GOSFORD HIGH SCHOOL

2012
TRIAL HSC EXAMINATION

MATHEMATICS

General Instructions:

- Reading time: 5minutes.
- Working time: 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in Questions 11-16.

Total marks: - 100

Section I (10 marks)
Attempt Questions 1- 10.
Answer on the Multiple Choice answer sheet provided.
Allow about 15 minutes for this section.

Section II (90 marks)
Attempt Questions 11-16
Start each question in a separate writing booklet.
Allow about 2 hours 45 minutes for this section.

Section I

Total marks (10)

Attempt Questions 1-10

Allow about 15 minutes for this section

Answer on the multiple choice answer sheet provided. Select the alternative A, B, C, D that best answers the question. Fill in the response oval completely.

1. After recent rainfall brought an increase of 8% to the water storage of a dam, its capacity rose to 61020 megalitres (ML). The amount of water stored in the dam before the increase (to the nearest 100 ML) was

A. 53000 ML B. 56100 ML C. 56500 ML D. 65900ML

2. The equation of the directrix of the parabola $y^2 = -8x$ is

A. $x = 2$ B. $y = 2$ C. $x = -2$ D. $y = -2$

3. A coin is tossed three times. What is the probability that the outcome of the last toss is the same as that of the first toss?

A. $\frac{1}{8}$ B. $\frac{1}{4}$ C. $\frac{3}{8}$ D. $\frac{1}{2}$

4. If $t = \sqrt{6} - 2$, $t + t^{-1} =$

A. $\sqrt{6}$ B. $2\sqrt{6} - 2$ C. $3\sqrt{6} - 2$ D. $\frac{3\sqrt{6}-2}{2}$

5. If $y = xe^x$, $\frac{dy}{dx} =$

A. $1 + e^x$ B. $e^x(1 + x)$ C. e^x D. x^2e^x

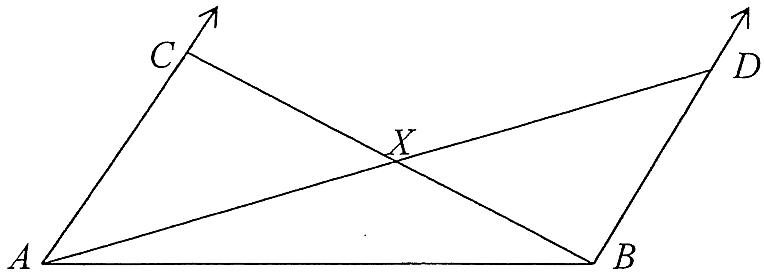
6. A particle is moving along a straight line so that its displacement, x metres, from a fixed point O is given by $x = 1 + 2\cos 3t$, where t is measured in seconds. The initial displacement of the particle to the right of the point O is

A. 1 metre B. 2 metres C. 3 metres D. $\sqrt{3}$ metres

7. $\int \cos 2x \, dx =$

A. $\frac{1}{2} \sin 2x + c$ B. $2 \sin 2x + c$ C. $-\frac{1}{2} \sin 2x + c$ D. $-2 \sin 2x + c$

8.



NOT TO SCALE

In the diagram above $AC \parallel BD, \angle CAX = 2\angle BAX, \angle DBX = 2\angle ABX.$

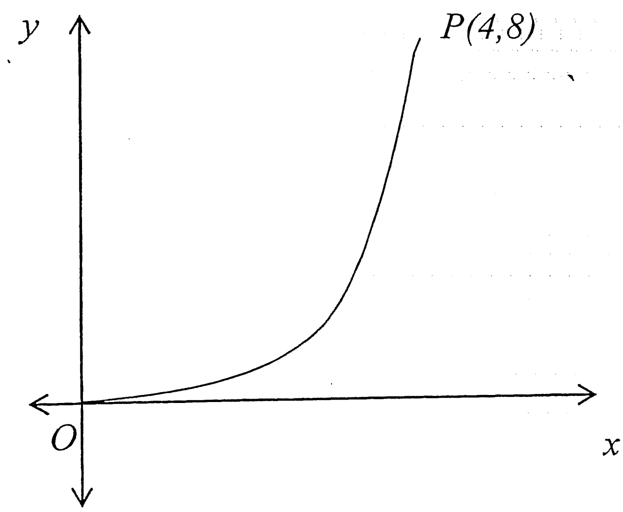
$\angle AXB =$

A. 150^0 B. 120^0 C. 90^0 D. 60^0

9. If α and β are the solutions of the equation $x^2 + 4x + 1 = 0$, then $\alpha + \frac{1}{\alpha} =$

A. -1 B. 1 C. -4 D. 4

10.



