Student Number

2020

Trial Higher School Examination

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2-4)

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 5-24)

- Attempt Questions 11-32
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. What is the value of $\operatorname{cosec}\dfrac{\pi}{3}$ to three significant figures?

(A) 1.00
(B) 1.15
(C) 1.41
(D) 2.00

2. What is the value of c for which the circle $(x-3)^2+(y-2)^2=c$ touches the x axis?

(A) 2
(B) 3
(C) 4
(D) 9

3. What is the equation of the tangent to $y=x^2-3$ at $x=-1$?

(A) $y=-2x-4$
(B) $y=2x-4$
(C) $y=\dfrac{x}{2}-\dfrac{3}{2}$
(D) $y=-\dfrac{x}{2}-\dfrac{3}{2}$

4. Which statement is true for an ungrouped data set with no outliers?
   (A) The largest possible range is 2 times the interquartile range.
   (B) The largest possible range is 3 times the interquartile range.
   (C) The largest possible range is 4 times the interquartile range.
   (D) The largest possible range is 5 times the interquartile range.

5. Which one of the following is the set of all solutions to $2x^2 - 5x + 2 \geq 0$?
   (A) $\left[\frac{1}{2}, 2\right]$
   (B) $\left(\frac{1}{2}, 2\right)$
   (C) $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$
   (D) $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$

6. The graph of $y = f(x)$ has a stationary point at $(2, -3)$.
   Which one of the following is a guaranteed stationary point of $y = -f\left(\frac{x}{2}\right) - 5$?
   (A) $(1, -2)$
   (B) $(1, 2)$
   (C) $(4, -2)$
   (D) $(4, 2)$

7. What is the period and amplitude for the curve $y = \sin \pi x$?

(A) Amplitude = 1; Period = 2

(B) Amplitude = π; Period = 2

(C) Amplitude = 1; Period = 2π

(D) Amplitude = π; Period = 2π

8. If the z scores on an examination are normally distributed and $P(z < N) = 0.6$ for some number N, what is the value of $P(-N < z < N)$?

(A) 0.1

(B) 0.2

(C) 0.3

(D) 0.4

9. Which one of the following equations is NOT correct?

(A) $\int x\left(x^2-1\right)^2 dx = \dfrac{\left(x^2-1\right)^3}{6} + c$

(B) $\int_{-3}^{3} \sqrt{9-x^2}\, dx = \dfrac{9\pi}{2}$

(C) $\int_{-1}^{1} 3^x\, dx = \dfrac{1}{\ln 3}\left(3 - \dfrac{1}{3}\right)$

(D) $\int_{-5}^{5} 4x^4 - x^3 + \cos x\, dx = 0$

10. Consider the series $\sqrt{5} + \sqrt{45} + \sqrt{125} + \ldots + z = 225\sqrt{5}$, the value of z is:

(A) $25\sqrt{5}$

(B) $29\sqrt{5}$

(C) $30\sqrt{5}$

(D) $35\sqrt{5}$

Section II

90 marks
Attempt all questions
Allow about 2 hours and 45 minutes for this section

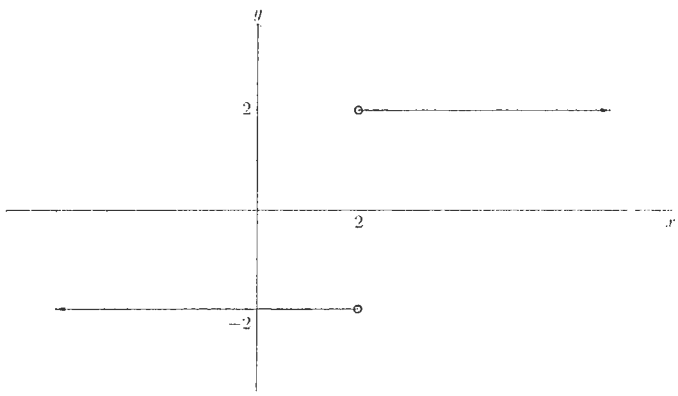
Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the back of the examination paper.

Question 11 (2 marks) **Marks**

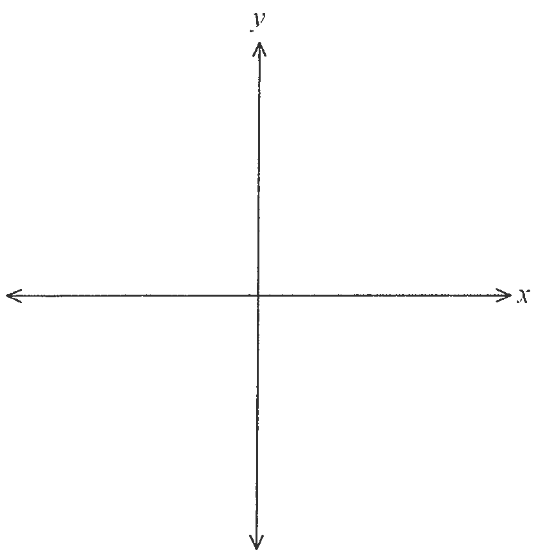
What angle does the line $2x+3y+6=0$ make with the positive x-axis? Round to the nearest minute. 2

Question 12 (2 marks)

Sketch a possible function which could have the gradient function as graphed below.

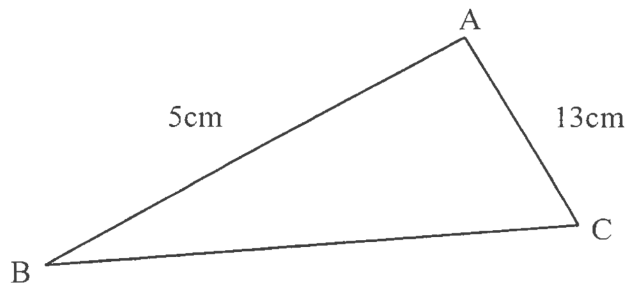


2



Question 13 (3 marks) **Marks**

In triangle ABC, the length of AB=5cm, AC=13cm and $\cos\langle BAC = \frac{1}{8}$



(a) Find the exact value of $\sin\langle BAC$ **2**

(b) Find the area of triangle ABC **1**

Question 14 (3 marks)

Solve $2\log x = \log(5x+6)$ **3**

Question 15 (3 marks) **Marks**

Solve $\left|1-2\cos^2 x\right|=1$ for $0 \le x \le 2\pi$ **3**

Question 16 (5 marks)

