HORNSBY GIRLS HIGH SCHOOL

Mathematics

Year 12 Higher School Certificate
Trial Examination Term 3 2018

STUDENT NUMBER: ______________________________

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- NESA-approved calculators and drawing templates may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room

Total marks – 100

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 18

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

---

1 What is the value of $\dfrac{(\pi+2)^3}{11}$, correct to 3 significant figures.

(A) 12.357

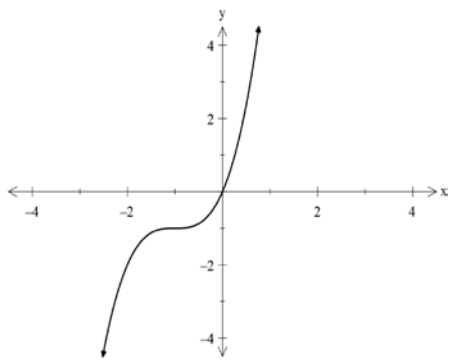
(B) 12.356

(C) 12.4

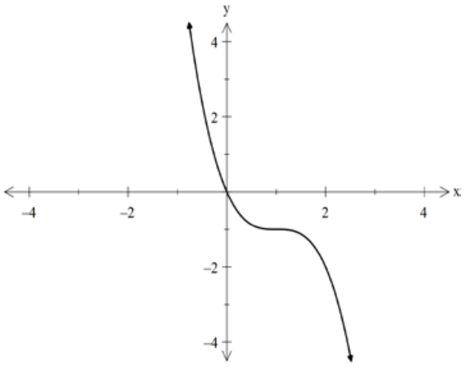
(D) 3.87

2 Which graph best represents $y=\left(x+1\right)^3-1$?

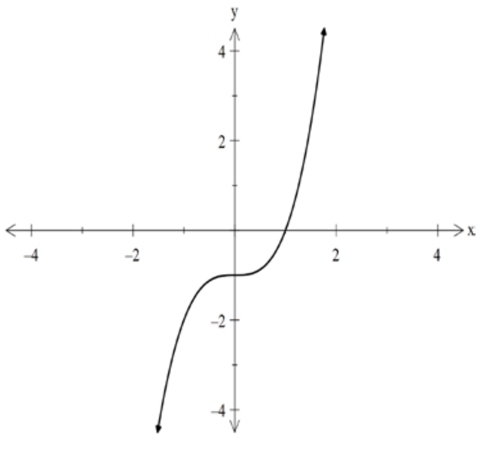
(A)



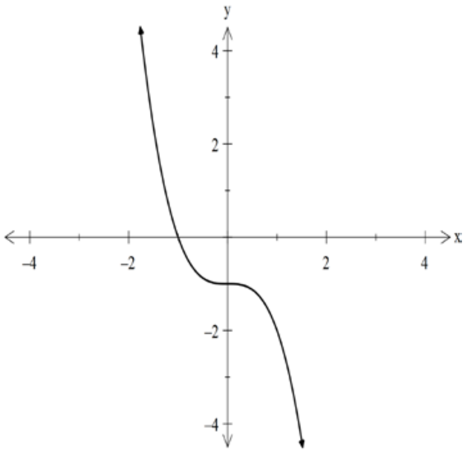
(B)



(C)



(D)



3 The derivative of $\dfrac{\ln x}{x^2}$ is

(A) $\dfrac{x-2\ln x}{x^4}$

(B) $\dfrac{1-\ln x}{x^3}$

(C) $\dfrac{1-2\ln x}{x^3}$

(D) $\dfrac{1-2x\ln x}{x^4}$

4 Which expression is equal to $\int e^{4x}\,dx$?

(A) $4e^{4x}+c$

(B) $\dfrac{e^{4x}}{4}+c$

(C) $\dfrac{e^{4x+1}}{4x+1}+c$

(D) $\dfrac{e^{5x}}{5}+c$

5 Which equation represents the line perpendicular to $3x+2y=5$ and intersecting y-axis at $(0,-1)$?

(A) $3x-2y=-2$

(B) $2x-3y=3$

(C) $2x+3y=3$

(D) $3x+2y=-2$

6 Which expression is a factorisation of $27x^3-1$?

(A) $(3x+1)(9x^2-3x+1)$

(B) $(3x-1)(9x^2-6x+1)$

(C) $(3x-1)(9x^2+3x+1)$

(D) $(3x-1)(9x^2+6x+1)$

7 The maximum value of the graph $y = 1 - 2\cos x$ is:

(A) -1

(B) 1

(C) 2

(D) 3

8 Which of the following is a term of the geometric sequence

$$2p, -4p^2, 8p^3, -16p^4, ... ?$$

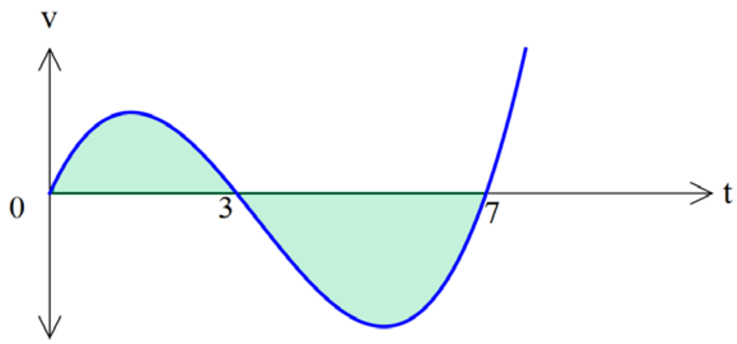
(A) $256p^8$

(B) $512p^8$

(C) $-256p^8$

(D) $-512p^8$

9



The graph above represents the velocity *v* m/s of a particle after *t* seconds. The particle is moving in a straight line from rest.

Which of the following describes the area of the shaded region?

(A) The acceleration of the particle after 7 seconds.

(B) The velocity of the particle in the first 7 seconds.

(C) The displacement of the particle after 7 seconds.

(D) The distance travelled by the particle in the first 7 seconds.

10 A raffle consists of thirty tickets in which there are two prizes. First prize is two book vouchers and second prize is one book voucher. Emma buys five tickets.

The probability that Emma wins at least one book voucher is

(A) $\dfrac{9}{29}$

(B) $\dfrac{11}{36}$

(C) $\dfrac{3}{10}$

(D) $\dfrac{49}{174}$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet.

(a) Rationalise the denominator of $\dfrac{1}{2\sqrt{3}+5}$. **2**

(b) Factorise $6x^2 - 5x + 1$. **2**

(c) Differentiate $x\sqrt{x+1}$. **2**

(d) Find $\displaystyle\int \frac{1}{(2x-5)^3}\,dx$. **2**

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$. **3**

(f) The gradient function of a curve $y = f(x)$ is given by $f'(x) = e^{2x+1}$, **2**

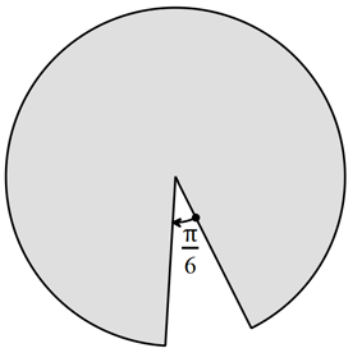
and $f(0) = \dfrac{-e}{2}$, find the equation of $f(x)$.

Question 11 continues on page 8

Question 11 (continued)

(g) The area of the shaded major sector of the circle is $\dfrac{44\pi}{3}$. **2**

Find the radius of the sector.

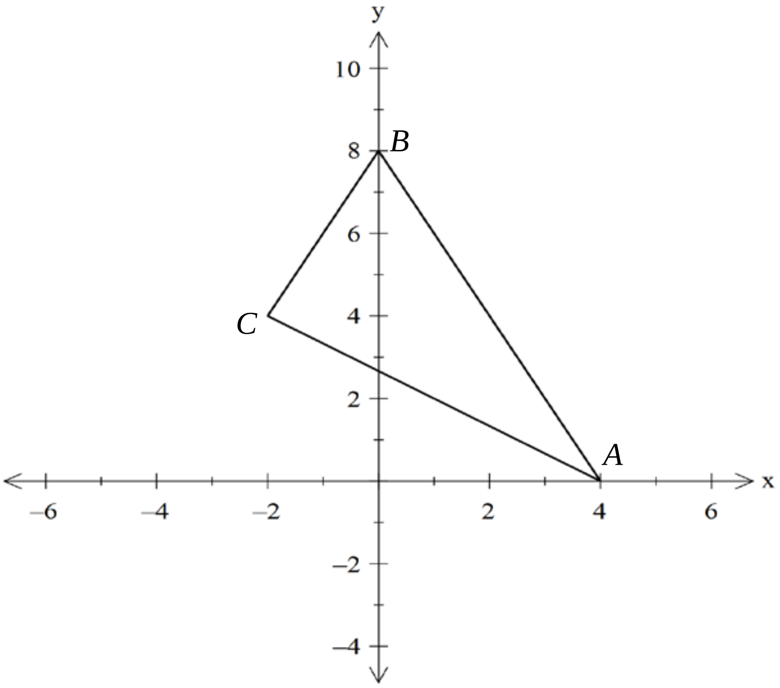


End of Question 11

Question 12 (15 marks) Start a new writing booklet.