NOT TO SCALE

OP is an arc of the curve $y^2 = x^3$. The volume of the solid of revolution formed when the region bounded by the arc OP and the y - axis is rotated about the y - axis is given by

A. $\pi \int_0^4 y^{\frac{2}{3}} \, dy$ B. $\pi \int_0^8 y^{\frac{2}{3}} \, dy$ `C. $\pi \int_0^4 y^{\frac{4}{3}} \, dy$ D. $\pi \int_0^8 y^{\frac{4}{3}} \, dy$

Section II

Total marks (90)

Attempt Questions 11-16

Allow about 2 hours 45 minutes for this section

Answer all questions, starting each question in a separate writing booklet.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $|5 - 2x| < 1$ (2)

(b) If $(\sqrt{3} - 2)^2 = a - b\sqrt{3}$, find a and b. (2)

(c) Find the equation of the tangent to the curve $y = 2x^2 + 3x - 5$ at the point on the curve where $x = 1$. (3)

(d) Find the exact length of the arc subtended by an angle of 54^0 in a circle of radius 15 centimetres. (2)

(e) Solve $3tan^2 2x = 1$, where $0 \leq x \leq \pi$. (3)

(f) Differentiate with respect to x.

(i) $2x + \frac{3}{x}$. (1)

(ii) $\frac{x}{sin\, x}$. (2)

Question 12 (15 marks) Use a SEPARATE writing booklet.

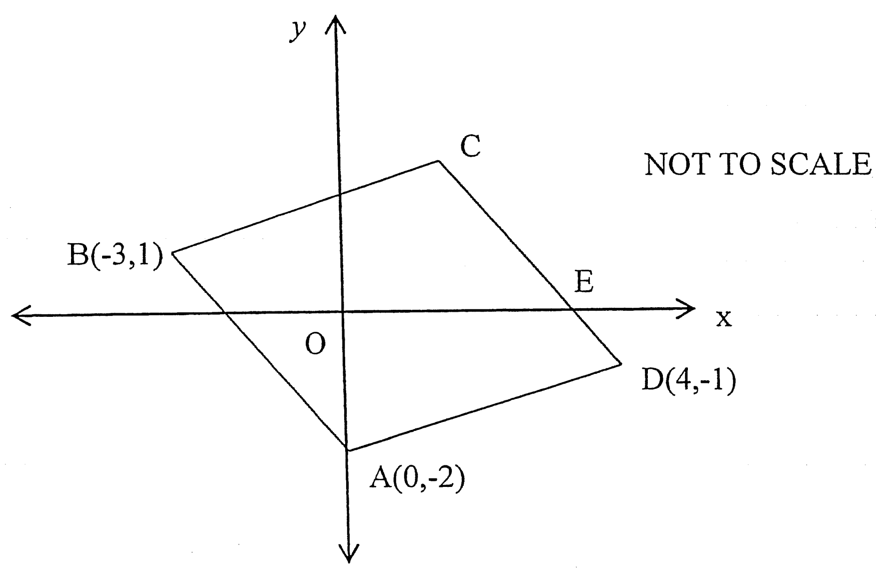
(a) Solve $2log_e x = log_e(5x + 6)$. (2)

(b) Find (i) $\int \frac{dx}{4x^2}$. (1)

(ii) $\int \frac{4x}{4-x^2}\,dx$ (2)

(c) Evaluate $\int_{\pi/4}^{\pi/3} sec^2 x\,dx$ (2)

(d) In the diagram below, $A(0,-2), B(-3,1)$ and $D(4,-1)$ are three vertices of a parallelogram $ABCD$ in which $AB \parallel DC$ and $BC \parallel AD$. E is the point of intersection of CD and the x-axis.



(i) Find the coordinates of the point C. (1)

(ii) Calculate the size of $\angle CEO$ where O is the origin. (1)

(iii) Show that the equation of CD is $x + y - 3 = 0$. (2)

(iv) Calculate the perpendicular distance from the point A to CD. (2)

(v) Hence, or otherwise, calculate the area of parallelogram $ABCD$. (2)

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In a class of students there are 18 boys and 12 girls and in another class there are 16 boys and 14 girls.

