefore \triangle ABC \equiv \triangle DCB$ (SAS) (3m)

13)(d)(iii) $\angle ABC = \angle DCB$

(Matching angles in congruent triangles are equal)

i.e. $\angle ABD + \angle DBC = \angle DCA + \angle ACB$

But $\angle DBC = x° = \angle ACB$ (given)

$\therefore \angle ABD = \angle DCA$ (2m)

Question 14 (15 Marks)

14)(a)(i) $1 + (\sqrt{5}-2) + (\sqrt{5}-2)^2 + \ldots$

Told this is geometric series:

have $a = 1$, $r = \sqrt{5} - 2$

Limiting sum exists if $|r| < 1$

in this case $r = \sqrt{5} - 2 = 0.236\ldots$

i.e. $r < 1$

$\therefore$ limiting sum exists. (1m)

14)(a)(ii) $S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-(\sqrt{5}-2)}$

$= \frac{1}{3-\sqrt{5}}$

$= \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$

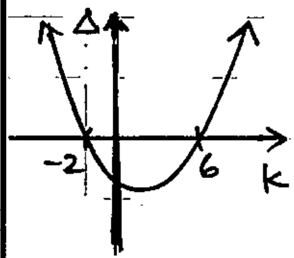
$= \frac{3+\sqrt{5}}{9-5}$

$\therefore S_\infty = \frac{3+\sqrt{5}}{4}$

Question 14 continued:

14)(b) $x^2 + (k-2)x + 4 = 0$

$\Delta = b^2 - 4ac$
$= (k-2)^2 - 4 \times 1 \times 4$
$= k^2 - 4k + 4 - 16$
$\Delta = k^2 - 4k - 12$
$\Delta = (k-6)(k+2)$



for no real roots,
$\Delta < 0$
From sketch,
$\Delta < 0$ when
$-2 < k < 6$

(2m)

14)(c)(i) $x^2 + y^2 = 9$ is a circle centre $(0,0)$ radius 3.
For $3x - 4y = 15$ to be a tangent to circle, $\perp$ distance from $(0,0)$ to $3x - 4y - 15 = 0$ must be 3 units.

$d = \left| \dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$
$= \left| \dfrac{3 \times 0 - 4 \times 0 - 15}{\sqrt{3^2 + 4^2}} \right|$
$= \left| -\dfrac{15}{\sqrt{25}} \right|$
$= \dfrac{15}{5}$
$\therefore d = 3$

(2m)

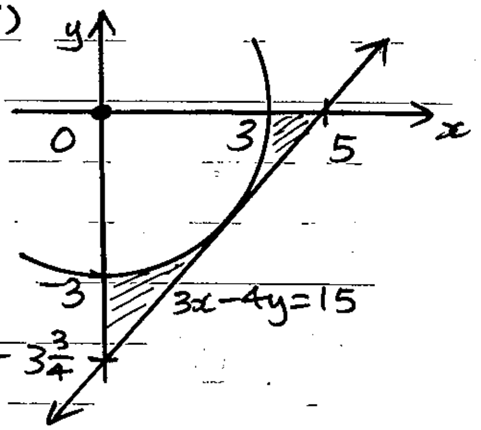
So perpendicular distance is 3 units
$\therefore$ $3x - 4y = 15$ is tangent to circle $x^2 + y^2 = 9$

14)(c)(ii) $3x - 4y = 15$ crosses:
x-axis: when $y = 0$: $3x = 15$
$x = 5$
x-intercept is $(5, 0)$

y-axis: when $x = 0$: $-4y = 15$
$y = -\dfrac{15}{4}$
$y = -3\frac{3}{4}$

Question 14(c)(ii) continued:

14)(c)(ii)



Shaded Area
$= (\text{Area triangle}) - (\text{Area Quadrant})$
$= \left(\frac{1}{2} \times 5 \times \frac{15}{4}\right) - \left(\frac{1}{4} \times \pi \times 3^2\right)$
$= \dfrac{75}{8} - \dfrac{9\pi}{4}$
$= \dfrac{75 - 18\pi}{8}$
$= \dfrac{3(25 - 6\pi)}{8}$ square units