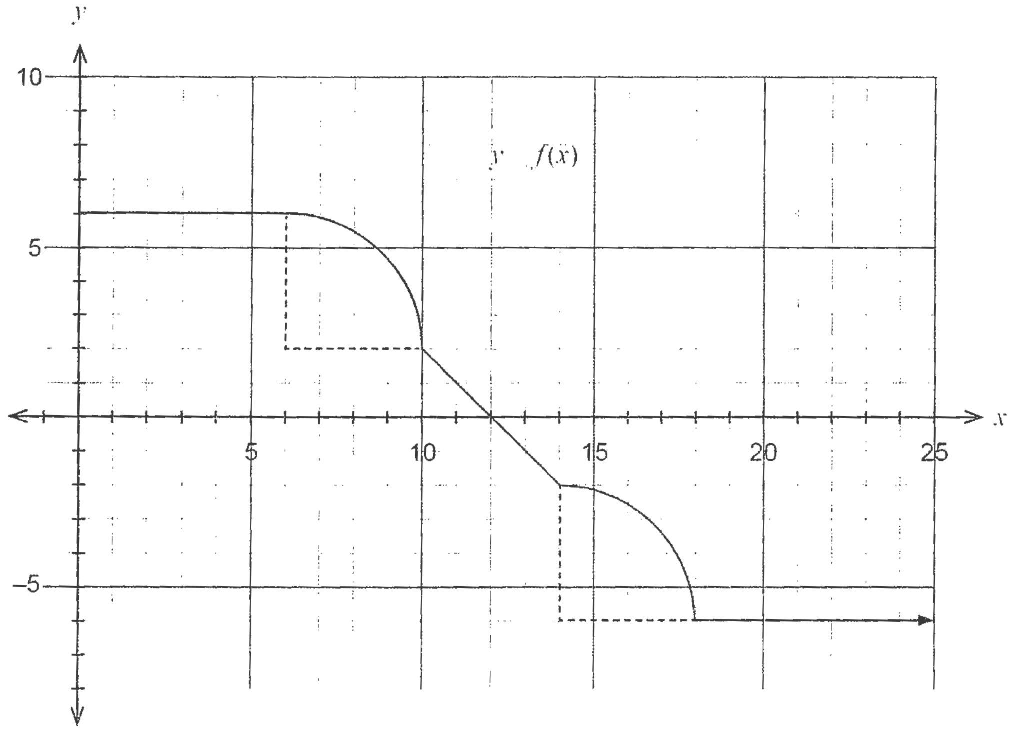
Differentiate the following expressions. **Marks**

(a) $\log_5(\tan x)$ **2**

(b) $\dfrac{2^x}{e^x}$ **3**

Question 17 (5 marks) **Marks**

Consider the graph $y = f(x)$. Both arcs have a radius of four units.



Using the graph of $y = f(x)$, $x \geq 0$, evaluate exactly the following integrals.

(a) $\displaystyle\int_0^{12} f(x)\,dx$ **3**

(b) $\displaystyle\int_0^{18} f(x)\,dx$ **2**

Question 18 (5 marks) **Marks**

The discrete random variable X has probability distribution shown in the table below

x	-1	0	1	2	3
$P(X=x)$	a	b	0.2	0.15	0.13

and $E(X)=0.55$

(a) By forming a pair of simultaneous equations, or otherwise, find the values of *a* and *b*. **3**

(b) Calculate $\mathrm{Var}(X)$ **2**

Question 19 (2 marks) **Marks**

The length of steel rods produced by a machine is normally distributed with a standard deviation of 3 mm. It is found that 2.5% of all rods are less than 25 mm long. Find the mean length of rods produced by the machine. **2**

Question 20 (8 marks)

Consider the function $f(x) = x^3 + 6x^2 + 9x + 4$ in the domain $-4 \le x \le 1$

(a) Find the coordinates of any stationary points and determine their nature. **3**

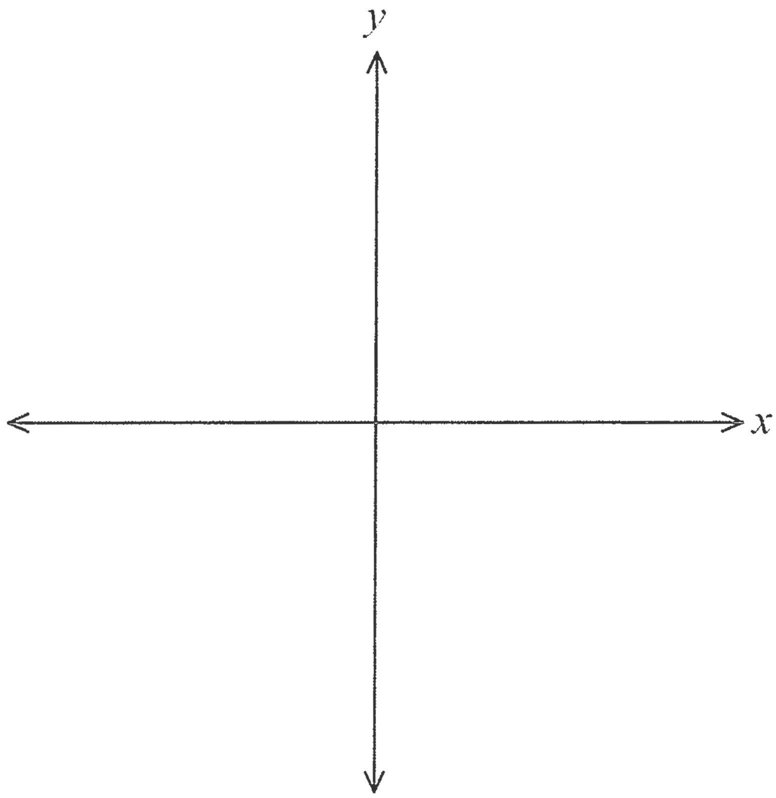
(b) Determine the coordinates of its point(s) of inflexion. **2**

Question 20 continued on next page

Question 20 continued

Marks

(c) Draw a sketch of the curve $y = f(x)$ in the domain $-4 \le x \le 1$ clearly showing all essential features. **2**

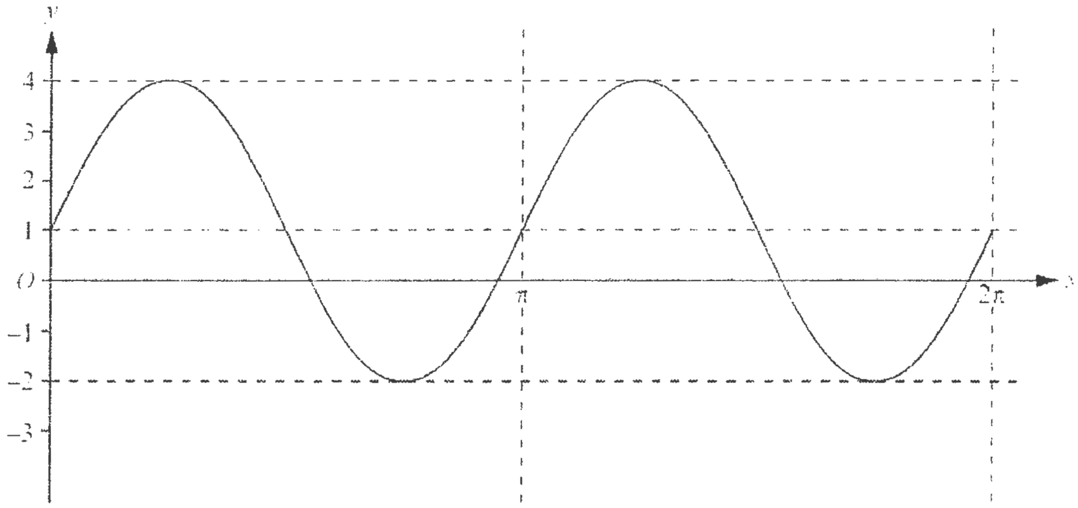


(d) What is the global maximum value of the curve in the domain $-4 \le x \le 1$ **1**

Marks

Question 21 (2 marks)

The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \leq x \leq 2\pi$, where a, b and c are positive integers.



2

Find the values of a, b and c.

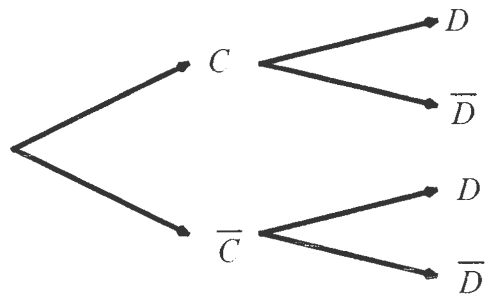
Question 22 (5 marks) **Marks**

A pet ownership survey resulted in the following results:

$P(C)=\frac{3}{7}$, $P(D|\overline{C})=\frac{2}{5}$, and $P(\overline{D}|C)=\frac{3}{4}$.