(i) One student is selected at random from each class. What is the probability that both are boys? (2)

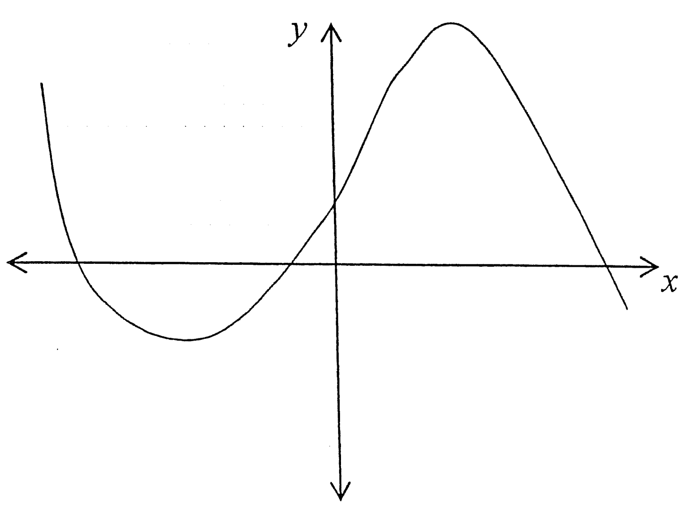
(ii) If one class is selected at random and two students are selected at random from that class, what is the probability that both are boys? (2)

(iii) If the two classes are combined and two students are selected at random from the combined class, what is the probability that at least one of the students selected is a girl? (2)

(b) If $y = x^2 e^{3x}$ show that $\frac{dy}{dx} - \frac{2y}{x} = 3y$. (3)

(c) Use Simpson's rule with 5 function values to find an approximation (correct to 3 d.p.) to the value of $\int_{0.5}^{1.5} \frac{dx}{\sqrt{x}}$ (3)

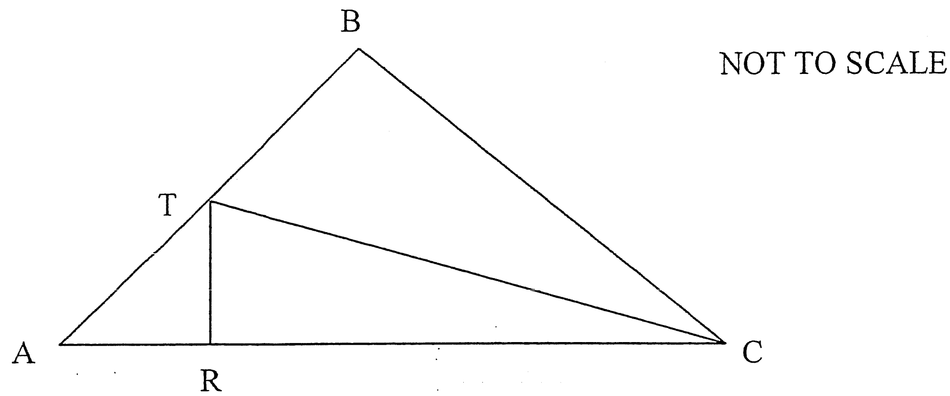
(d) The diagram below shows the graph of $y = f(x)$. Stationary points occur at $x = -1$ and $x = 1$. A point of inflexion occurs at $x = 0$.



Draw a sketch of the gradient function $y = f'(x)$ given that $f'(0) = 2$. (3)

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram below ABC is a right isosceles triangle with $\angle ABC = 90^0$ and $AB = AC$.



Given that TC bisects $\angle ACB$ and $TR \perp AC$,

(i) Prove that $\Delta BTC \equiv \Delta RTC$. (3)

(ii) Hence or otherwise, prove that $AR = TB$. (2)

(b) Let $f(x) = x^4 - 2x^3$.

(i) Find the coordinates of the points where the curve $y = f(x)$ crosses the coordinate axes. (1)

(ii) Find the coordinates of the stationary points and determine their nature. (4)

(iii) Find the coordinates of any points of inflexion. (3)

(iv) Sketch the graph of $y = f(x)$ clearly indicating these points. (2)

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Find the equation of the normal to the curve $y = \ln x + x$ at the point on the curve where $x = 1$. (2)

(b) A particle is moving along a straight line so that at any time t seconds its acceleration a is given by $a = 6t + 1$. Initially the particle is at the origin and its initial velocity is $-2m/s$.