(2m)

14)(d) $y = x^3 + x^2 - x + 2$
$\dfrac{dy}{dx} = 3x^2 + 2x - 1$

A and B are stationary points, so need to solve $\dfrac{dy}{dx} = 0$:

$3x^2 + 2x - 1 = 0$ ($\times$: -3; $+$: $+2$; $+3$, -1)
$3x^2 + 3x - x - 1 = 0$
$3x(x+1) - 1(x+1) = 0$
$(3x - 1)(x + 1) = 0$
$x = \frac{1}{3}$ or $x = -1$

when $x = -1$: $y = (-1)^3 + (-1)^2 - (-1) + 2$
$= -1 + 1 + 1 + 2$
$y = 3$

when $x = \frac{1}{3}$: $y = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 2$
$= \dfrac{49}{27}$
$y = 1\frac{22}{27}$

$\therefore A(-1, 3)$, $B\left(\frac{1}{3}, 1\frac{22}{27}\right)$

(3m)

Question 14 (d) continued:

(14)(d)(ii) $\frac{dy}{dx} = 3x^2 + 2x - 1$

$\frac{d^2y}{dx^2} = 6x + 2$

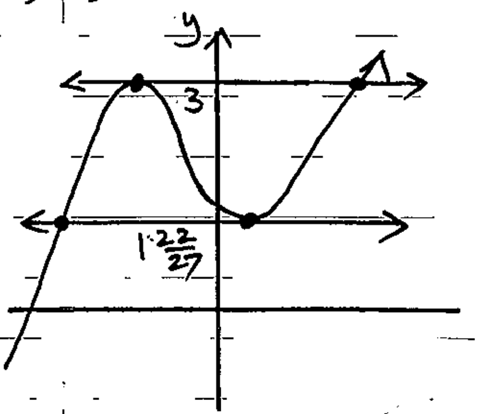
concave up when $\frac{d^2y}{dx^2} > 0$

i.e. when $6x + 2 > 0$

$6x > -2$

$\therefore$ concave up when $x > -\frac{1}{3}$ (2m)

(14)(d)(iii)



$x^3 + x^2 - x + 2 = k$ represents the intersection of $y = x^3 + x^2 - x + 2$ and $y = k$.

There will be 3 real solutions for

$1\frac{22}{27} < k < 3$ (1m)

(Note: only 2 roots if $k = 1\frac{22}{27}$ or $k = 3$)

Question 15 (15 Marks)

a)(i) After first card is drawn, 51 cards remain of which 3×13 are of a different suit

$\therefore$ P (2nd card different suit)

$= \frac{39}{51}$

$= \frac{13}{17}$ (1m)

Question 15 (a) continued:

(15)(a)(ii)

P(all 4 different suits)

$= P\left(\begin{matrix}\text{card 1}\\ \text{any suit}\end{matrix}\right) \times P\left(\begin{matrix}\text{card 2}\\ \text{any of 3 suits}\end{matrix}\right) \times P\left(\begin{matrix}\text{card 3}\\ \text{any of 2 suits}\end{matrix}\right) \times P\left(\begin{matrix}\text{card 4}\\ \text{last remaining suit}\end{matrix}\right)$

$= \left(\frac{52}{52}\right) \times \left(\frac{39}{51}\right) \times \left(\frac{26}{50}\right) \times \left(\frac{13}{49}\right)$

$= \frac{685\,464}{6\,497\,400}$

$= \frac{2197}{20\,825}$

$= 0.10549981993\ldots$ (2m)

$\doteqdot 0.105$ (3 decimal places)

(15)(b)(i) $\ddot{x} = 4\sin 2t$

$\dot{x} = \int \ddot{x}\, dt$

$= \int 4 \sin 2t$

$\dot{x} = -2\cos 2t + c_1$

when $t = 0$, $\dot{x} = 0$: $0 = -2\cos 2(0) + c_1$

$0 = -2 + c_1$

$\therefore c_1 = 2$

$\therefore \dot{x} = -2\cos 2t + 2$ (2m)