(a) A 35 cm seedling grows by $16\frac{2}{7}$ cm in the first week, and then keeps growing by $\frac{4}{7}$ of its previous weeks growth. What is the maximum height the seedling can grow to? **2**

(b) The points $A(4,0), B(0,8)$ and $C(-2,4)$ form a triangle, as shown in the diagram.



(i) Show that the equation of the line AB is $2x + y - 8 = 0$ **2**

(ii) Find the perpendicular distance from C to AB. **2**

(iii) Hence, or otherwise, find the area of ΔABC. **2**

(c) A bag contains 5 red, 4 green and 3 blue marbles. A marble is selected at random without replacement, after which, 2 blue marbles are added to the bag. Then a second marble is selected at random.

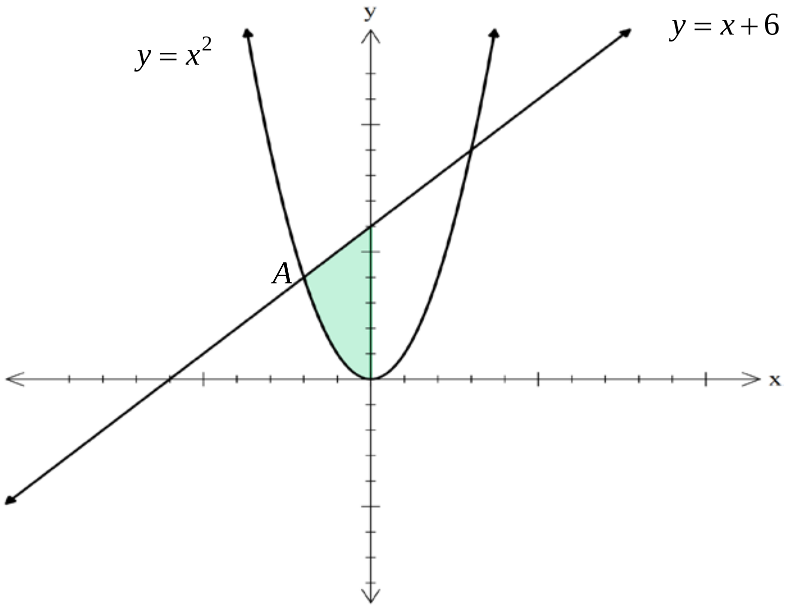
(i) Find the probability that both marbles are red. **1**

(ii) Find the probability that the first marble is not blue but the second marble is blue. **2**

Question 12 continues on page 10

Question 12 (continued)

(d) The parabola $y = x^2$ and the line $y = x + 6$ intersect at the point A.



(i) Find the coordinates of the point A. **1**

(ii) Calculate the shaded area. **3**

End of Question 12

Question 13 (15 marks)   Start a new writing booklet.

(a) Find the primitive of $\sin x + \cos x$. **2**

(b) Differentiate $\dfrac{1}{\tan x}$ and give your answer in the simplest form. **2**

(c) A 150 mg tablet is dissolved in a glass of water. After t minutes the amount of undissolved tablet, M in mg, is given by

$$M = 150 \ e^{-kt} \ ,$$

where k is a constant.

(i) Given that 70 mg of the tablet remains after 10 minutes, find the value of k, correct to 4 decimal places. **2**

(ii) Find the rate at which the tablet is dissolving in the glass of water after 15 minutes. Give your answer correct to 2 decimal places. **2**

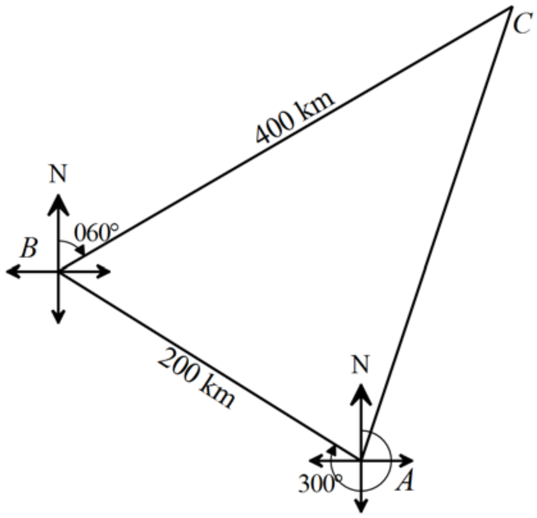
(d) The acceleration of a particle at any time t seconds is given by $\dfrac{dv}{dt} = 6.$ **2**

If x m is the displacement at any time t seconds. Given that at $t = 0, x = 1$ and at $t = 1,\ x = 2.$ Show that $x = 3t^2 - 2t + 1.$

Question 13 continues on page 12

Question 13 (continued)

(e) In the diagram, A ,B and C are 3 towns such that Town B is 200 km and 400 km from Town A and Town C respectively. The bearing of B from A is 300° T and bearing of C From B is 060° T .



Not To Scale

Copy this diagram into your writing booklet.

(i) Show that $\angle ABC$ is 60° . **1**

(ii) Show that the exact distance of AC is $200\sqrt{3}$ km. **2**

(iii) Find the bearing of C from A. **2**

End of Question 13

Question 14 (15 marks) Start a new writing booklet.

(a) For the graph $y = \dfrac{x}{\ln 2} - e^x$,

(i) find the x -coordinate of the stationary point. **2**

(ii) Determine the nature of the stationary pint. **1**

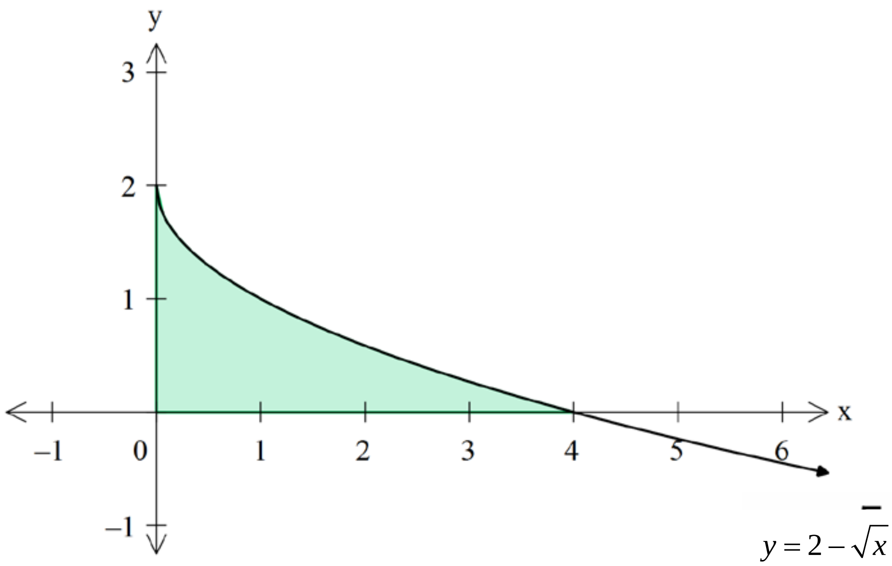
(b) The roots of the quadratic equation $4x^2 - kx + 8 = 0$ are α and β.

(i) Find the value of $\alpha\beta$. **1**

(ii) Given that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 6$, find the value of k . **2**

(c) The region bounded by the curve $y = 2 - \sqrt{x}$ and the x-axis between $x = 0$ **3**

and $x = 4$ is rotated about the x- axis to form a solid.



Find the volume of the solid.