(i) Show that the velocity of the particle v is given by $v = 3t^2 + t - 2$. (2)

(ii) Determine the time when the particle is at rest. (1)

(iii) Calculate the distance travelled by the particle during the first second. (3)

(c)

(i) Show that if $y = \frac{-\cos x}{\sin x + \cos x}$, then $\frac{dy}{dx} = \frac{1}{(\sin x + \cos x)^2}$ (2)

(ii) The region bounded by the curve $y = \frac{1}{\sin x + \cos x}$, the x axis, and between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x axis. Calculate the volume of the solid of revolution generated. (3)

(d) An Olympic swimmer knows that the probability of equalling or bettering their personal best time in any race is 0.2. Calculate the probability that the swimmer does not equal or better their personal best time in three successive races. (2)

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Given that the roots of the quadratic equation $2x^2 + 3x - 8 = 0$ are α and β.

(i) Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$. (1)

(ii) Find the quadratic equation whose roots are 2α and 2β. (2)

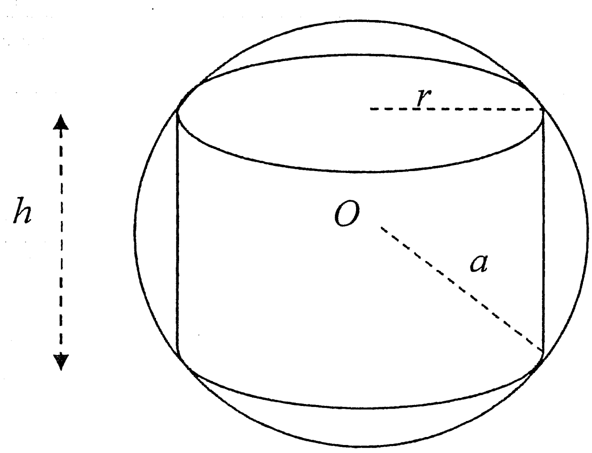
(b) The mass M of bacteria present, after t hours, in a medium with ample food increases at a rate proportional to M as given by the equation $M = M_0 e^{kt}$, where M_0 and k are constants. After 1 hour the mass increases from 1000 micrograms (μg) to 1200 μg

(i) Find the value of k. (2)

(ii) What will the mass of the bacteria be after 10 hours? (2)

(iii) How long will it take for the initial mass to increase to 20000 μg (2)

(c) A right cylinder of radius r units and height h units is inscribed in a sphere of radius a units centred at O as shown below.



(i) Write an expression for r^2 in terms of a and h. (1)

(ii) Show that the volume of the cylinder is given by $V = \pi(a^2 h - \frac{h^3}{4})$. (1)

(iii) Hence or otherwise, show that if the size of the sphere remains constant, the maximum volume of the cylinder is $\frac{4\sqrt{3}a^3\pi}{9}$ $units^3$. (4)

SOLUTIONS

SECTION I

1. 108% of x = 61020 mL

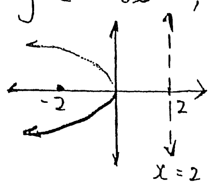
1% of x = $\frac{61020}{108}$ mL

100% of x = $\frac{61020}{108} \times 100$ mL

= 56500 mL

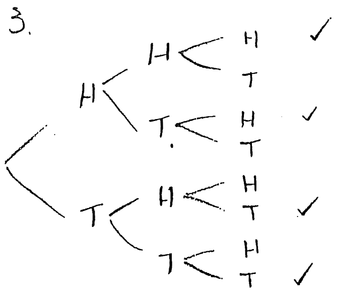
∴ C

2. If $y^2 = -8x$, $a = 2$



∴ A

3.



∴ D

4. $t + t^{-1} = \sqrt{6} - 2 + \frac{1}{\sqrt{6}-2} \times \frac{\sqrt{6}+2}{\sqrt{6}+2}$

$= \sqrt{6} - 2 + \frac{\sqrt{6}+2}{2}$

$= \frac{2\sqrt{6} - 4 + \sqrt{6} + 2}{2}$

$= \frac{3\sqrt{6} - 2}{2}$

∴ D

$\frac{dy}{dx} = e^x \times 1 + x \times e^x$

$= e^x(1+x)$

∴ B

6. If $x = 1 + 2\cos 3t$

When $t = 0$

$x = 1 + 2\cos 0$

$= 1 + 2(1)$

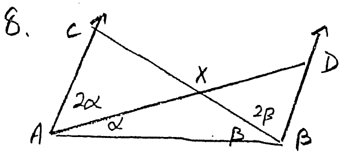
$= 3$

∴ C

7. $\int \cos 2x \, dx = \frac{1}{2}\sin 2x + c$

∴ A

8.