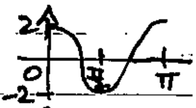
(15)(b)(ii) $\dot{x} = 2 - 2\cos 2t$

Period $= \frac{2\pi}{2}$

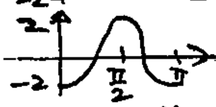
$= \pi$

Amplitude $= 2$

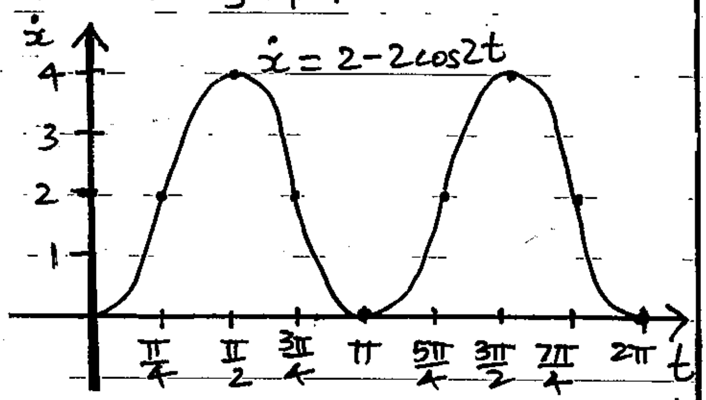
$\dot{x} = 2\cos 2t$



$\dot{x} = -2\cos 2t$



then translate UP 2 units to get final sketch of graph:



Particle first comes to rest (after $t = 0$) at $t = \pi$ (3m)

Question 15(b) continued:

15)(b)(iii) Distance travelled between $t=0$ and $t=\pi$

$$x = \int_0^{\pi} (2 - 2\cos 2t)\,dt$$

$$= \left[2t - \sin 2t \right]_0^{\pi}$$

$$= (2\pi - \sin 2\pi) - (0 - \sin 0)$$

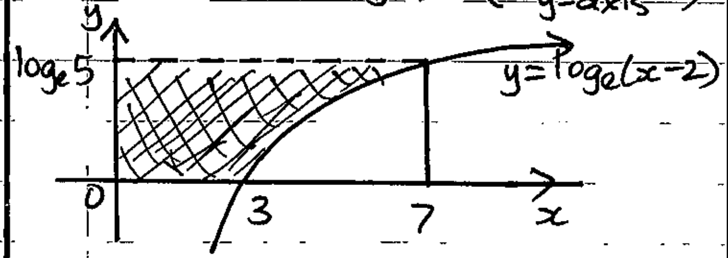
$$= (2\pi - 0) - (0 - 0)$$

Distance $= 2\pi$ (2m)

$\therefore$ distance travelled is 2π units

15)(c) Required area needs to be found by subtraction

$$\text{Area} = \begin{pmatrix}\text{area of}\\ \text{rectangle}\end{pmatrix} - \begin{pmatrix}\text{area between}\\ \text{curve and}\\ \text{y-axis}\end{pmatrix}$$



Shaded area about y-axis needed:

$$y = \log_e(x-2)$$

From definition of logarithm:

$$e^y = x - 2$$

$$\therefore x = 2 + e^y$$

$$\text{Area} = \int_0^{\log_e 5} (2 + e^y)\,dy$$

$$= \left[2y + e^y \right]_0^{\log_e 5}$$

$$= (2\log_e 5 + e^{\log_e 5}) - (2\times 0 + e^0)$$

$$= 2\log_e 5 + 5 - 1$$

Area $= 2\log_e 5 + 4$

Now Area Rectangle $= 7 \times \log_e 5$
$= 7\log_e 5$

So required Area

$$= (\text{Area rectangle}) - \begin{pmatrix}\text{area between}\\ \text{curve \& y-axis}\end{pmatrix}$$

$$= 7\log_e 5 - (2\log_e 5 + 4)$$

$$= 5\log_e 5 - 4 \text{ square units}$$ (5m)