Where C is the event that "a person has a cat" and D is the event that "a person has a dog"

(a) Complete the probability tree by marking a probability on each branch. **2**



(b) If one person is chosen at random, find the probability that the person has:

i) a cat and a dog **1**

ii) at least one pet (cat or dog) **2**

Question 23 (5 marks) **Marks**

The function f is defined by $f(x)=2+\sqrt{x-3}$ for $x \geq 3$.

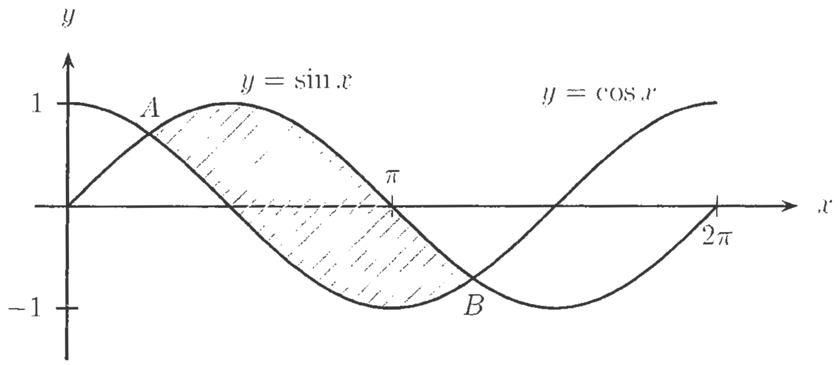
The function g is defined by $g(x)=\dfrac{12}{x}+2$ for $x>0$

(a) Write the domain and range of the function f using interval notation. **3**

(b) Write an expression for the composite function $h(x)=g(f(x))$ and hence find a value for $g(f(12))$ **2**

Question 24 (5 marks)

The diagram shows the graphs $y=\sin x$ and $y=\cos x$, $0 \leq x \leq 2\pi$. The graphs intersect at A and B.



(a) Show that A has coordinates $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$. **2**

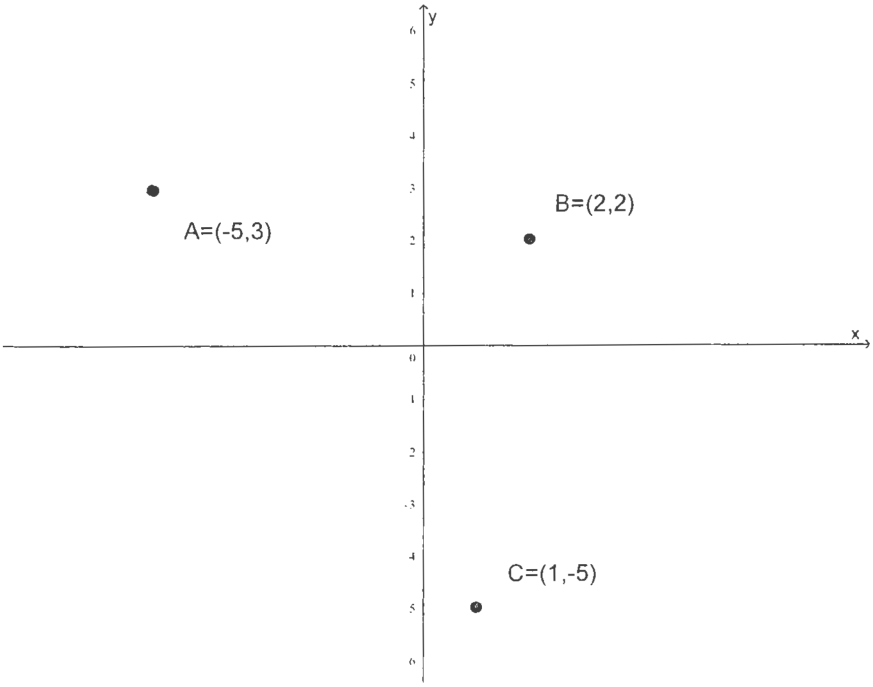
Question 24 continued on next page

Question 24 continued

(b) Find the area enclosed by the two graphs. **3**

Question 25 (3 marks) **Marks**

In the diagram, the points $A(-5,3)$, $B(2,2)$ and $C(1,-5)$ are shown.



(a) Calculate the gradient of AC. **1**

(b) Find the coordinates of X, the midpoint of AC. **1**

(c) Find the coordinates of D if X is also the midpoint of BD. **1**

Question 26 (5 marks) **Marks**

A particle is moving in a straight line. Its velocity for $t \geq 0$ is given by $v = \dfrac{4}{t+1} - 2t$, where time is in seconds and displacement in metres.

(a) Find when the particle changes direction. **2**

(b) Find the exact distance travelled in the first 2 seconds. **3**

Question 27 (6 marks) **Marks**

The continuous random variable X has probability density function $f(x)$ given by

$$f(x)=\begin{cases} k\left(x^2-2x+2\right) & 1\le x\le 4 \\ 0 & \textit{otherwise} \end{cases}$$

Where k is a constant

(a) Show that $k=\dfrac{1}{12}$ **2**

(b) Fully define the cumulative distribution function $F(x)$. **3**

Question 27 continued on next page

Question 27 continued

(c) Show that the median of X lies between $x = 3.2$ and $x = 3.3$ **1**

Question 28 (3 marks) **Marks**

Consider the geometric series $1+\frac{4}{3}\sin^2 x+\frac{16}{9}\sin^4 x+\frac{64}{27}\sin^6 x+....$

(a) When the limiting sum exists, find an expression for its value. **1**

(b) For what values of x in the interval $0<x<\frac{\pi}{2}$ does the limiting sum of this series exist? **2**

Question 29 (4 marks)

(a) Find $\int \sec^2(2x)\tan^4(2x)dx$ **2**

(b) Find $\int \frac{5x^2}{x^3+1}dx$ **2**

Question 30 (4 marks) **Marks**

Luke suspects that the rate at which he spends cash is affected by the amount of cash he withdrew at his previous visit to an ATM.

The table below shows the amount of cash withdrawn, \$$x$, from an ATM, and the time, y hours, until Luke's next withdrawal from an ATM.

Withdrawal	1	2	3	4	5	6	7	8	9	10
x	40	10	100	110	120	150	20	90	80	130
y	56	62	195	330	94	270	48	196	214	286

(a) Find the equation of the least squares regression line for y in terms of x, for the withdrawals 1 to 10 and hence estimate how much cash (to the nearest \$10) Luke would need to withdraw from the ATM at his previous visit in order to not need to visit an ATM again for 120 hours. **2**

(b) Calculate the correlation coefficient between x and y for the withdrawals 1 to 10. Describe the nature of the correlation. **2**

Question 31 (4 marks) **Marks**

The table below shows the future values of an annuity, for different rates of interest and for different numbers of compounding periods, where contributions of $1 are made at the end of each compounding period.