Let $\angle CAX = 2\alpha$, $\angle BAX = \alpha$

& $\angle DBX = 2\beta$, $\angle ABX = \beta$

$3\alpha + 3\beta = 180°$ (co-int. ∠'s)

$\alpha + \beta = 60°$

∴ $\angle AXB = 120°$ (∠ sum of a Δ)

∴ B

9. $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

$\alpha + \beta = -4$ $\alpha\beta = 1$

∴ $\beta = \frac{1}{\alpha}$

Since $\beta = \frac{1}{\alpha}$

$\alpha + \frac{1}{\alpha} = -4$

∴ C

If $y^2 = x^3$, $x = y^{2/3}$

∴ $x^2 = y^{4/3}$

∴ $V = \pi \int_0^8 y^{4/3} \, dy$

∴ D

SECTION II

Q11

a) $|5 - 2x| < 1$

$-1 < 5 - 2x < 1$

$-6 < -2x < -4$

$3 > x > 2$

i.e. $2 < x < 3$ (2)

b) $(\sqrt{3} - 2)^2 = 3 - 4\sqrt{3} + 4$

$= 7 - 4\sqrt{3}$

∴ $a = 7$, $b = 4$ (2)

c) $y = 2x^2 + 3x - 5$

$y' = 4x + 3$

When $x = 1$, $y' = 7$, $y = 0$

∴ Eqⁿ is given by

$y - 0 = 7(x - 1)$

$y = 7x - 7$ (3)

d) $l = r\theta$

If $\theta = 54°$

$= 54 \times \frac{\pi}{180}$

$= \frac{3\pi}{10}$

$= \frac{9\pi}{2}$ cm (2)

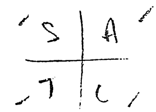
e) $3\tan^2 2x = 1$

$\tan^2 2x = \frac{1}{3}$

$\tan 2x = \pm\frac{1}{\sqrt{3}}$

If $0 \le x \le \pi$

$0 \le 2x \le 2\pi$



$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ (2

f) (i) Let $y = 2x + 3x^{-1}$

$y' = 2 - 3x^{-2}$

$= 2 - \frac{3}{x^2}$ (1)

(ii) Let $y = \frac{x}{\sin x}$

$y' = \frac{\sin x \times 1 - x \times \cos x}{(\sin x)^2}$

$= \frac{\sin x - x\cos x}{\sin^2 x}$ (2

Q12

a) $2\log_e x = \log_e(5x+6)$

$\log_e x^2 = \log_e(5x+6)$

∴ $x^2 = 5x + 6$

$x^2 - 5x + 6 = 0$

$(x - 6)(x + 1) = 0$

$x = 6$ or -1 (2

Solⁿ is $x = 6$ as $x \neq -1$

i) $\int 4x^{-2}$

$= \frac{1}{4}\frac{x^{-1}}{-1} + c$

$= \frac{-1}{4x} + c$ (1)

(ii) $\int \frac{4x}{4-x^2}\,dx = -2\int \frac{-2x}{4-x^2}\,dx$

$= -2\ln(4-x^2) + c$ (2)

) $\int_{\pi/4}^{\pi/3} \sec^2 x \, dx$

$= [\tan x]_{\pi/4}^{\pi/3}$

$= \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$ (2)

$= \sqrt{3} - 1$.

) (i) By inspection C is (1,2) (1)

(ii) m of CD $= \frac{2--1}{1-4}$

$= \frac{3}{-3}$

$= -1$

$\therefore \angle CEx = 135^\circ$ (1)

$\therefore \angle CEO = 45^\circ$

(iii) Eqn is given by

$y-2 = -1(x-1)$

$y-2 = -x+1$ (2)

ie $x+y-3=0$

$\sqrt{A^2+B^2}$

$\therefore \frac{|1(0)+1(-2)-3|}{\sqrt{1^2+1^2}}$

$= \frac{|-5|}{\sqrt{2}}$ (2)

$= \frac{5}{\sqrt{2}}$ units or $\frac{5\sqrt{2}}{2}$ units

(v) Area = bh

where b = CD

$CD = \sqrt{(4-1)^2+(-1-2)^2}$

$= \sqrt{9+9}$

$= \sqrt{18}$

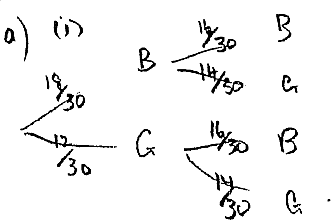
$= 3\sqrt{2}$ units

$\therefore$ Area $= 3\sqrt{2} \times \frac{5}{\sqrt{2}}$ (2)