Question 14 continues on page 14

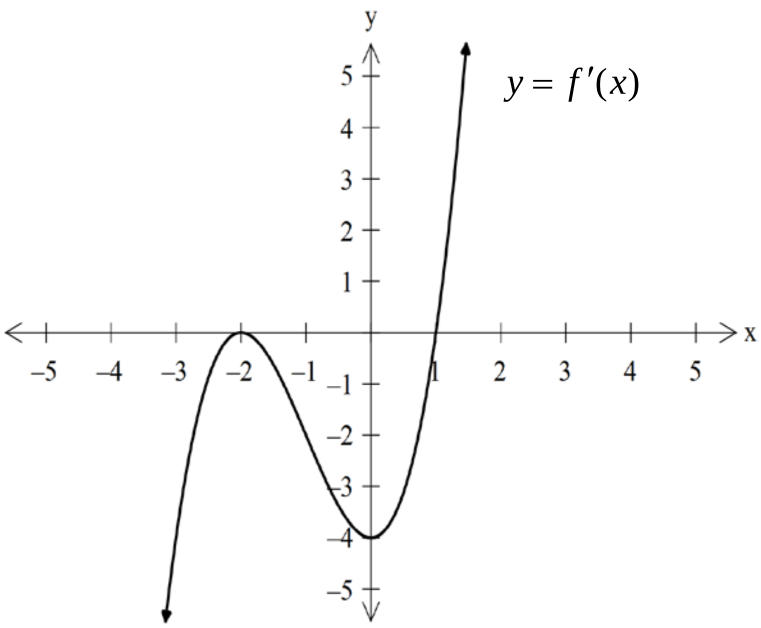
Question 14 (continued)

(d) A super bouncy ball is dropped from the top of a 100m tall building. After its first bounce it reaches a height of 75 m before it falls back down again. The ball continues to bounce, each time reaching a height that is $\frac{3}{4}$ of the height of each preceding bounce.

(i) Show that the ball has travelled a total distance of 250 m when it hits the ground the second time. **1**

(ii) Find the expression of the total distance travelled by the ball when it hits the 10th time. **2**

(e) The diagram shows the graph of a function $y = f'(x)$.



(i) Explain why there is a horizontal point of inflexion at $x = -2$. **1**

(ii) Sketch a possible graph for $y = f(x)$, given $f(0) = 2$. **2**

End of Question 14

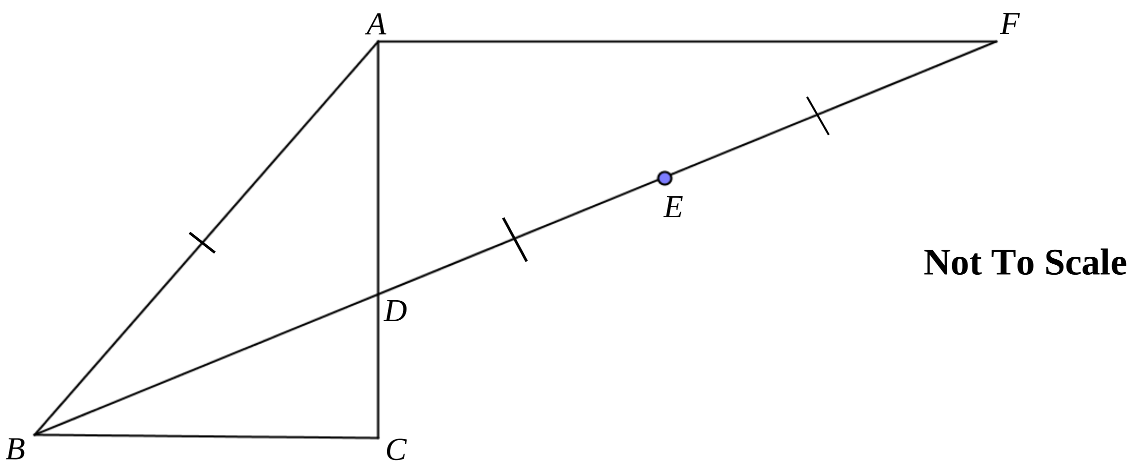
Question 15 (15 marks) Start a new writing booklet

(a) Find all solutions of $\sec^2 x + \tan x - 3 = 0$, where $0 \le x \le 2\pi$. **2**

Give x to 3 decimal places where necessary.

(b) In the diagram below $AC \perp BC$, $AC \perp AF$ and $AB = DE = EF$

Copy or Trace the diagram into your workbook.



(i) Show that $\angle DBC = \angle DFA$. **2**

(ii) By constructing the line EG, such that G lies on AF and $EG \parallel AC$, **3**

show $\Delta AGE \equiv \Delta FGE$.

(iii) Prove that $\angle ABD = 2\angle DBC$. **2**

Question 15 continues on page 16

Question 15 (continued)

(c) The line $y = mx$ is a tangent to the curve $y = \ln(2x-1)$ at a point P.

(i) Sketch the line and the curve on the same diagram, clearly indicating the point P. **2**

(ii) Show the coordinates of P are $\left(\dfrac{2+m}{2m}, \dfrac{2+m}{2}\right)$. **2**

(iii) Hence, show that $2 + m = \ln\left(\dfrac{4}{m^2}\right)$. **2**

End of Question 15

Question 16 (15 marks)   Start a new writing booklet

(a) Use Simpson's Rule with five function values to show that **3**

$$\int_0^1 e^{4x}\,dx \approx \frac{1}{12}\left[1+4\left(e+e^3\right)+2e^2+e^4\right]$$

(b) A university student received a scholarship of \$25 000 in her bank account.

The account earns interest at 6% per annum, compounded monthly.

At the end of each month interest is added to the account balance and then the student withdraws \$1000. Let $\$A_n$ be the amount of money remaining in the account at the end of the nth month, following the student's withdrawal.

(i) Show that $A_2 = 25000(1.005)^2 - 1000(1+1.005)$. **2**

(ii) Show that $A_n = 200000 - 175000 \times 1.005^n$. **2**

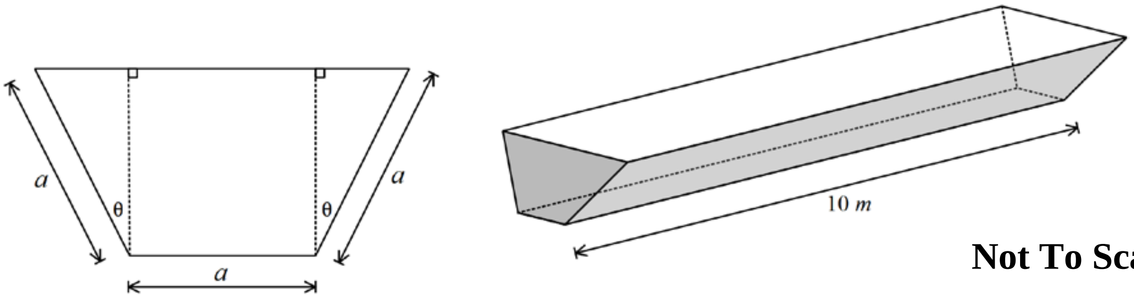
(iii) After how many months will the account have a balance of zero dollars? Give your answer to the nearest month. **2**

Question 16 continues on page 18

Question 16 (continued)

c) A trough in the figure is to be made to the dimensions shown below.

The cross section is a trapezium. Only the angle θ can be varied.



Not To Scale

(i) Show that the area of the trapezoidal cross section is $A = a^2 \cos\theta(1+\sin\theta)$. **2**

(ii) Show that the trough's volume is maximised at $\theta = \dfrac{\pi}{6}$. **3**

(iii) Find the maximum exact volume of the trough in terms of a. **1**

End of Question 16

End of Paper

HORNSBY GIRLS HIGH SCHOOL

Mathematics

Year 12 Higher School Certificate
Trial Examination Term 3 2018

STUDENT NUMBER: ____________SOLUTION____________________________

General Instructions

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Total marks – 100

Section I Pages 3 – 6

10 marks

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Answer on the Objective Response Answer Sheet provided

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90 marks

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Question Total	***1-10***	***11***	***12***	***13***	***14***	***15***	***16***	***Total***
	/10	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

Section I

10 marks
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1 What is the value of $\frac{(\pi+2)^3}{11}$, correct to 3 significant figures.

(A) 12.357

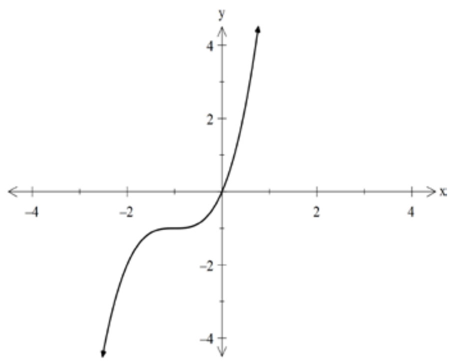
(B) 12.356

(C) 12.4

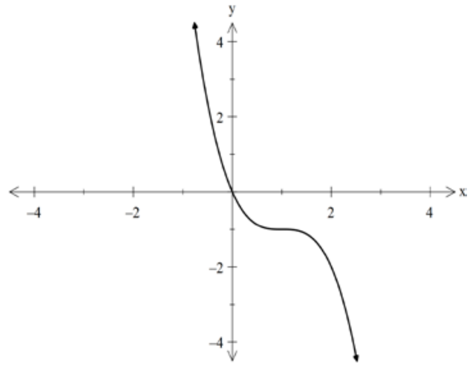
(D) 3.87

2 Which graph best represents $y=(x+1)^3-1$?

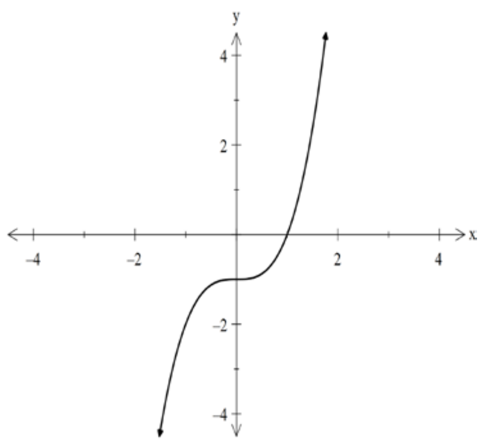
(A)



