James Ruse Agricultural High School

2020 **TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Advanced

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- A table of z-scores is provided
- Show relevant mathematical reasoning and/or calculations

Total Marks: 100

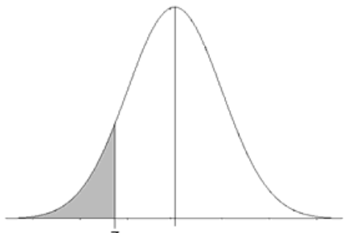
Section I – 10 marks

- Answer on the Multiple Choice answer sheet provided on the back of this page
- Allow about 15 minutes for this section

Sections II, III, IV – 90 marks in total

- Attempt Questions 11–32
- Answer in the space provided in the question booklet.
- If you need extra space use the extra writing pages provided at the end of Section II. Clearly label the question number you are completing.
- Allow about 2 hours and 45 minutes for this section

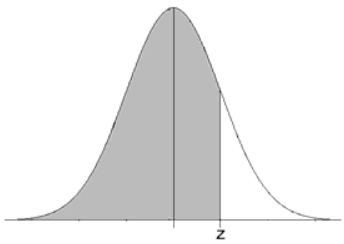
Standard Normal Cumulative Probability Table



Cumulative probabilities for NEGATIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Section I
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

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1. For the series,

$$2 - 1 + \frac{1}{2} - \frac{1}{4} + \cdots$$

what is the sum of the first eight terms, correct to two decimal places?

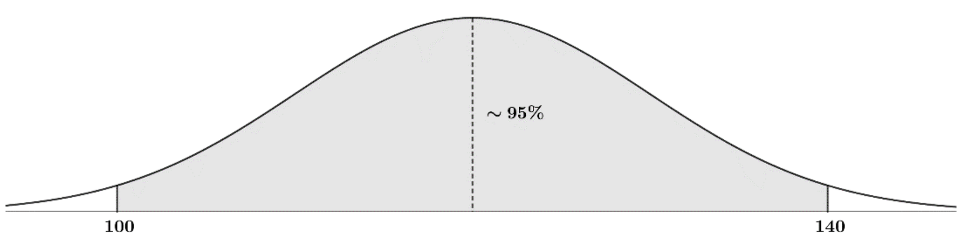
A. 0.68

B. 1.31

C. 1.33

D. 4.00

2. Consider the following graph of a normal distribution, with approximately 95% of the area bounded by the curve, the horizontal axis and $x = 100, x = 140$.



Which of the following parameters best describes the curve?

A. $\mu = 100,\ \sigma^2 = 140$

B. $\mu = 120,\ \sigma^2 = 20$

C. $\mu = 120,\ \sigma^2 = 10$

D. $\mu = 120,\ \sigma^2 = 100$

3. An integer n is chosen at random from the set $\{5, 7, 9, 11\}$. An integer p is also chosen at random from the set $\{2, 6, 10, 14, 18\}$. What is the probability that $n + p = 23$?

A. 0.1

B. 0.2

C. 2.5

D. 0.3

4. If $\ln 3a = \ln b - 2\ \ln c$ where $a, b, c > 0$, which of the following is true?

A. $a = \frac{b-c^2}{3}$

B. $a = \frac{b}{3c^2}$

C. $\ln 3a = \frac{b}{c^2}$

D. $\ln 3a = \frac{\ln b}{\ln c^2}$

5. What is the x coordinate of the point on the curve $y = e^{2x}$ where the tangent is parallel to the line $y = 4x - 1$?

A. $x = \frac{1}{2}\ln 2$

B. $x = \ln 2$

C. $x = -\frac{1}{2}\ln 2$

D. $x = 2$

6. A discrete random variable X has the following probability distribution:

$\boldsymbol{x}$	0	1	2	3
$\boldsymbol{P(X = x)}$	$2q$	$6q$	$3q$	$4q$

The mean of X is which of the following?

A. 1

B. $\frac{8}{5}$

C. $\frac{1}{15}$

D. 2

7. The integral

$$\int_0^6 |x-2|\,dx$$

evaluates to which of the following?

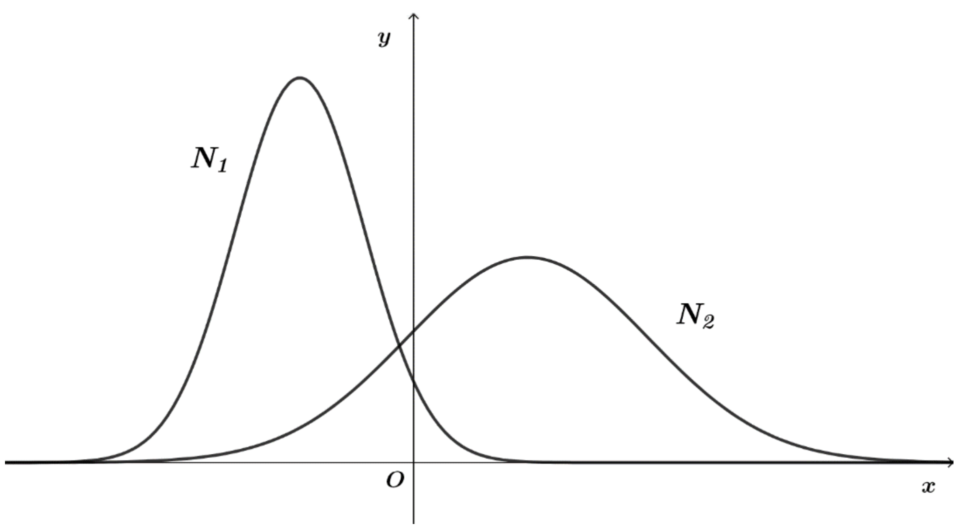
A. 10

B. 20

C. 30

D. None of the above.

8. Consider the following graphs of two normal distributions, N_1 and N_2.



Assuming μ_1, σ_1 are the mean and standard deviation of N_1, and μ_2, σ_2 are the mean and standard deviation of N_2, which of the following statements is true?