$= 15$ units^2

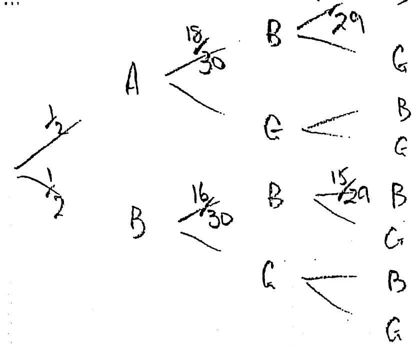
13

a) (i)



$P(B,B) = \frac{18}{30} \times \frac{16}{30}$

$= \frac{8}{25}$ (2)

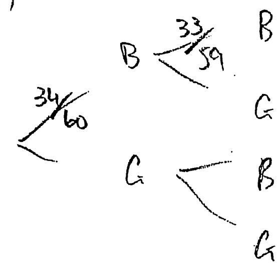


$P(B,B) = \frac{1}{2} \times \frac{18}{30} \times \frac{17}{29} + \frac{1}{2} \times \frac{16}{30} \times \frac{15}{29}$

$= \frac{51}{290} + \frac{4}{29}$

$= \frac{91}{290}$ (2)

(iii)



$P(B,B) = \frac{34}{60} \times \frac{33}{59}$

$= \frac{187}{590}$

$\therefore P(\text{at least } 1G) = 1 - \frac{187}{590}$

$= \frac{403}{590}$ (2)

b) If $y = x^2 e^{3x}$

$\frac{dy}{dx} = e^{3x} \times 2x + x^2 \times 3e^{3x}$

$= xe^{3x}(2+3x)$

LHS $= \frac{dy}{dx} - \frac{2y}{x}$

$= xe^{3x}(2+3x) - \frac{2x^2e^{3x}}{x}$

$= 2xe^{3x} + 3x^2e^{3x} - 2xe^{3x}$

$\therefore 3x^2e^{3x}$

$= 3y$ (3)

$=$ RHS.

c) $h = \frac{1.5-0.5}{4} = 0.25$

0.5	0.75	1	1.25	1.5
1.4142	1.1547	1	0.8944	0.8165

$\int_{0.5}^{1.5} \frac{dx}{\sqrt{x}} \doteq \frac{0.25}{3}\left[1.4142 + 0.816\ldots + 2\times1 + 4(1.1547+0.\ldots\right.$

$= 1.03559\ldots$

$= 1.036$ (3 d.p) (3

d) If $x < -1$, $y' < 0$

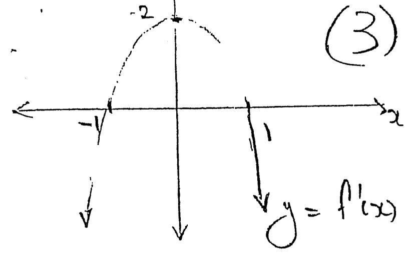
If $x = -1$, $y' = 0$

If $-1 < x < 1$, $y' > 0$

If $x = 1$, $y' = 0$

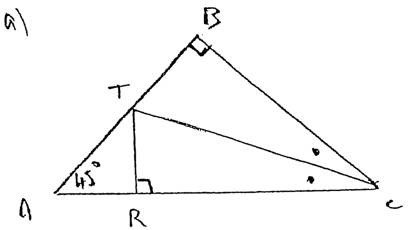
If $x > 1$, $y' < 0$

$y' = 2$ when $x = 0$.



(3)

7/ a)



In Δ's TBC & TRC

TC is a common side

∠BCT = ∠RCT (given TC bisects ∠ACB)

∠TBC = ∠TRC (given both 90°)

∴ Δ TBC ≡ Δ TRC (AAS test)

∴ TB = TR (corresponding sides in congruent Δ's are equal) *

Also, ∠TAR = 45° since Δ ABC is a right isosceles Δ)

Hence ΔTAR is a right isosceles Δ.

∴ AR = TR (equal sides of an isosc. Δ) *

AR = TB. (5)

b) (i) If $f(x) = 0$

$x^4 - 2x^3 = 0$

$x^3(x-2) = 0$

$x = 0$ or 2 (1)

∴ Coordinates are (0,0) & (2,0)

(ii) For stat. pts $f'(x) = 0$

$f'(x) = 4x^3 - 6x^2$

So $4x^3 - 6x^2 = 0$

$2x^2(2x-3) = 0$

$x = 0$ or $\frac{3}{2}$

If $x = 0$, $f(0) = 0$

If $x = \frac{3}{2}$, $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3$

$= \frac{81}{16} - 2\left(\frac{27}{8}\right)$

$= -\frac{27}{16}$

Stat pts at (0,0) & $\left(\frac{3}{2}, -\frac{27}{16}\right)$

$f''(x) = 12x^2 - 12x$

When $x = \frac{3}{2}$, $f''\left(\frac{3}{2}\right) = 12\left(\frac{3}{2}\right)^2 - 12\left(\frac{3}{2}\right)$