Table of future value interest factors

n	1%	2%	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753

(a) An annuity account is opened and contributions of $1200 are made at the end of each half year for 3 years at an interest rate of 4% p.a. compounding half yearly. Calculate the final amount in the account immediately after the last contribution is made. **2**

(b) Calculate the single lump sum amount that would need to be invested at the start to reach the same final amount at the end of the 3 years with the same interest rate of 4% compounding half yearly. **2**

Question 32 (6 marks) **Marks**

An open cone, of radius r cm and height h cm is made from a sector of a circle. The area of the sector used is 300 cm²

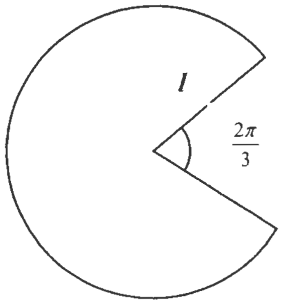


Figure 1

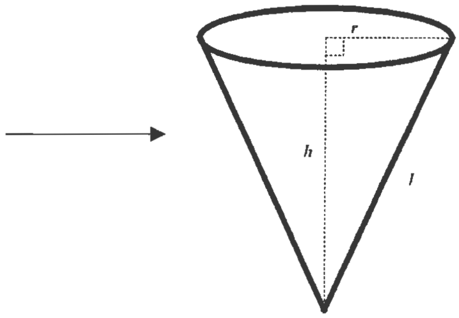


Figure 2

(a) Show from Figure 1 that the slant height l is given by $l^2 = \dfrac{450}{\pi}$ **2**

(b) In Figure 2 it is given that $h = \sqrt{l^2 - r^2}$ (do not prove this). **2**

Show that the volume of the cone is given by $V = \dfrac{1}{3}r^2\sqrt{450\pi - \pi^2 r^2}$

Question 32 continued on next page

Question 32 continued

(c) It is known that $\frac{dV}{dr} = \frac{300\pi r - \pi^2 r^3}{\sqrt{450\pi - \pi^2 r^2}}$ (do not prove this). **2**

Hence or otherwise, find the value of r for the volume of the cone to be a maximum.

End of paper

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1-10

1. What is the value of $\operatorname{cosec}\frac{\pi}{3}$ to three significant figures?

(A) 1.00
(B) 1.15
(C) 1.41
(D) 2.00

2. What is the value of c for which the circle $(x-3)^2+(y-2)^2=c$ touches the x axis?

(A) 2
(B) 3
(C) 4
(D) 9

3. What is the equation of the tangent to $y=x^2-3$ at $x=-1$?

(A) $y=-2x-4$
(B) $y=2x-4$
(C) $y=\frac{x}{2}-\frac{3}{2}$
(D) $y=-\frac{x}{2}-\frac{3}{2}$

4. Which statement is true for an ungrouped data set with no outliers?

(A) The largest possible range is 2 times the interquartile range.
(B) The largest possible range is 3 times the interquartile range.
(C) The largest possible range is 4 times the interquartile range.
(D) The largest possible range is 5 times the interquartile range.

5. Which one of the following is the set of all solutions to $2x^2 - 5x + 2 \geq 0$?

(A) $\left[\frac{1}{2}, 2\right]$

(B) $\left(\frac{1}{2}, 2\right)$

(C) $\left(-\infty, \frac{1}{2}\right) \cup (2, \infty)$

(D) $\left(-\infty, \frac{1}{2}\right] \cup [2, \infty)$

6. The graph of $y = f(x)$ has a stationary point at $(2, -3)$.

Which one of the following is a guaranteed stationary point of $y = -f\left(\frac{x}{2}\right) - 5$?

(A) $(1, -2)$

(B) $(1, 2)$

(C) $(4, -2)$

(D) $(4, 2)$

7. What is the period and amplitude for the curve $y = \sin \pi x$?
   (A) Amplitude = 1; Period = 2
   (B) Amplitude = π; Period = 2
   (C) Amplitude = 1; Period = 2π
   (D) Amplitude = π; Period = 2π

8. If the z scores on an examination are normally distributed and $P(z < N) = 0.6$ for some number N, what is the value of $P(-N < z < N)$?
   (A) 0.1
   (B) 0.2
   (C) 0.3
   (D) 0.4

9. Which one of the following equations is NOT correct?
   (A) $\int x\left(x^2-1\right)^2 dx = \dfrac{\left(x^2-1\right)^3}{6} + c$
   (B) $\int_{-3}^{3} \sqrt{9-x^2}\, dx = \dfrac{9\pi}{2}$
   (C) $\int_{-1}^{1} 3^x\, dx = \dfrac{1}{\ln 3}\left(3 - \dfrac{1}{3}\right)$
   (D) $\int_{-5}^{5} 4x^4 - x^3 + \cos x\, dx = 0$

10. Consider the series $\sqrt{5} + \sqrt{45} + \sqrt{125} + \ldots + z = 225\sqrt{5}$, the value of z is:
   (A) $25\sqrt{5}$
   (B) $29\sqrt{5}$
   (C) $30\sqrt{5}$
   (D) $35\sqrt{5}$

Section II

90 marks
Attempt all questions
Allow about 2 hours and 45 minutes for this section

Answer each question in the spaces provided.
Your responses should include relevant mathematical reasoning and/or calculations.
Extra writing space is provided at the back of the examination paper.

Question 11 (2 marks) **Marks**

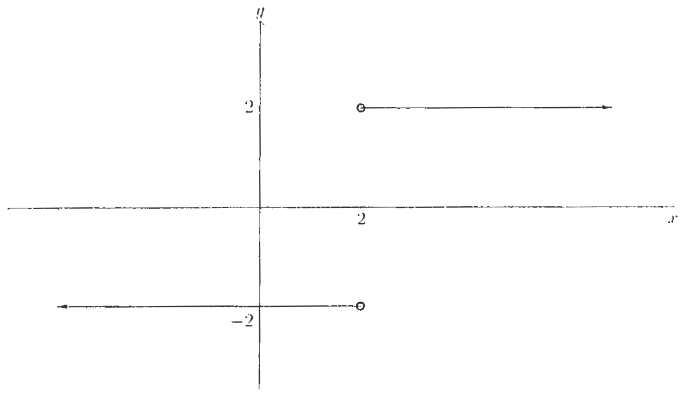
What angle does the line $2x + 3y + 6 = 0$ make with the positive x-axis? Round to the nearest minute. 2

$m = -\frac{2}{3}$ $\therefore$ $\tan \alpha = -\frac{2}{3}$

$\alpha = 146°19'$

Question 12 (2 marks)

Sketch a possible function which could have the gradient function as graphed below.



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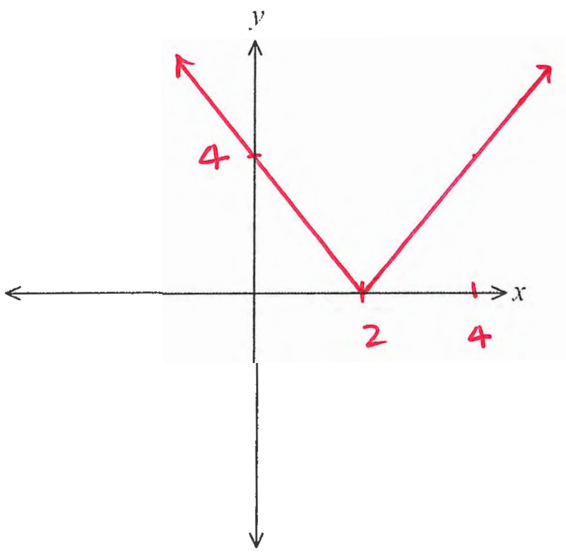
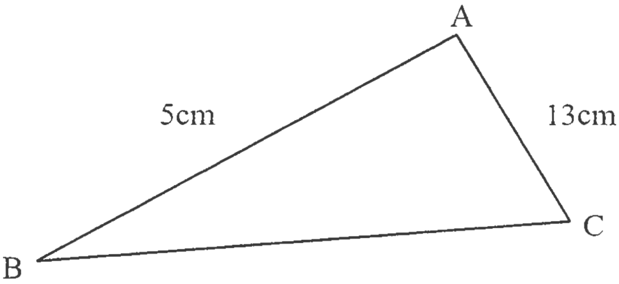
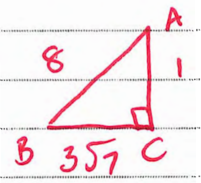