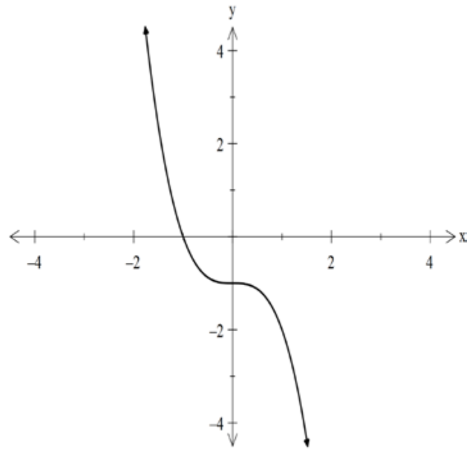
(B)



(C)



(D)



3 The derivative of $\dfrac{\ln x}{x^2}$ is

(A) $\dfrac{x-2\ln x}{x^4}$

(B) $\dfrac{1-\ln x}{x^3}$

(C) $\dfrac{1-2\ln x}{x^3}$

(D) $\dfrac{1-2x\ln x}{x^4}$

$$\frac{d}{dx}\left(\frac{\ln x}{x^2}\right)=\frac{x^2\frac{1}{x}-2x\ln(x)}{(x^2)^2}$$
$$=\frac{x-2x\ln x}{x^4}$$
$$=\frac{1-2\ln x}{x^3}$$

4 Which expression is equal to $\int e^{4x}\,dx$?

(A) $4e^{4x}+c$

(B) $\dfrac{e^{4x}}{4}+c$

(C) $\dfrac{e^{4x+1}}{4x+1}+c$

(D) $\dfrac{e^{5x}}{5}+c$

$$\int e^{4x}\,dx=e^{4x}\cdot\frac{1}{4}+C$$
$$=\frac{e^{4x}}{4}+C$$

5 Which equation represents the line perpendicular to $3x+2y=5$ and intersecting y-axis at $(0,-1)$?

(A) $3x-2y=-2$

(B) $2x-3y=3$

(C) $2x+3y=3$

(D) $3x+2y=-2$

$m=\dfrac{2}{3}$, y-intercept is -1

$y=\dfrac{2}{3}x-1$

$\therefore 2x-3y=3$

6 Which expression is a factorisation of $27x^3-1$?

(A) $(3x+1)(9x^2-3x+1)$

(B) $(3x-1)(9x^2-6x+1)$

(C) $(3x-1)(9x^2+3x+1)$

(D) $(3x-1)(9x^2+6x+1)$

$$\begin{aligned}27x^3-1&=(3x)^3-1^3\\&=(3x-1)\left[(3x)^2+3x+1^2\right]\\&=(3x-1)(9x^2+3x+1)\end{aligned}$$

7 The maximum value of the graph $y = 1 - 2\cos x$ is:

(A) -1

(B) 1

(C) 2

(D) 3

maximum value occurs when

$\cos x = -1$

$$y = 1 - 2\cos x$$
$$= 1 - 2(-1)$$
$$= 3$$

8 Which of the following is a term of the geometric sequence

$$2p, -4p^2, 8p^3, -16p^4, ... ?$$

(A) $256p^8$

(B) $512p^8$

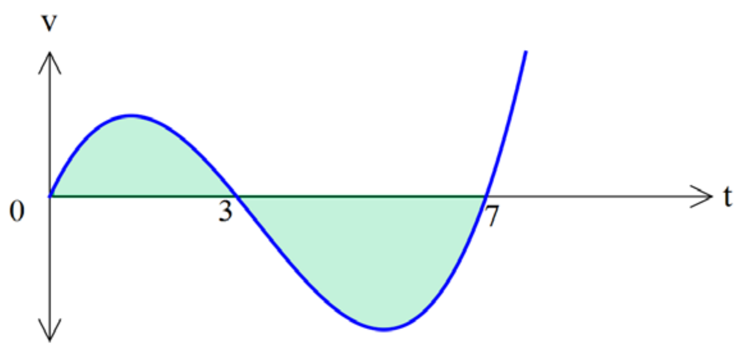
(C) $-256p^8$

(D) $-512p^8$

in the G.P: $2p, -4p^2, 8p^3, -16p^4 ....$

$$a = 2p,\ r = \frac{-4p^2}{2p} = -2p$$
$$T_n = (2p)(-2p)^{n-1}$$
$$= (-1)^{n-1} \cdot 2^n p^n$$
$$n = 8, T_8 = (-1)^{8-1} \cdot 2^8 p^8$$
$$= -256p^8$$

9



The graph above represents the velocity v m/s of a particle after t seconds. The particle is moving in a straight line from rest. Which of the following describes the area of the shaded region?

(A) The acceleration of the particle after 7 seconds.

(B) The velocity of the particle in the first 7 seconds.

(C) The displacement of the particle after 7 seconds.

(D) The distance travelled by the particle in the first 7 seconds.

The area of the shaded region = $\int_0^3 v\ dt + \left| \int_3^7 v\ dt \right|$

It represents the distance of the particle after 7 seconds.

10 A raffle consists of thirty tickets in which there are two prizes. First prize is two book vouchers and second prize is one book voucher. Emma buys five tickets.

The probability that Emma wins at least one book voucher is

(A) $\dfrac{9}{29}$

(B) $\dfrac{11}{36}$

(C) $\dfrac{3}{10}$

(D) $\dfrac{49}{174}$

P (wins at least 1 voucher)

= 1 – P (not winning any)

= 1 – P (not winning 1st prize) × P (not winning 2rd prize)

$$=1-\frac{25}{30}\times\frac{24}{29}$$

$$=1-\frac{20}{29}$$

$$=\frac{9}{29}$$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)   Start a new writing booklet.

(a) Rationalise the denominator of $\dfrac{1}{2\sqrt{3}+5}$. **2**

$$\begin{aligned}\frac{1}{2\sqrt{3}+5} &= \frac{(2\sqrt{3}-5)}{(2\sqrt{3}+5)(2\sqrt{3}-5)}\\ &= \frac{(2\sqrt{3}-5)}{12-25}\\ &= \frac{2\sqrt{3}-5}{-13}\\ \text{or } &= \frac{5-2\sqrt{3}}{13}\end{aligned}$$

(b) Factorise $6x^2-5x+1$. **2**

$$6x^2-5x+1=(2x-1)(3x-1)$$

(c) Differentiate $x\sqrt{x+1}$. **2**

$$\begin{aligned}\frac{d}{dx}x\sqrt{x+1} &= \frac{d}{dx}x(x+1)^{\frac{1}{2}}\\ &= x\cdot\frac{1}{2}(x+1)^{-\frac{1}{2}}+(x+1)^{\frac{1}{2}}\cdot 1\\ &= \frac{x}{2\sqrt{x+1}}+\sqrt{x+1}\\ &= \frac{x+2(x+1)}{2\sqrt{x+1}}\\ &= \frac{3x+2}{2\sqrt{x+1}}\end{aligned}$$

(d) Find $\int \frac{1}{(2x-5)^3}\,dx$. **2**

$$\int \frac{1}{(2x-5)^3}\,dx = \int (2x-5)^{-3}\,dx$$
$$= \frac{(2x-5)^{-2}}{2\times -2} + C$$
$$= \frac{-1}{4(2x-5)^2} + C$$

(e) Evaluate $\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$. **3**

$$\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx = \frac{1}{2}\left[\tan 2x\right]_0^{\frac{\pi}{8}}$$
$$= \frac{1}{2}\left[\tan 2\left(\frac{\pi}{8}\right) - \tan 2(0)\right]$$
$$= \frac{1}{2}(1-0)$$
$$= \frac{1}{2}$$

(f) The gradient function of a curve $y = f(x)$ is given by $f'(x) = e^{2x+1}$, **2**

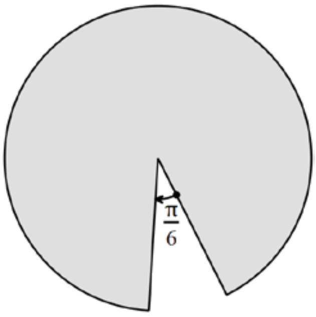
and $f(0) = \frac{-e}{2}$, find the equation of $f(x)$.