$= 9$

> 0

∴ $\left(\frac{3}{2}, \frac{-27}{16}\right)$ is a min t.p. (4)

When $x = 0$

x	0^-	0	0^+
$f'(x)$	$-$	0	$-$

∴ (0,0) is a horizontal pt of inflexion

(iii) $f''(x) = 0$

$12x^2 - 12x = 0$

$12x(x-1) = 0$

$x = 0$ or 1

When $x = 0$, $f(0) = 0$

When $x = 1$, $f(1) = (1)^4 - 2(1)^3$

$= -1$

Possible pts of inflexion are (0,0) & (1,-1)

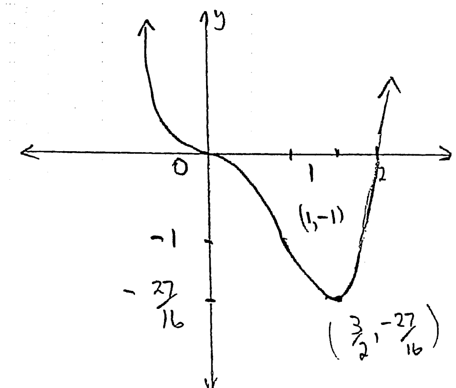
From (ii) (0,0) is a horizontal point of inflexion

When $x = 1$ (3)

x	1^-	1	1^+
$f''(x)$	$-$	0	$+$

∴ (1,-1) is a point of inflexion.

iv)



(2)

8/ a) If $y = \ln x + x$

$y' = \frac{1}{x} + 1$

When $x = 1$, $y = 1$; $y' = 2$

∴ the eqⁿ is given by

$y - 1 = -\frac{1}{2}(x-1)$

$2y - 2 = -x + 1$

$x + 2y - 3 = 0$ (2)

b) (i) $a = 6t + 1$

$v = \int 6t + 1 \, dt$

$v = 3t^2 + t + c$

If $t = 0$, $v = -2$

$-2 = 3(0)^2 + (0) + c$

$c = -2$ (2)

∴ $v = 3t^2 + t - 2$.

(ii) If $v = 0$

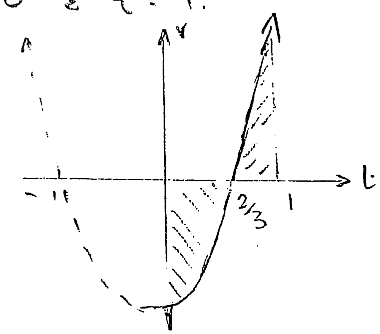
$3t^2 + t - 2 = 0$

$(3t-2)(t+1) = 0$

$t = \frac{2}{3}$ or -1

∴ The particle is at rest (1) when $t = \frac{2}{3}$ sec since $t = -1$ is meaningless.

The area bounded by the velocity time graph between $t=0$ & $t=1$.



$$d = \left|\int_0^{2/3} 3t^2+t-2\,dt\right| + \int_{2/3}^{1} 3t^2+t-2\,dt$$

$$= \left|\left[t^3+\frac{t^2}{2}-2t\right]_0^{2/3}\right| + \left[t^3+\frac{t^2}{2}-2t\right]_{2/3}^{1}$$

$$= \left|\left(\frac{8}{27}+\frac{4}{18}-\frac{4}{3}\right) - 0\right|$$

$$+ \left(1+\frac{1}{2}-2\right) - \left(\frac{8}{27}+\frac{4}{18}-\frac{4}{3}\right)$$

$$= \frac{22}{27} + \frac{17}{54}$$

$$= \frac{61}{27} \text{ metres}$$

(3)

$\sin x + \cos x$

$$\frac{dy}{dx} = \frac{(\sin x+\cos x)\sin x - \cos x(\cos x-\sin x)}{(\sin x+\cos x)^2}$$

$$= \frac{\sin^2 x + \sin x\cos x + \cos^2 x - \sin x \cos x}{(\sin x+\cos x)^2}$$

$$= \frac{\sin^2 x+\cos^2 x}{(\sin x+\cos x)^2}$$

$$= \frac{1}{(\sin x+\cos x)^2}$$

(2)