Question 13 (3 marks) **Marks**

In triangle ABC, the length of AB=5cm, AC=13cm and $\cos\langle BAC = \frac{1}{8}$



(a) Find the exact value of $\sin\langle BAC$ **2**



$\sin \angle BAC = \frac{3\sqrt{7}}{8}$

(b) Find the area of triangle ABC **1**

$A = \frac{1}{2} \times 5 \times 13 \times \sin \angle BAC$

$= \frac{1}{2} \times 5 \times 13 \times \frac{3\sqrt{7}}{8}$

$= \frac{195\sqrt{7}}{16}$

$\doteqdot 32.2 \ u^2$

Question 14 (3 marks)

Solve $2\log x = \log(5x+6)$ **3**

$\log x^2 = \log(5x+6)$

$\therefore x^2 = 5x + 6$

$x^2 - 5x - 6 = 0$

$(x-6)(x+1) = 0$

$x = 6, -1$ but $x > 0$

$\therefore x = 6$

Question 15 (3 marks) **Marks**

Solve $\left|1-2\cos^2 x\right|=1$ for $0\le x\le 2\pi$ 3

$1-2\cos^2 x=1$
$2\cos^2 x=0$
$\cos^2 x=0$
$x=\frac{\pi}{2},\frac{3\pi}{2}$

$1-2\cos^2 x=-1$
$2\cos^2 x=2$
$\cos^2 x=1$
$\cos x=\pm 1$
$x=0,\pi,2\pi$

$\therefore\ x=0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi$

Question 16 (5 marks)

Differentiate the following expressions. **Marks**

(a) $\log_5(\tan x)$ 2

$\frac{d}{dx}\left(\log_5(\tan x)\right)=\frac{\sec^2 x}{\ln 5\tan x}$

(b) $\frac{2^x}{e^x}$ 3

$u=2^x$ $\quad v=e^x$
$u'=\ln 2\,.\,2^x$ $\quad v'=e^x$

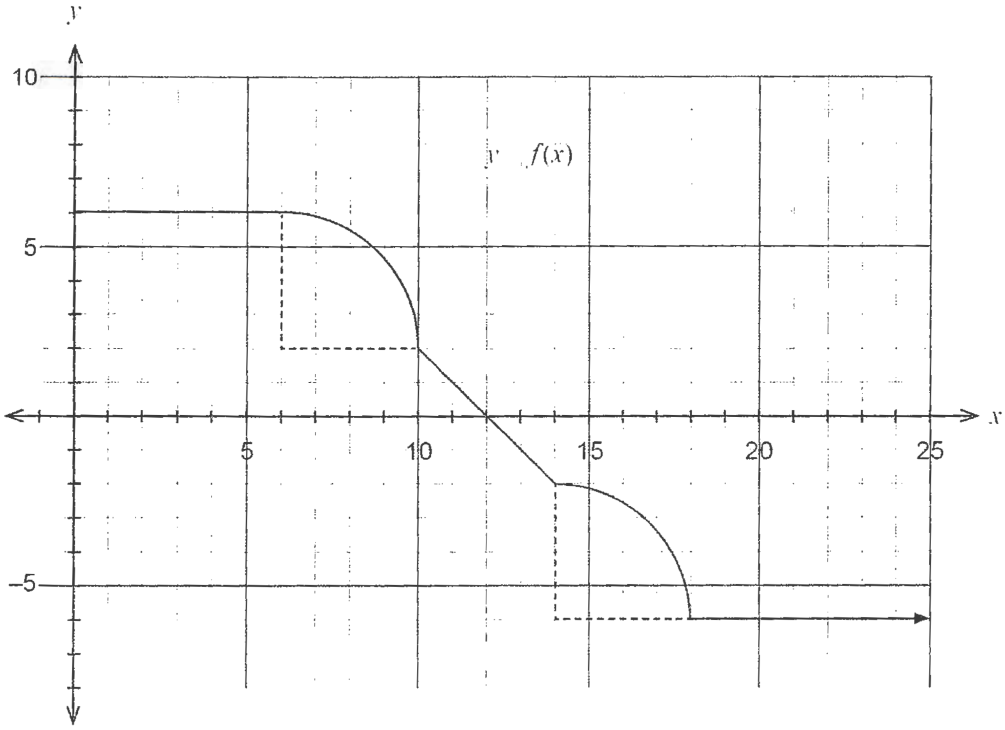
$\frac{d}{dx}\left(\frac{2^x}{e^x}\right)=\frac{\ln 2.\,2^x.\,e^x-2^x.\,e^x}{(e^x)^2}$

$=\frac{e^x 2^x(\ln 2-1)}{e^{2x}}$

$=\frac{2^x(\ln 2-1)}{e^x}$

Question 17 (5 marks) **Marks**

Consider the graph $y = f(x)$. Both arcs have a radius of four units.



Using the graph of $y = f(x), x \geq 0$, evaluate exactly the following integrals.

(a) $\int_0^{12} f(x)dx$ **3**

$= (5\times 6) + (6+8) + \frac{1}{4}\times\pi\times 4^2 + \frac{1}{2}\times 2\times 2$

$= 46 + 4\pi$

(b) $\int_0^{18} f(x)dx$ **2**

$= \int_0^{12} f(x)\,dx + \int_{12}^{18} f(x)\,dx$

$= 46 + 4\pi - \left(\frac{1}{2}\times 2\times 2 + \left(4\times 6 - \frac{1}{4}\times\pi\times 4^2\right)\right)$

$= 46 + 4\pi - (26 - 4\pi)$

$= 20 + 8\pi$

Question 18 (5 marks) **Marks**

The discrete random variable X has probability distribution shown in the table below

x	-1	0	1	2	3
$P(X=x)$	a	b	0.2	0.15	0.13

and $E(X)=0.55$

(a) By forming a pair of simultaneous equations, or otherwise, find the values of a and b. **3**

$E(x) = \sum x_i p_i$

$= -1(a) + 0(b) + 1(0.2) + 2(0.15) + 3(0.13)$

$\therefore 0.55 = -a + 0.89$

$a = 0.34$

& $0.34 + b + 0.2 + 0.15 + 0.13 = 1$

$\therefore b = 0.18$

(b) Calculate $\text{Var}(X)$ **2**

$Var(x) = E(x^2) - \mu^2$

$E(x^2) = 1(a) + 0(b) + 1(0.2) + 4(0.15) + 9(0.13)$

$= 2.31$

$\therefore Var(x) = 2.31 - 0.55^2$

$= 2.0075$

Question 19 (2 marks) **Marks**

The length of steel rods produced by a machine is normally distributed with a standard deviation of 3 mm. It is found that 2.5% of all rods are less than 25 mm long. Find the mean length of rods produced by the machine. **2**

$\sigma = 3 \quad x = 25 \quad z = -2$

$\therefore -2 = \frac{25-\mu}{3}$

$25 - \mu = -6$

$\mu = 31$

Question 20 (8 marks)

Consider the function $f(x) = x^3 + 6x^2 + 9x + 4$ in the domain $-4 \le x \le 1$

(a) Find the coordinates of any stationary points and determine their nature. **3**