Take the primitive of $f'(x) = e^{2x+1}$,

$$f(x) = \frac{1}{2}e^{2x+1} + C$$
$$f(0) = \frac{1}{2}e^{2(0)+1} + C = -\frac{e}{2}$$
$$C = -e$$
$$\therefore\ f(x) = \frac{1}{2}e^{2x+1} - e$$

(g) The area of the shaded major sector of the circle is $\frac{44\pi}{3}$. **2**

Find the radius of the sector.



$$\frac{44\pi}{3} = \frac{1}{2}r^2\left(2\pi - \frac{\pi}{6}\right)$$

$$\frac{44\pi}{3} = \frac{11\pi r^2}{12}$$

$$r^2 = 16$$

$$r = 4 \quad (r > 0)$$

End of Question 11

Question 12 (15 marks)   Start a new writing booklet.

(a) A 35 cm seedling grows by $16\frac{2}{7}$ cm in the first week, and then keeps growing by $\frac{4}{7}$ of its previous weeks growth. What is the maximum height the seedling can grow to? **2**

$a = 16\frac{2}{7}, \quad r = \frac{4}{7}$

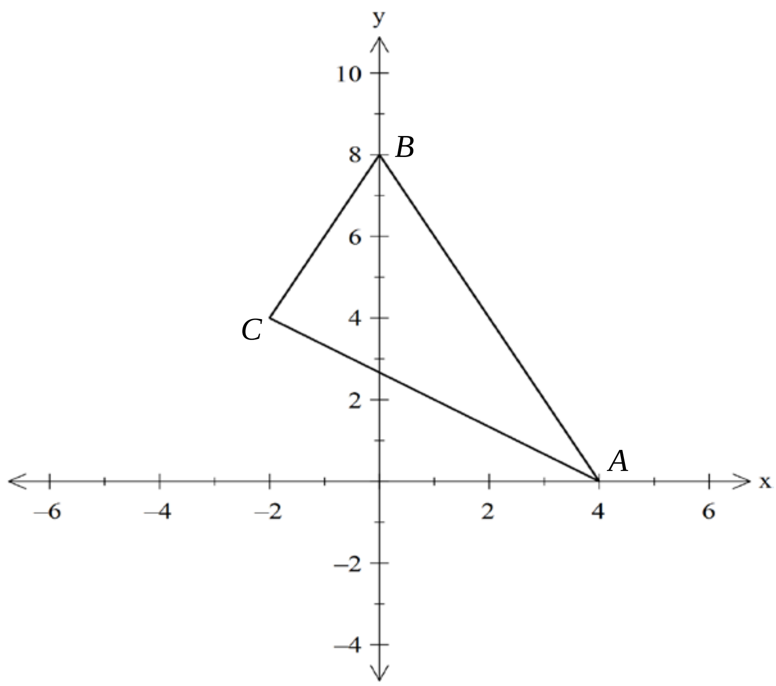
Sum of total growth (limiting sum) $= \dfrac{16\frac{2}{7}}{1-\frac{4}{7}}$

$= 38$ cm

Maximum height = 35 + 38

= 73 cm

(b) The points $A(4,0), B(0,8)$ and $C(-2,4)$ form a triangle, as shown in the diagram.



(i) Show that the equation of the line AB is $2x + y - 8 = 0$ **2**

Equation of AB:

$\dfrac{y-0}{x-4} = \dfrac{8-0}{0-4}$

$y = -2(x-4)$

$= -2x + 8$

$\therefore 2x + y - 8 = 0$

(ii) Find the perpendicular distance from C to AB. **2**

$$\text{D} = \frac{|2(-2)+4-8|}{\sqrt{2^2+1^2}}$$
$$= \frac{8}{\sqrt{5}}$$
$$= \frac{8\sqrt{5}}{5} \text{ units}$$

(iii) Hence, or otherwise, find the area of ΔABC. **2**

$$\text{AB} = \sqrt{(4-0)^2+(0-8)^2}$$
$$= 4\sqrt{5} \text{ units}$$
$$\text{Area of } \triangle\text{ABC} = \frac{1}{2}\cdot 4\sqrt{5}\cdot\frac{8\sqrt{5}}{5}$$
$$= 16 \text{ unit}^2$$

(c) A bag contains 5 red, 4 green and 3 blue marbles. A marble is selected at random without replacement, after which, 2 blue marbles are added to the bag. Then a second marble is selected at random.

(i) Find the probability that both marbles are red. **1**

$$\text{P(both are red)} = \frac{5}{12}\times\frac{4}{13}$$
$$= \frac{5}{39}$$

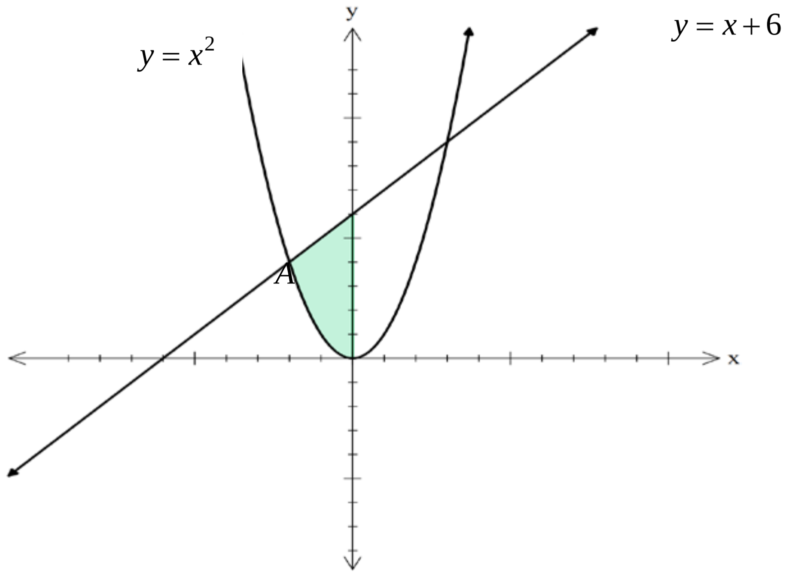
(ii) Find the probability that the first marble is not blue but the second marble is blue. **2**

P(first is not blue, second is blue)

$$= \frac{9}{12}\times\frac{5}{13}$$
$$= \frac{15}{52}$$

Question 12 (continued)

(d) The parabola $y = x^2$ and the line $y = x + 6$ intersect at the point A.



(i) Find the coordinates of the point A. **1**

$solve \begin{cases} y = x^2 \\ y = x + 6 \end{cases}$ simultaneously,

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3, \ -2$

$x = 3$ is rejected as A is in quadrant 2.

sub $x = -2$ into $y = x^2$, $y = (-2)^2 = 4$

$\therefore$ A is $(-2, 4)$.

(ii) Calculate the shaded area. **3**

$$\text{The shaded area} = \int_{-2}^{0} \left[(x+6) - x^2 \right] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^{0}$$

$$= \left[\frac{0^2}{2} - 6(0) + \frac{0^3}{3} \right] - \left[\frac{(-2)^2}{2} + 6(-2) + \frac{(-2)^3}{3} \right]$$

$$= 0 - \left[\frac{4}{2} - 12 + \frac{8}{3} \right]$$

$$= 7\frac{1}{3} \ \text{unit}^2$$

Question 13 (15 marks)   Start a new writing booklet.

(a) Find the primitive of $\sin x + \cos x$. **2**

$$\int (\sin x + \cos x)\, dx = -\cos x + \sin x + C$$

(b) Differentiate $\dfrac{1}{\tan x}$ and give your answer in the simplest form. **2**

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{\tan x}\right) &= \frac{d}{dx}(\tan x)^{-1}\\ &= -(\tan x)^{-2} \cdot \sec^2 x\\ &= -\frac{\sec^2 x}{\tan^2 x}\\ &= -\frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}\\ &= -\frac{1}{\sin^2 x}\\ &= -\text{cosec}^2 x\end{aligned}$$

(c) A 150 mg tablet is dissolved in a glass of water. After t minutes the amount of undissolved tablet, M in mg, is given by

$$M = 150\ e^{-kt}\ ,$$

where k is a constant.