Question 16 (15 Marks)

16)(a)(i) $Q = Q_0 e^{-kt}$

when $t=0$, $Q=6$: $6 = Q_0 e^0$

$\therefore Q_0 = 6$

so $Q = 6e^{-kt}$

when $t=15$, $Q = \frac{Q_0}{2}$

$$\therefore \frac{Q_0}{2} = Q_0 e^{-15k}$$

$$\frac{1}{2} = e^{-15k}$$

Taking logarithms base e of both sides:

$$\log_e\left(\tfrac{1}{2}\right) = -15k$$

$$k = -\frac{1}{15}\log_e\left(\tfrac{1}{2}\right)$$

$$= 0.04620981204\ldots$$

$\therefore k \doteqdot 0.046$ (3 decimal places)

$\therefore Q_0 = 6,\ k \doteqdot 0.046$ (2m)

16)(a)(ii) Require $\frac{1}{8}$ of initial dose Q_0

i.e. require $\frac{Q_0}{8}$

Solving $\frac{Q_0}{8} = Q_0 e^{-kt}$

$$\frac{1}{8} = e^{-kt}$$

Taking logarithms base e of both sides:

$$\log_e\left(\tfrac{1}{8}\right) = -kt$$

$$t = \frac{\log_e\left(\frac{1}{8}\right)}{-k}$$

$$= \frac{\log_e\left(\frac{1}{8}\right)}{-0.04620981204}$$

$$= 45$$

$\therefore$ After 45 hours, there will be $\frac{1}{8}$ of initial dose remaining (2m)

ALTERNATIVE METHOD 16(a)(ii)

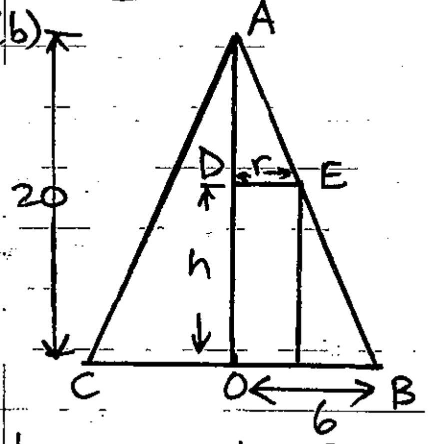
15 hours $\rightarrow$ half life

$\frac{1}{8} \rightarrow$ 3 "half lives" $\rightarrow \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

So $\frac{1}{8}$ of initial dose $= 3 \times 15$
$= 45$ hours

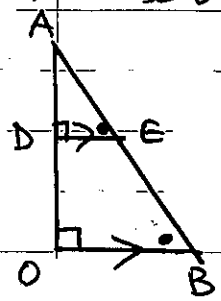
Question 16 continued:

16)(b)



(i) Take cross-section of cone.

Since top of cylinder (DE) is parallel to base of cone (OB),



$\triangle ADE \mathbin{|||} \triangle AOB$ (equiangular)

$\therefore \dfrac{AD}{AO} = \dfrac{DE}{OB}$ (matching sides similar triangles are in same ratio)

i.e. $\dfrac{20-h}{20} = \dfrac{r}{6}$

$6(20-h) = 20r$

$120 - 6h = 20r$

$6h = 120 - 20r$

$h = \dfrac{120-20r}{6}$

$h = \dfrac{20(6-r)}{6}$

$\therefore h = \dfrac{10(6-r)}{3}$

$\therefore$ Volume of cylinder

$= \pi r^2 h$

$= \pi r^2 . \dfrac{10(6-r)}{3}$

$\therefore V = \dfrac{10\pi r^2(6-r)}{3}$ (2m)

Question 16(b) continued:

16)(b)(ii) $V = \dfrac{10\pi r^2(6-r)}{3}$

i.e. $V = \dfrac{10\pi . 6r^2}{3} - \dfrac{10\pi r^3}{3}$

$V = 20\pi r^2 - \dfrac{10\pi r^3}{3}$

$\therefore \dfrac{dV}{dr} = 40\pi r - \dfrac{3.10\pi r^2}{3}$

$\dfrac{dV}{dr} = 40\pi r - 10\pi r^2$

For stationary points, solve $\dfrac{dV}{dr} = 0$

$40\pi r - 10\pi r^2 = 0$

$10\pi r(4-r) = 0$

$\therefore r = 0$ or $r = 4$

Require maximum volume, so $r \neq 0$

Need to check $r = 4$ is a maximum:

$\dfrac{dV}{dr} = 40\pi r - 10\pi r^2$

$\dfrac{d^2V}{dr^2} = 40\pi - 20\pi r$

when $r = 4$: $\dfrac{d^2V}{dr^2} = 40\pi - 80\pi$

$= -40\pi$

i.e. $\dfrac{d^2V}{dr^2} < 0$ when $r = 4$

i.e. concave down

$\therefore$ maximum when $r = 4$.

from (i): $h = \dfrac{10(6-r)}{3}$

$= \dfrac{10 \times 2}{3}$

$h = \dfrac{20}{3}$ or $6\frac{2}{3}$

$\therefore$ Maximum volume when

$r = 4$ cm, $h = \frac{20}{3}$ cm or $6\frac{2}{3}$ cm

(2m)

Question 16 continued:

16) (c) A_n is amount owed at end of n months

P is principal, amount borrowed.

in this case P = 80 000.

(i) $A = P(1+\frac{r}{100})^n$

$A_1 = 80\,000 \times (1+\frac{2}{100})^1$

ie. $A_1 = 80\,000 \times 1.02$

$= \$81\,600$ (1m)

(ii) $A_2 = A_1(1+\frac{2}{100})$

$= A_1 \times 1.02$

$= (80\,000 \times 1.02) \times 1.02$

$A_2 = 80\,000 \times 1.02^2$

$A_3 = A_2 \times 80\,000$

$= (80\,000 \times 1.02^2) \times 1.02$

$A_3 = 80000 \times 1.02^3$

Following this pattern for first 12 months, then:

$A_{12} = 80\,000 \times 1.02^{12}$

So after 1st instalment of \$M,

Amount owing after 12 months

$= 80\,000 \times 1.02^{12} - M$ (2m)

(iii) 1st repayment made after A_{12}.

But 2nd repayment not made until after A_{24}.

$A_{13} = A_{12}(1+\frac{2}{100})$

$= A_{12} \times 1.02$

$A_{13} = [80\,000 \times 1.02^{12} - M] \times 1.02$

$A_{14} = [80\,000 \times 1.02^{12} - M] \times 1.02^2$

$A_{15} = [80\,000 \times 1.02^{12} - M] \times 1.02^3$

Following this pattern:

$A_{24} = [80\,000 \times 1.02^{12} - M] \times 1.02^{12}$

Question 16(c)(iii) continued:

2nd (Final) repayment of \$M is paid after A_{24}:

ie. $[80\,000 \times 1.02^{12} - M] \times 1.02^{12} - M$

But after 2nd/Final repayment there is no money owing.

$\therefore [80\,000 \times 1.02^{12} - M] \times 1.02^{12} - M = 0$

$80\,000 \times 1.02^{24} - 1.02^{12}M - M = 0$

$80\,000 \times 1.02^{24} - M(1.02^{12}+1) = 0$

ie. $M(1.02^{12}+1) = 80\,000 \times 1.02^{24}$

$\therefore M = \frac{80\,000 \times 1.02^{24}}{(1.02^{12}+1)}$

ie. $M = \frac{80\,000 \times (1.02)^{24}}{(1.02^{12})+1}$ (2m)

(iv) From calculator

$M = \frac{80\,000 \times 1.02^{24}}{1.02^{12}+1} = 56728.95203$

ie. each instalment M = \$56 728.95

$\therefore$ Total repaid in the 2 instalments

$= 2 \times \$56728.95$

$= \$113\,457.90$

$\therefore$ Interest = (Total repaid) − (original loan of \$80 000)

$= \$113\,457.90 - \$80\,000$

$= \$33\,457.90$