$f'(x) = 3x^2 + 12x + 9 \qquad f''(x) = 6x + 12$

stat. pts at $f'(x) = 0$

$3(x^2 + 4x + 3) = 0$

$(x+3)(x+1) = 0$

$x = -3 \qquad x = -1$

$y = 4 \qquad y = 0$

$f''(-3) = -18 + 12 < 0 \qquad f''(-1) = -6 + 12 > 0$

∩ ∴ max turning pt at $(-3, 4)$

∪ ∴ min turn. pt at $(-1, 0)$

(b) Determine the coordinates of its point(s) of inflexion. **2**

poss. pt. inflex at $y'' = 0$

$\therefore 6x + 12 = 0$

$x = -2$

$y = 2$

x	-1	-2	-3
$f''(x)$	6	0	-6

∴ change in concavity

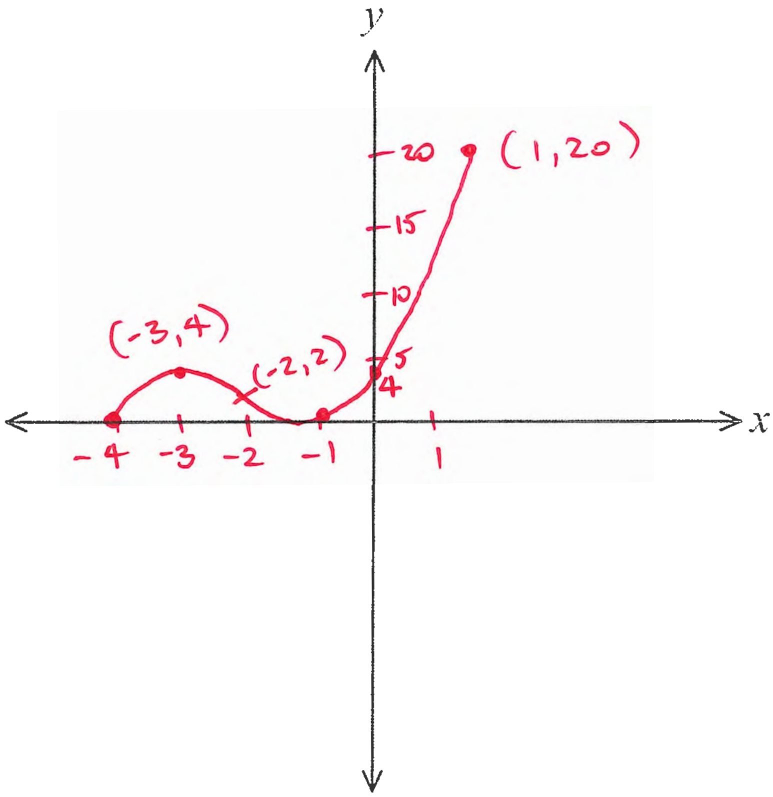
∴ pt. inflex at $(-2, 2)$

Question 20 continued on next page

Question 20 continued

Marks

(c) Draw a sketch of the curve $y = f(x)$ in the domain $-4 \le x \le 1$ clearly showing all essential features. **2**



endpoints $(-4,0)$ & $(1,20)$

y-int = 4

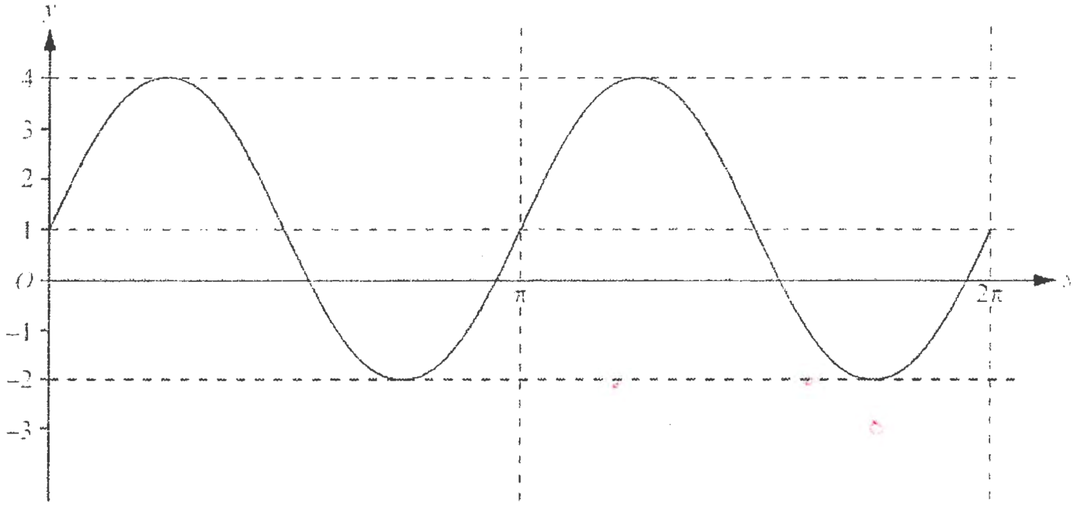
(d) What is the global maximum value of the curve in the domain $-4 \le x \le 1$ **1**

20

Marks

Question 21 (2 marks)

The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \le x \le 2\pi$, where a, b and c are positive integers. **2**



Find the values of a, b and c.

$a = 3$

$c = 1$

$T = \pi$

$\therefore \dfrac{2\pi}{b} = \pi$

$\therefore b = 2$

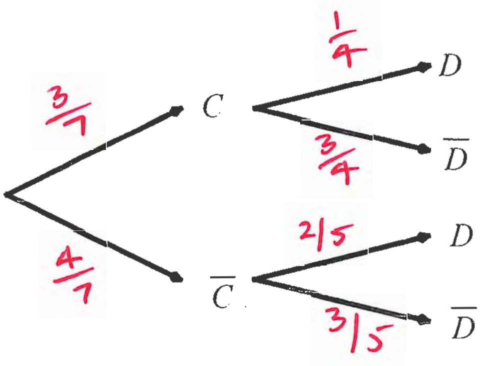
Question 22 (5 marks) **Marks**

A pet ownership survey resulted in the following results:

$P(C)=\frac{3}{7}$, $P(D\,|\,\overline{C})=\frac{2}{5}$, and $P(\overline{D}\,|\,C)=\frac{3}{4}$.

Where C is the event that "a person has a cat" and D is the event that "a person has a dog"

(a) Complete the probability tree by marking a probability on each branch. **2**



(b) If one person is chosen at random, find the probability that the person has:

i) a cat and a dog **1**

$P(C\cap D)=\frac{3}{7}\times\frac{1}{4}$

$=\frac{3}{28}$

ii) at least one pet (cat or dog) **2**

$P(\text{at least 1 pet})=1-P(\overline{D}\,\overline{C})$

$=1-\frac{4}{7}\times\frac{3}{5}$

$=\frac{23}{35}$

Question 23 (5 marks) **Marks**

The function f is defined by $f(x) = 2 + \sqrt{x-3}$ for $x \geq 3$.