(i) Given that 70 mg of the tablet remains after 10 minutes, find the value of k, correct to 4 decimal places. **2**

$$\begin{aligned}&t = 10,\ M = 70\\ &70 = 150e^{-10k}\\ &e^{-10k} = \frac{7}{15}\\ &-10k = \ln\frac{7}{15}\\ &k = \frac{\ln\frac{7}{15}}{-10}\\ &\quad = 0.076214...\\ &\quad \approx 0.0762 (4\text{ d.p.})\end{aligned}$$

Question 13 (continued)

(ii) Find the rate at which the tablet is dissolving in the glass of water after 15 minutes. Give your answer correct to 2 decimal places. **2**

$$M = 150e^{-kt}$$
$$\frac{dM}{dt} = -150ke^{-kt}$$
$$t = 15, k = 0.0762 \ ,$$
$$\frac{dM}{dt} = -150 \times 0.0762 \times e^{-0.0762\times 15}$$
$$= -3.6445...$$
$$= -3.64 \ \text{mg/min (2 d.p.)}$$
$$\left(\text{using } k = \frac{\ln \frac{7}{15}}{-10} \text{ is also acceptable.} \right)$$

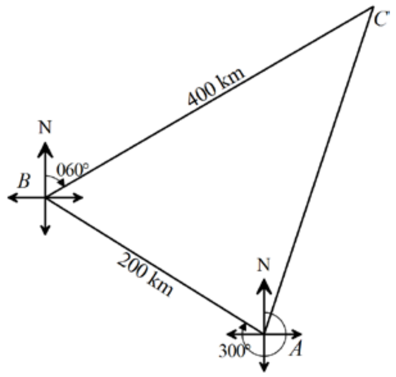
(d) The acceleration of a particle at any time t seconds is given by $\frac{dv}{dt} = 6.$ **2**

If x m is the displacement at any time t seconds. Given that at $t = 0, x = 1$ and at $t = 1,\ x = 2.$ Show that $x = 3t^2 - 2t + 1.$

$$\frac{dv}{dt} = 6$$
$$v = \int 6 \ dt$$
$$v = \frac{dx}{dt} = 6t + C_1$$
$$x = \int (6t + C_1) dt$$
$$= 3t^2 + C_1 t + C_2$$
At $t = 0,\ x = 1,$
$$1 = 0 + 0 + C_2$$
$$\therefore C_2 = 1$$
At $t = 1, x = 2,$
$$2 = 3 + C_1 + 1$$
$$C_1 = -2$$
$$\therefore x = 3t^2 - 2t + 1$$

Question 13 (continued)

(e) In the diagram, A, B and C are 3 towns such that Town B is 200 km and 400 km from Town A and Town C respectively. The bearing of B from A is 300° T and bearing of C from B is 060° T.



Not To Scale

Copy this diagram into your writing booklet.

(i) Show that $\angle ABC$ is 60°. **1**

$$\begin{aligned}&\text{At A, } \angle \text{BAN} = 360^\circ - 300^\circ \quad (\angle\text{s at a point})\\&\qquad\qquad = 60^\circ\\&\text{At B, } \angle \text{NBA} = 180^\circ - 60^\circ = 120^\circ \quad (\text{co-interior } \angle\text{s, } \| \text{ lines})\\&\qquad \angle ABC = \angle \text{NBA} - \angle NBC\\&\qquad\qquad = 120^\circ - 60^\circ\\&\qquad\qquad = 60^\circ\end{aligned}$$

(ii) Show that the exact distance of AC is $200\sqrt{3}$ km. **2**

$$\begin{aligned}&\text{In } \triangle ABC,\ AC^2 = AB^2 + BC^2 - 2\cdot AB\cdot BC\cdot \cos\angle ABC\\&AB = 200,\ BC = 400,\ \angle ABC = 60^\circ,\\&AC^2 = 200^2 + 400^2 - 2\times 200\times 400\cos 60^\circ\\&\qquad = 3\times 200^2\\&AC = 200\sqrt{3}\ \text{km}\end{aligned}$$

(iii) Find the bearing of C from A. **2**

$$\begin{aligned}&In \Delta ABC, \angle BAC = 60^\circ + \angle NAC\\&\angle ABC = 60^\circ,\ AC = 200\sqrt{3},\ BC = 400,\\&\frac{\sin\angle BAC}{BC} = \frac{\sin\angle ABC}{AC}\\&\frac{\sin(60^\circ + \angle NAC)}{400} = \frac{\sin 60^\circ}{200\sqrt{3}}\\&\sin(60^\circ + \angle NAC) = \frac{400\sin 60^\circ}{200\sqrt{3}} = 1\\&\therefore 60^\circ + \angle NAC = 90^\circ\\&\qquad \angle NAC = 30^\circ\\&\therefore \text{The bearing of C from A is } 030^\circ T.\end{aligned}$$

End of Question 13

Question 14 (15 marks) Start a new writing booklet.

(a) For the graph $y = \dfrac{x}{\ln 2} - e^x$,

(i) find the x -coordinate of the stationary point. **2**

$$\frac{dy}{dx} = \frac{1}{\ln 2} - e^x$$

Let $\dfrac{dy}{dx} = 0$, $e^x = \dfrac{1}{\ln 2}$

$e^x = (\ln 2)^{-1}$

$x = -\ln(\ln 2)$

$\therefore$ x-cooridnate of the stationary point is: $-\ln(\ln 2)$.

(ii) Determine the nature of the stationary point.

1

$$\frac{dy}{dx} = \frac{1}{\ln 2} - e^x$$

$$\frac{d^2y}{dx^2} = 0 - e^x$$

$$= -e^x$$

At $x = -\ln(\ln 2)$, $\dfrac{d^2y}{dx^2} = -e^{-\ln(\ln 2)}$

$$= -\frac{1}{\ln 2} = -1.4426... < 0$$

$\therefore$ The stationary point at $x = -\ln(\ln 2)$ is a maximum turning point.

(b) The roots of the quadratic equation $4x^2 - kx + 8 = 0$ are α and β.

(i) Find the value of $\alpha\beta$. **1**

$$\alpha\beta = \frac{c}{a} = \frac{8}{4} = 2$$

(ii) Given that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 6$, find the value of k . **2**

$$\alpha + \beta = -\frac{b}{a} = -\frac{-k}{4} = \frac{k}{4}$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 6$$

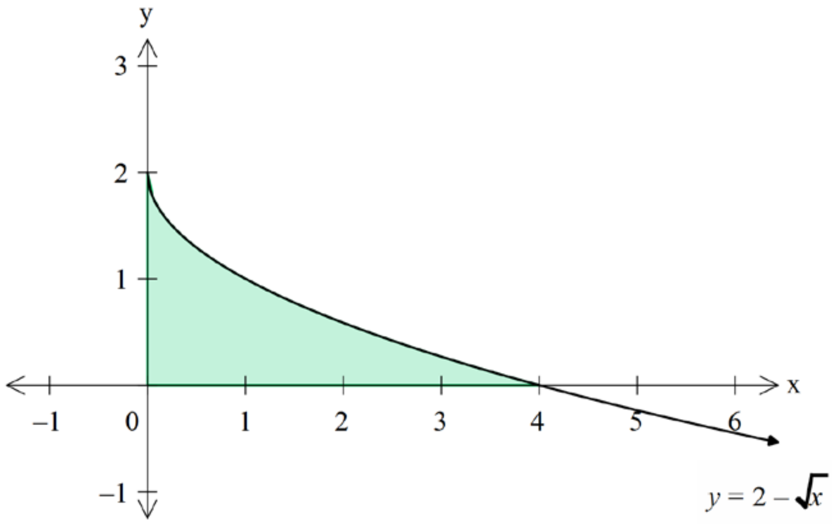
$$\frac{\alpha + \beta}{\alpha\beta} = 6$$

$$\frac{k}{4} = 6 \times 2$$

$\therefore k = 48$

Question 14 (continued)

(c) The region bounded by the curve $y = 2 - \sqrt{x}$ and the x-axis between $x = 0$ and $x = 4$ is rotated about the x- axis to form a solid. **3**



Find the volume of the solid.

$$
\begin{aligned}
V &= \int_0^4 \pi y^2 dx, \quad \text{where } y = 2 - \sqrt{x} \\
&= \int_0^4 \pi (2 - \sqrt{x})^2 dx \\
&= \int_0^4 \pi (4 + x - 4\sqrt{x})\ dx \\
&= \pi \left[4x + \frac{x^2}{2} - \frac{8}{3} x^{\frac{3}{2}} \right]_0^4 \\
&= \pi \left[\left(4(4) + \frac{(4)^2}{2} - \frac{8}{3}(4)^{\frac{3}{2}} \right) - \left(4(0) + \frac{(0)^2}{2} - \frac{8}{3}(0)^{\frac{3}{2}} \right) \right] \\
&= \pi \left[16 + 8 - \frac{8}{3} \times 8 - 0 \right] \\
&= \frac{8\pi}{3} \ \text{unit}^3
\end{aligned}
$$

Question 14 (continued)

(d) A super bouncy ball is dropped from the top of a 100m tall building. After its first bounce it reaches a height of 75 m before it falls back down again. The ball continues to bounce, each time reaching a height that is $\frac{3}{4}$ of the height of each preceding bounce.