(ii) $$V = \pi\int_a^b y^2\,dx$$

$$= \pi\int_0^{\pi/4} \frac{1}{(\sin x+\cos x)^2}\,dx$$

$$= \pi\left[\frac{-\cos x}{\sin x+\cos x}\right]_0^{\pi/4}$$

$$= \pi\left\{\left(\frac{-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}+\cos\frac{\pi}{4}}\right) - \left(\frac{-\cos 0}{\sin 0+\cos 0}\right)\right\}$$

$$= \pi\left\{\left(\frac{-1/\sqrt{2}}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}\right) - \left(\frac{-1}{0+1}\right)\right\}$$

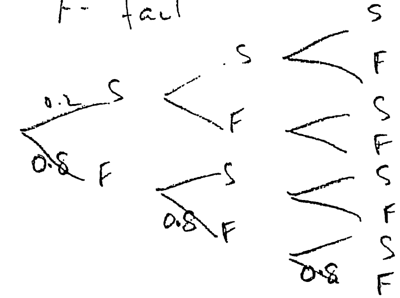
$$= \pi\left\{\frac{-\frac{1}{\sqrt{2}}}{2/\sqrt{2}} + 1\right\}$$

$$= \pi\left\{-\tfrac{1}{2}+1\right\}$$

$$= \frac{\pi}{2} \text{ units}^3$$

(3)

F - fail



$$P(F,F,F) = (0.8)^3 = 0.512$$

(2)

16/ a)

(i) $$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \frac{-b/a}{c/a} = \frac{-3/2}{-4} = \frac{3}{8}$$

(1)

(ii) The required eqn is of the form

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

If the roots are 2α & 2β

$2\alpha + 2\beta = 2(\alpha+\beta) = -3$

$2\alpha \times 2\beta = 4\alpha\beta = -16$

∴ The eqn is

$$x^2 + 3x - 16 = 0.$$

(2)

If $t=0$, $M=1000$

$1000 = M_0 e^0$

$M_0 = 1000$

$\therefore M = 1000e^{kt}$

If $t=1$, $M=1200$

$\therefore 1200 = 1000e^k$

$1.2 = e^k$

$\ln(1.2) = k$

$k = \ln(1.2)$

(2)

(ii) $M = 1000e^{10\times\ln(1.2)}$

$= 1000e^{\ln(1.2)^{10}}$

$= 1000 \times 1.2^{10}$

$= 6192\,\mu g$ (to the nearest μg.)

(iii) If $M = 20000$

$20000 = 1000e^{\ln(1.2)\times t}$

$20 = e^{\ln(1.2)\times t}$

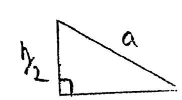
$\ln 20 = \ln(1.2)\times t$

$t = \frac{\ln 20}{\ln(1.2)}$

$= 16.4310\ldots$ (2)

$= 16$ hrs 26 mins (to the nearest minute)

c) (i)



$r^2 = a^2 - \left(\frac{h}{2}\right)^2$

$r^2 = a^2 - \frac{h^2}{4}$

$\therefore = \pi\left(a^2 - \frac{h^2}{4}\right)h$

$= \pi\left(a^2h - \frac{h^3}{4}\right)$ (1)

ii) If the size of the sphere remains constant 'a' is a constant.

For a max $\frac{dV}{dh} = 0$

$\frac{dV}{dh} = \pi\left(a^2 - \frac{3h^2}{4}\right)$

$\therefore \pi\left(a^2 - \frac{3h^2}{4}\right) = 0$

$a^2 - \frac{3h^2}{4} = 0$

$4a^2 - 3h^2 = 0$

$3h^2 = 4a^2$

$h^2 = \frac{4}{3}a^2$

$h = \pm \frac{2}{\sqrt{3}}a$

$h = \frac{2}{\sqrt{3}}a$ since $h > 0$

$\frac{d^2V}{dh^2} = \pi\left(0 - \frac{6h}{4}\right)$

$= -\frac{6\pi h}{4}$

< 0 when $h = \frac{2a}{\sqrt{3}}$

$\therefore$ Max volume occurs when $h = \frac{2}{\sqrt{3}}a$

$\therefore V = \pi\left(a^2 - \frac{\frac{4}{3}a^2}{4}\right) \cdot \frac{2}{\sqrt{3}}a$

$\pi\left(\quad \frac{}{3}\right) \frac{}{\sqrt{3}}$

$= \pi \times \frac{2a^2}{3} \times \frac{2a}{\sqrt{3}}$

$= \frac{4\pi a^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{4\sqrt{3}\pi a^3}{9}$ units^3

(4)