The function g is defined by $g(x) = \frac{12}{x} + 2$ for $x > 0$

(a) Write the domain and range of the function f using interval notation. **3**

domain: $x \geq 3$ $[3, \infty)$

range: $y \geq 2$ $[2, \infty)$

(b) Write an expression for the composite function $h(x) = g(f(x))$ and hence find a value for $g(f(12))$ **2**

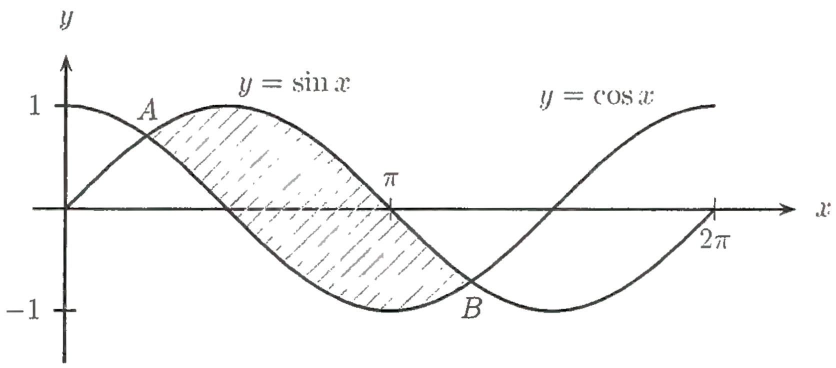
$g(f(x)) = \frac{12}{2+\sqrt{x-3}} + 2$

$g(f(12)) = \frac{12}{2+\sqrt{12-3}} + 2$

$= \frac{22}{5}$

Question 24 (5 marks)

The diagram shows the graphs $y = \sin x$ and $y = \cos x$, $0 \leq x \leq 2\pi$. The graphs intersect at A and B.



(a) Show that A has coordinates $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$. **2**

pt. int at $\sin x = \cos x$ $(\div \cos x)$

$\tan x = 1$

$x = \frac{\pi}{4}$ $0 < x < \frac{\pi}{2}$

sub in $y = \sin x$

$\therefore y = \sin \frac{\pi}{4}$

$y = \frac{1}{\sqrt{2}}$ $\therefore A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

Question 24 continued on next page

Question 24 continued

(b) Find the area enclosed by the two graphs. **3**

Solve for B. $\tan x = 1$ $\pi < x < \frac{3\pi}{2}$

$\therefore x = \frac{5\pi}{4}$

$A = \int_{\frac{\pi}{4}}^{5\pi/4} (\sin x - \cos x)\, dx$

$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

$= -\left[\cos x + \sin x\right]_{\frac{\pi}{4}}^{5\pi/4}$

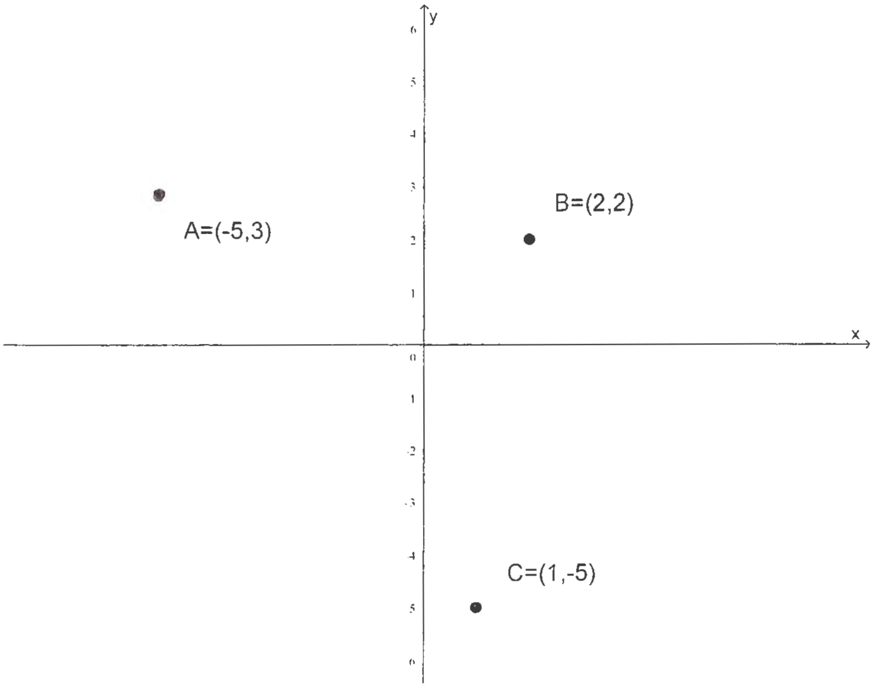
$= -\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right)$

$= -\left(-\frac{4}{\sqrt{2}}\right.$

$= 2\sqrt{2}$ u^2

Question 25 (3 marks) **Marks**

In the diagram, the points $A(-5,3)$, $B(2,2)$ and $C(1,-5)$ are shown.



(a) Calculate the gradient of *AC*. **1**

$m(AC) = \frac{-5-3}{1+5}$

$= -\frac{8}{6}$

$= \frac{-4}{3}$

(b) Find the coordinates of *X*, the midpoint of *AC*. **1**

$M = \left(\frac{-5+1}{2}, \frac{3-5}{2}\right)$

$= (-2, -1)$

(c) Find the coordinates of *D* if *X* is also the midpoint of *BD*. **1**

$-2 = \frac{2+x}{2}$ $\quad$ $-1 = \frac{2+y}{2}$

$-4 = 2+x$ $\quad$ $2+y = -2$

$x = -6$ $\quad$ $y = -4$

$\therefore D(-6, -4)$

Question 26 (5 marks) **Marks**

A particle is moving in a straight line. Its velocity for $t \geq 0$ is given by $v = \dfrac{4}{t+1} - 2t$, where time is in seconds and displacement in metres.

(a) Find when the particle changes direction. **2**

Change direction at $v = 0$

$\therefore \dfrac{4}{t+1} - 2t = 0$

$4 - 2t^2 - 2t = 0$

$t^2 + t - 2 = 0$

$(t+2)(t-1) = 0$

$t = -2, 1$, $t > 0$

$\therefore t = 1$

(b) Find the exact distance travelled in the first 2 seconds. **3**

$$d = \left| \int_0^1 \left(\frac{4}{t+1} - 2t \right) dt \right| + \left| \int_1^2 \left(\frac{4}{t+1} - 2t \right) dt \right|$$

$$= \left| \left[4\ln|t+1| - t^2 \right]_0^1 \right| + \left| \left[4\ln|t+1| - t^2 \right]_1^2 \right|$$

$$= |4\ln 2 - 1| + |4\ln 3 - 3 - 4\ln 2|$$

$$= 4\ln 2 - 1 - (4\ln 3 - 3 - 4\ln 2)$$

$$= 8\ln 2 + 2 - 4\ln 3$$

Question 27 (6 marks) **Marks**

The continuous random variable X has probability density function $f(x)$ given by

$$f(x)=\begin{cases} k(x^2-2x+2) & 1\le x\le 4 \\ 0 & otherwise \end{cases}$$

Where k is a constant

(a) Show that $k=\frac{1}{12}$ **2**

$F(x) = k\int_1^4 (x^2-2x+2)\,dx$

$\therefore k\left[\frac{x^3}{3} - x^2 + 2x\right]_1^4 = 1$

$\therefore k\left(\left(\frac{64}{3} - 16 + 8\right) - \left(\frac{1}{3} - 1 + 2\right)\right) = 1$

$\therefore 12k = 1, \quad k = \frac{1}{12}$ ✓

(b) Fully define the cumulative distribution function $F(x)$. **3**

For $1 \le x \le 4$

$F(x) = \frac{1}{12}\int_1^x (x^2 - 2x + 2)\,dx$

$= \frac{1}{12}\left[\frac{x^3}{3} - x^2 + 2x\right]_1^x$

$= \frac{1}{12}\left[\frac{x^3}{3} - x^2 + 2x - \left(\frac{1}{3} - 1 + 2\right)\right]$