(i) Show that the ball has travelled a total distance of 250 m when it hits the ground the second time. **1**

Total distance travelled when the ball hits the ground the second time
= 100 + 75 + 75
= 250 m

(ii) Find the expression of the total distance travelled by the ball when it hits the 10th time. **2**

Total distance travelled when it hits the ground the 10^{th} time

$$=\left[100+2\left(\frac{3}{4}\times 100\right)+2\left(\left(\frac{3}{4}\right)^2\times 100\right)+2\left(\left(\frac{3}{4}\right)^3\times 100\right)+\ldots+2\left(\left(\frac{3}{4}\right)^9\times 100\right)\right]$$

$$=100+200\left[\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots+\left(\frac{3}{4}\right)^9\right]$$

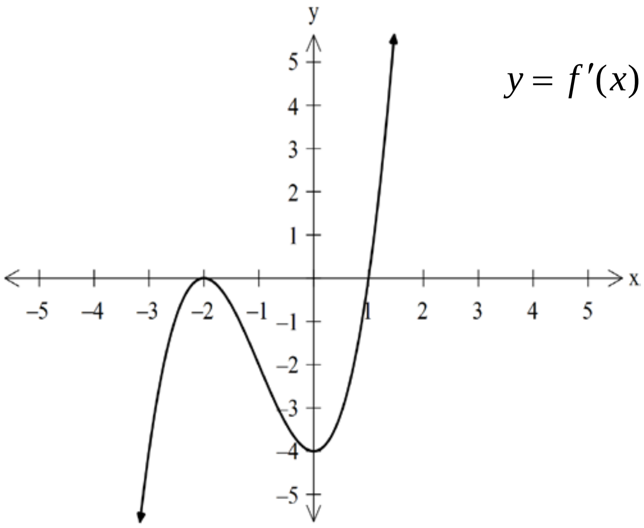
$$=100+200\left[\frac{\frac{3}{4}\left(1-\left(\frac{3}{4}\right)^9\right)}{1-\frac{3}{4}}\right]$$

$$=100+800\left(\frac{3}{4}-\left(\frac{3}{4}\right)^{10}\right)$$

$$=700-800\left(\frac{3}{4}\right)^{10}\ \text{m}$$

Question 14 (continued)

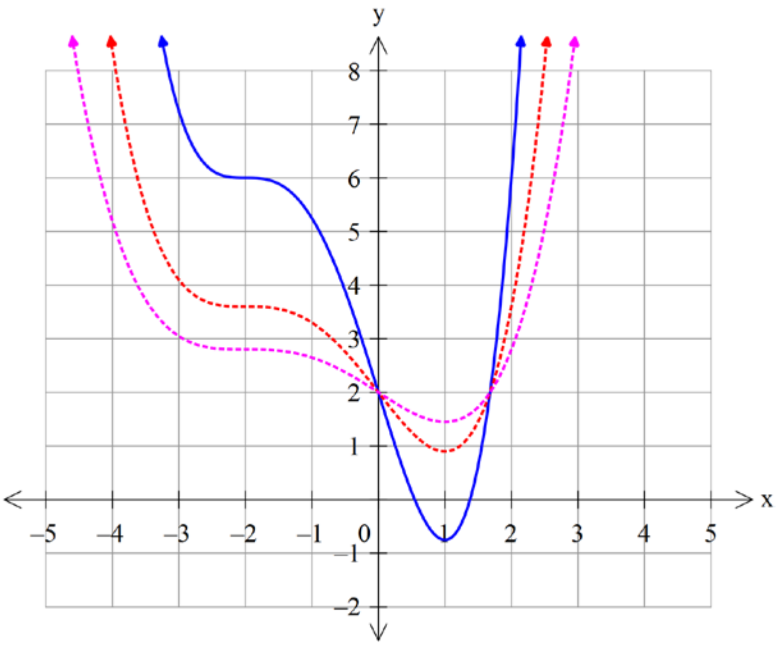
(e) The diagram shows the graph of a function $y = f'(x)$.



(i) Explain why there is a horizontal point of inflexion at $x = -2$. **1**

At $x = -2,\ f'(x) = 0$ $\therefore$ there is a stationary point at $x = -2$. To the left hand side of $x = -2$, $f''(x)$ is positive, $f(x)$ is concave up. To the right hand side of $x = -2$, $f''(x)$ is negative, $f(x)$ is concave down. $\therefore$ there is a horizontal point of inflexion at $x = -2$.

(ii) Sketch a possible graph for $y = f(x)$, given $f(0) = 2$. **2**



Different curves are acceptable as long as the critical points are correct.

End of Question 14

Question 15 (15 marks) Start a new writing booklet

(a) Find all solutions of $\sec^2 x + \tan x - 3 = 0$, where $0 \le x \le 2\pi$. **2**

Give x to 3 decimal places where necessary.

$\sec^2 x + \tan x - 3 = 0$

$1 + \tan^2 x + \tan x - 3 = 0$

$\tan^2 x + \tan x - 2 = 0$

$(\tan x + 2)(\tan x - 1) = 0$

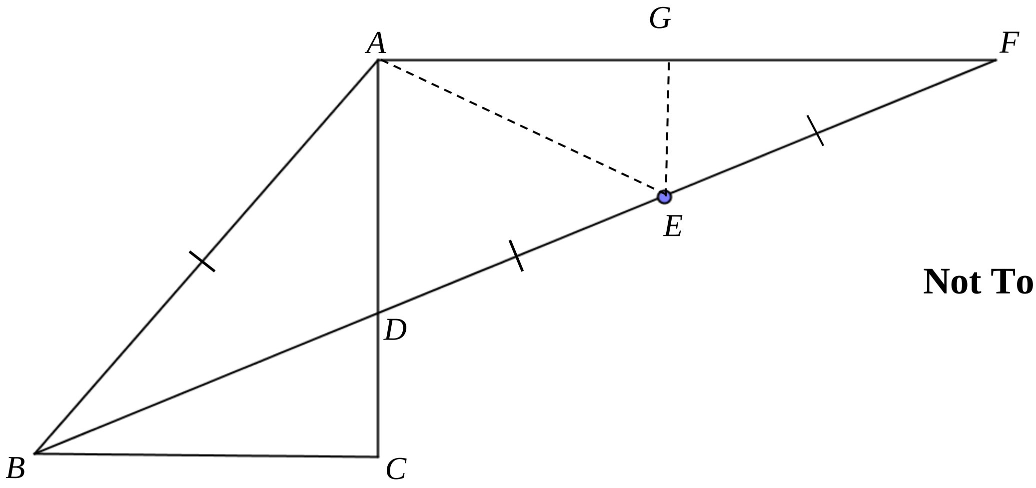
$\tan x = -2$ (quadrant 2 & quadrant 4), $\tan x = 1$ (quadrant 1 & quadrant 3) where $0 \le x \le 2\pi$

$x = \pi - 1.1071,\ \ 2\pi - 1.1071,\ \ \frac{\pi}{4},\ \ \pi + \frac{\pi}{4}$

$x = 2.034,\ \ 5.176,\ \ \frac{\pi}{4},\ \ \frac{5\pi}{4}$

(b) In the diagram below $AC \perp BC$, $AC \perp AF$ and $AB = DE = EF$

Copy or trace the diagram into your workbook.



Not To Scale

(i) Show that $\angle DBC = \angle DFA$. **2**

$\angle FAC = 90°(AC \perp AF)$

$\angle ACB = 90°(AC \perp BC)$

$\therefore \angle FAC = \angle ACB$

Hence $AF \parallel BC$ (alternate $\angle$s are equal)

$\therefore \angle DBC = \angle DFA$ (alternate $\angle s$, $AF \parallel BC$)

Question 15 (continued)

(ii) By constructing the line EG, such that G lies on AF and $EG \parallel AC$, show $\Delta AGE \equiv \Delta FGE$. **3**

$EG \parallel HF \parallel AC$

$EF = DE(given)$

$\therefore AG = GF$ (join of midpoints)

$\angle$EGF=$\angle$CAF=90° (corresponding $\angle$s,EG || AC)

$\therefore \angle$EGF = $\angle$AGE = 90°

In $\triangle$AGE and $\triangle$FGE,

$AG = GF$

$\angle$EGF = $\angle$AGE = 90°

GE is common,

$\therefore \triangle$AGE $\equiv \triangle$FGE (SAS)

(iii) Prove that $\angle ABD = 2\angle DBC$. **2**

$\angle GAE = \angle GFE$ (corresponding $\angle s$, congruent Δs)

$\angle AEB = \angle GAE + \angle GFE$ (exterior $\angle$s of Δ)

$= 2 \times \angle GFE = 2 \times \angle AFD$

In $\triangle ABE, \angle ABE = \angle AEB$ (base $\angle$s, isoceles Δ)