$= \frac{1}{12}\left[\frac{x^3}{3} - x^2 + 2x - \frac{4}{3}\right]$

$= \frac{x^3}{36} - \frac{x^2}{12} + \frac{x}{6} - \frac{1}{9}$

$$\therefore F(x) = \begin{cases} 0 & x < 1 \\ \frac{x^3}{36} - \frac{x^2}{12} + \frac{x}{6} - \frac{1}{9} & 1 \le x \le 4 \\ 1 & x > 4 \end{cases}$$

Question 27 continued on next page

Question 27 continued

(c) Show that the median of X lies between $x = 3.2$ and $x = 3.3$ **1**

$F(3.2) = \frac{3.2^3}{36} - \frac{3.2^2}{12} + \frac{3.2}{6} - \frac{1}{9}$

$= 0.479 < 0.5$

$F(3.3) = \frac{3.3^3}{36} - \frac{3.3^2}{12} + \frac{3.3}{6} - \frac{1}{9}$

$= 0.5296 > 0.5$

$F(\text{median}) = 0.5$

$\therefore$ median lies between $x = 3.2$

& $x = 3.3$

Question 28 (3 marks) **Marks**

Consider the geometric series $1+\frac{4}{3}\sin^2 x+\frac{16}{9}\sin^4 x+\frac{64}{27}\sin^6 x+\ldots$

(a) When the limiting sum exists, find an expression for its value. **1**

$r = \frac{4}{3}\sin^2 x$

$S_\infty = \frac{1}{1-\frac{4}{3}\sin^2 x}$

$a = 1$

(b) For what values of x in the interval $0 < x < \frac{\pi}{2}$ does the limiting sum of this series exist? **2**

S_∞ exists for $|r| < 1$ ie $-1 < r < 1$

$\therefore \left|\frac{4}{3}\sin^2 x\right| < 1$ ie $-1 < \frac{4}{3}\sin^2 x < 1$

$0 < \frac{4}{3}\sin^2 x < 1$ in Q1

$0 < \sin^2 x < \frac{3}{4}$

$0 < \sin x < \frac{\sqrt{3}}{2}$

$\therefore 0 < x < \frac{\pi}{3}$

Question 29 (4 marks)

(a) Find $\int \sec^2(2x)\tan^4(2x)dx$ **2**

$= \frac{1}{2}\int 2\sec^2(2x)\tan^4(2x)\,dx$

$= \frac{1}{2}\frac{\tan^5(2x)}{5} + c$

$= \frac{\tan^5(2x)}{10} + c$

(b) Find $\int \frac{5x^2}{x^3+1}dx$ **2**

$= \frac{5}{3}\int \frac{3x^2}{x^3+1}\,dx$

$= \frac{5}{3}\ln|x^3+1| + c$

Question 30 (4 marks) **Marks**

Luke suspects that the rate at which he spends cash is affected by the amount of cash he withdrew at his previous visit to an ATM.

The table below shows the amount of cash withdrawn, $\$x$, from an ATM, and the time, y hours, until Luke's next withdrawal from an ATM.

Withdrawal	1	2	3	4	5	6	7	8	9	10
x	40	10	100	110	120	150	20	90	80	130
y	56	62	195	330	94	270	48	196	214	286

(a) Find the equation of the least squares regression line for y in terms of x, for the withdrawals 1 to 10 and hence estimate how much cash (to the nearest $10) Luke would need to withdraw from the ATM at his previous visit in order to not need to visit an ATM again for 120 hours. **2**

from calculator:
$A = 30.26$
$B = 1.7$

$y = A + Bx$
$y = 1.7x + 30.26$

Find x at $y = 120$
$\therefore 120 = 1.7x + 30.26$
$\therefore x = 52.79$

$\therefore$ withdrawal $= \$60$

(b) Calculate the correlation coefficient between x and y for the withdrawals 1 to 10. Describe the nature of the correlation. **2**

$r = 0.777$ {strong} positive correlation

Question 31 (4 marks) **Marks**

The table below shows the future values of an annuity, for different rates of interest and for different numbers of compounding periods, where contributions of $1 are made at the end of each compounding period.

Table of future value interest factors

n	1%	2%	3%	4%	5%	6%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753

(a) An annuity account is opened and contributions of $1200 are made at the end of each half year for 3 years at an interest rate of 4% p.a. compounding half yearly. Calculate the final amount in the account immediately after the last contribution is made. **2**

$FV = 1200 \times 6.3081$ $\quad r = 2\%$ / half year

$= \$7569.72$ $\quad n = 6$

(b) Calculate the single lump sum amount that would need to be invested at the start to reach the same final amount at the end of the 3 years with the same interest rate of 4% compounding half yearly. **2**

$\therefore \quad 7569.72 = P(1.02)^6$

$\therefore \quad P = \$6721.69$

Question 32 (6 marks) **Marks**

An open cone, of radius r cm and height h cm is made from a sector of a circle. The area of the sector used is 300 cm²

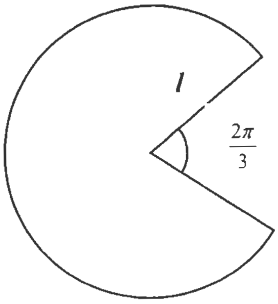


Figure 1

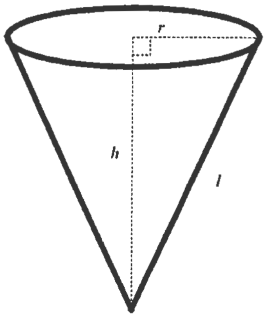


Figure 2

(a) Show from Figure 1 that the slant height l is given by $l^2 = \dfrac{450}{\pi}$ **2**

$A = \frac{1}{2} r^2 \theta$

$A = \frac{1}{2} \times l^2 \times \left(2\pi - \frac{2\pi}{3}\right)$

$= \frac{1}{2} l^2 \times \frac{4\pi}{3}$

$300 = \frac{2}{3} \pi l^2$

$\therefore l^2 = \frac{900}{2\pi}$

$l^2 = \frac{450}{\pi}$

(b) In Figure 2 it is given that $h = \sqrt{l^2 - r^2}$ (do not prove this). **2**

Show that the volume of the cone is given by $V = \dfrac{1}{3} r^2 \sqrt{450\pi - \pi^2 r^2}$

$V = \frac{1}{3} \pi r^2 h$

$V = \frac{1}{3} \pi r^2 \sqrt{l^2 - r^2}$

$= \frac{1}{3} \pi r^2 \sqrt{\frac{450}{\pi} - r^2}$

$= \frac{1}{3} r^2 \sqrt{\pi^2 \left(\frac{450 - \pi r^2}{\pi}\right)}$

$= \frac{1}{3} r^2 \sqrt{450\pi - \pi^2 r^2}$

Question 32 continued on next page

Question 32 continued

(c) It is known that $\dfrac{dV}{dr} = \dfrac{300\pi r - \pi^2 r^3}{\sqrt{450\pi - \pi^2 r^2}}$ (do not prove this). **2**

Hence or otherwise, find the value of r for the volume of the cone to be a maximum.

stat pt at $\frac{dV}{dr} = 0$

$\therefore 300\pi r - \pi^2 r^3 = 0$

$\therefore \pi r(300 - \pi r^2) = 0 \qquad r > 0$

$\pi r^2 = 300$

$r^2 = \frac{300}{\pi}$

$r = \sqrt{\frac{300}{\pi}} = \frac{10\sqrt{3}}{\sqrt{\pi}}$

r	$\sqrt{\frac{290}{\pi}}$	$\sqrt{\frac{300}{\pi}}$	$\sqrt{\frac{320}{\pi}}$
$\frac{dV}{dr}$	13.46	0	−31.38

/ — \

$\therefore$ max vol at

$r = \sqrt{\frac{300}{\pi}}$

End of paper