$\therefore \angle ABE = \angle ABD = 2 \times \angle AFD$

In ΔDBC and ΔDFA,

$\angle$BDC= $\angle$ADF

$\angle FAD = \angle BCD = 90°$

$\therefore \triangle DBC \parallel\!| \triangle D$FA

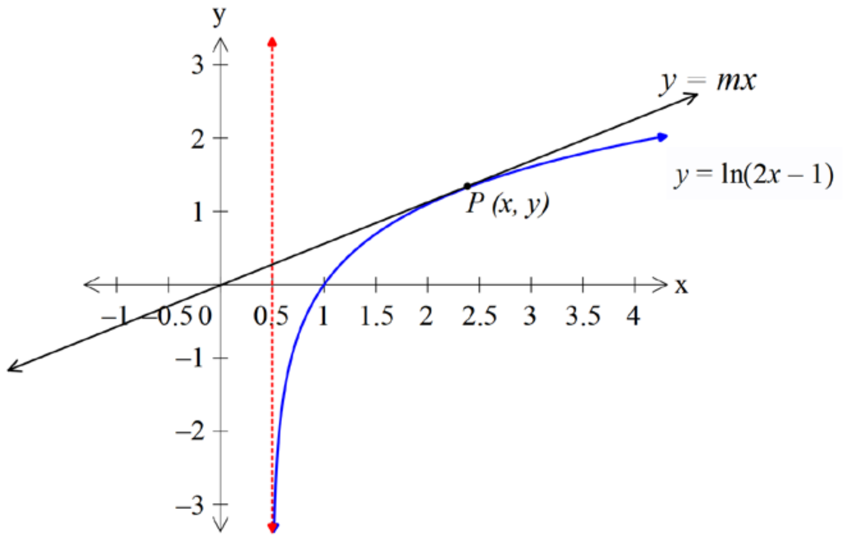
$\therefore \angle$AFD=$\angle$DBC

Hence $\angle$ABD=2$\times\angle$DBC

Question 15 (continued)

(c) The line $y = mx$ is a tangent to the curve $y = \ln(2x-1)$ at a point P.

(i) Sketch the line and the curve on the same diagram, clearly indicating the point P. **2**



(ii) Show the coordinates of P are $\left(\dfrac{2+m}{2m}, \dfrac{2+m}{2}\right)$. **2**

At P, $\dfrac{dy}{dx} = \dfrac{2}{2x-1} = m$

$(2x-1)m = 2$

$x = \dfrac{2+m}{2m}$

sub $x = \dfrac{2+m}{2m}$ in $y = mx$

$y = m\dfrac{2+m}{2m}$

$= \dfrac{2+m}{2}$

$\therefore P$ is $\left(\dfrac{2+m}{2m}, \dfrac{2+m}{2}\right)$

(iii) Hence, show that $2 + m = \ln\left(\dfrac{4}{m^2}\right)$. **2**

sub P into $y = \ln(2x-1)$,

$\dfrac{2+m}{2} = \ln\left(2\dfrac{2+m}{2m} - 1\right)$

$= \ln\left(\dfrac{2}{m}\right)$

$2 + m = 2\ln\left(\dfrac{2}{m}\right)$

$2 + m = \ln\left(\dfrac{4}{m^2}\right)$

End of Question 15

Question 16 (15 marks)   Start a new writing booklet

(a) Use Simpson's Rule with five function values to show that **3**

$$\int_0^1 e^{4x}\,dx \approx \frac{1}{12}\left[1+4\left(e+e^3\right)+2e^2+e^4\right]$$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$	1	e	e^2	e^3	e^4

$$\int_0^1 e^{4x}\,dx \approx \frac{\frac{1}{2}-0}{6}\left[1+4e+e^2\right]+\frac{1-\frac{1}{2}}{6}\left[e^2+4e^3+e^4\right]$$

$$=\frac{1}{12}\left[1+4e+e^2+e^2+4e^3+e^4\right]$$

$$=\frac{1}{12}\left[1+4\left(e+e^3\right)+2e^2+e^4\right]$$

(b) A university student received a scholarship of $25 000 in her bank account.

The account earns interest at 6% per annum, compounded monthly.

At the end of each month interest is added to the account balance and then the student withdraws $1000. Let $\$A_n$ be the amount of money remaining in the account at the end of the *n*th month, following the student's withdrawal.

(i) Show that $A_2 = 25000(1.005)^2 - 1000(1+1.005)$. **2**

Let $A_0 = \$25000, M = \$1000, R = 1+\frac{0.06}{12} = 1.005$

$A_1 = 25000(1.005) - 1000$

$A_2 = \left[25000(1.005)-1000\right](1.005) - 1000$

$\quad = 25000(1.005)^2 - 1000(1+1.005)$

(ii) Show that $A_n = 200000 - 175000 \times 1.005^n$. **2**

$$A_n = 25000\times 1.005^n - \frac{1000(1.005^n - 1)}{0.005}$$

$$= 25000\times 1.005^n - 200000(1.005^n-1)$$

$$= 200000 - 175000\times 1.005^n$$

Question 16 (continued)

(iii) After how many months will the account have a balance of zero dollars? Give your answer to the nearest month. **2**

Let $A_n = 0$,

$200000 - 175000 \times 1.005^n = 0$

$200000 = 175000 \times 1.005^n$

$1.005^n = \frac{8}{7}$

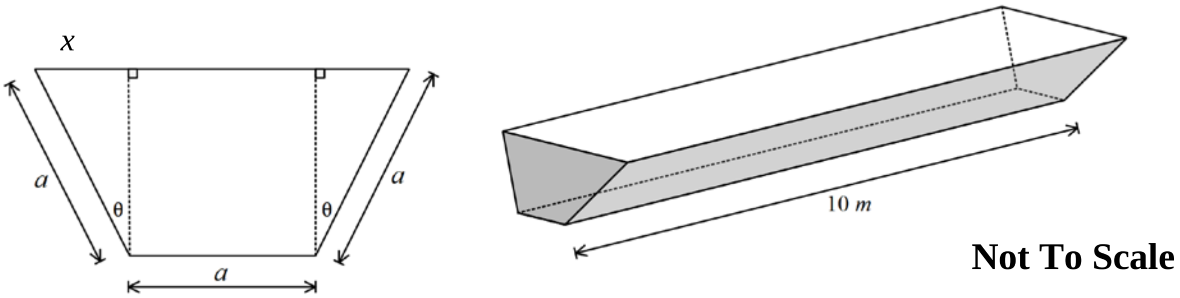
$n = \frac{\log\left(\frac{8}{7}\right)}{\log 1.005}$

$= 26.772...$

≈ 27 months

c) A trough in the figure is to be made to the dimensions shown below.

The cross section is a trapezium. Only the angle θ can be varied.



(i) Show that the area of the trapezoidal cross section is $A = a^2 \cos\theta(1+\sin\theta)$. **2**

$\sin\theta = \frac{x}{a}, \cos\theta = \frac{h}{a}$

$x = a\sin\theta, h = a\cos\theta$

Area of the trapezium cross section $= \frac{1}{2}a\cos\theta(2a + 2a\sin\theta)$

$= a^2\cos\theta(1+\sin\theta)$

Question 16 (continued)

(ii) Show that the trough's volume is maximised at $\theta = \dfrac{\pi}{6}$. **3**

$V = Ah = 10a^2 \cos\theta(1+\sin\theta)$

$\dfrac{dV}{d\theta} = 10a^2\left[-\sin\theta(1+\sin\theta)+\cos\theta(\cos\theta)\right]$

$= 10a^2(-\sin\theta - \sin^2\theta + \cos^2\theta)$

$= 10a^2(-\sin\theta - \sin^2\theta + 1 - \sin^2\theta)$

$= 10a^2(-2\sin^2\theta - \sin\theta + 1)$

For possible extreme values, $\dfrac{dV}{d\theta} = 0$

$10a^2(-2\sin^2\theta - \sin\theta + 1) = 0$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \dfrac{1}{2}, \sin\theta = -1$

Since θ is an acute angle, $\therefore \sin\theta = \dfrac{1}{2}, \theta = \dfrac{\pi}{6}$

To test the nature,

θ	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
$\frac{dV}{d\theta}$	$6.0720a^2$	0	$-17.6795a^2$
slope	+	0	-

Since there is a change of gradient from positive to negative at $\theta = \dfrac{\pi}{6}$,

Therefore, the volume is at its maximum when $\theta = \dfrac{\pi}{6}$.

(iii) Find the maximum exact volume of the trough in terms of a. **1**

At $\theta = \dfrac{\pi}{6}$,

$V = 10a^2\cos\theta(1+\sin\theta)$

$= 10a^2 \cos\dfrac{\pi}{6}(1+\sin\dfrac{\pi}{6})$

$= 10a^2 \times \dfrac{\sqrt{3}}{2}(1+\dfrac{1}{2})$

$= \dfrac{15\sqrt{3}}{2}a^2$

End of Question 16

End of Paper

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