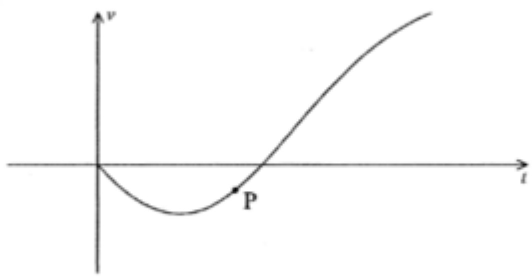
A. $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$

B. $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$

C. $\mu_1 < \mu_2$ and $\sigma_1 > \sigma_2$

D. $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$

9. The graph shows the velocity v of a particle moving along a straight line as a function of time t.



Taking the rightward direction as positive, which statement describes the motion of the particle at the point P?

A. The particle is moving left at increasing speed.

B. The particle is moving left at decreasing speed.

C. The particle is moving right at decreasing speed.

D. The particle is moving right at increasing speed.

10. A probability density function f is given by

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3) & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

The median m of this function satisfies which of the following equations?

A. $m^4 - 16m^2 = 0$

B. $m^4 - 16m^2 + 24 = 0.5$

C. $m^4 - 16m^2 + 24 = 0$

D. $16m^2 - m^4 - 6 = 0$

Mathematics Advanced
Sections II, III, IV Answer Booklet

90 marks
Attempt Questions 11–32
Allow about 2 hours and 45 minutes for this section.

Instructions

- At the beginning of each section, write your Student Number at the top of the page.
- Answer the questions in the spaces provided. These spaces provide guidance for the expected length of response.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Additional writing spaced is provided at the back of the booklet. If you use this space, clearly indicate which question you are answering.

Section II (22 marks) Student Number:__________________

11. Find the largest domain for which $\sqrt{x^2 - 2x - 8}$ is defined. **2**

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12. For x and y rational numbers and $\left(\sqrt{x} + \sqrt{y}\right)^2 = x + y + 2\sqrt{xy}$, write $\sqrt{16 + 2\sqrt{55}}$ in the form $\sqrt{x} + \sqrt{y}$. **2**

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13. Find the values of x and y if the first four terms of a geometric sequence are $3, x, y, 192$. **3**

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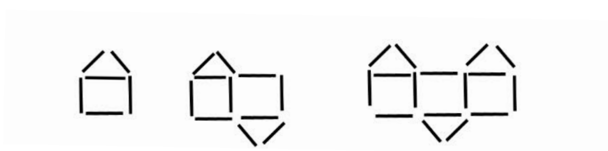
14. The patterns below are made using small matchsticks.



#1 #2 #3

Pattern #1 requires 6 matchsticks, pattern #2 requires 11 match sticks and pattern #3 requires 16 matchsticks.

a) Write a formula for the number of sticks, X_n, needed to construct pattern #*n*. **1**

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Question continues…

b) What is the largest pattern number that can be constructed using 200 matchsticks? **1**

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c) How many matchsticks would be needed to construct all patterns from pattern #1 to pattern #20? **1**

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15. A curve $y = f(x)$ passes through the point $(0, 7)$. Its gradient function is given by **2**

$$\frac{dy}{dx} = 1 - 6\sin 3x$$

Find the equation of the curve.

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16. Consider the curve $y = 3x^4 - 16x^3 + 24x^2 - 9$.

a) Show that the first and second derivatives are respectively **2**

$$y' = 12x(x-2)^2$$

$$y'' = 12(x-2)(3x-2)$$

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b) Find and classify all stationary points and points of inflexion. **4**

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Question continues…

c) Over which interval(s) is the curve decreasing? **1**

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d) Sketch the curve, ensuring you demonstrate all features found, including the intercept with the y-axis (you may ignore calculating any intercepts with the x-axis). **3**

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End Section II

Section III (35 marks) Student Number:__________________

17. A weightlifter in training becomes increasingly tired with each lift. Each time he lifts, he can only do so with 90% of the preceding weight.

a) If his first lift was 200 kg, what will be the weight lifted on the tenth lift, correct to two decimal places? **2**

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b) At the given rate, what would be the sum of all weights lifted by the time he were totally exhausted? **1**

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18. A pet shop has a tank of goldfish for sale. All fish in the tank may be taken to have their weights normally distributed with mean 100 g and standard deviation 10 g. Melanie is buying a goldfish and is invited to catch the one she wants. Sadly, the fish are too fast for Melanie to catch any particular fish and the one she eventually catches is done so at random.

Find the probability that the weight of the fish is:

a) over 120 g; **2**

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b) between 90 g and 120 g. **2**

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19.

a) Find $\tan A$ if $\operatorname{cosec} A = -\frac{13}{12}$ and $\sec A > 0$. **2**

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b) Solve **2**

$$\cos^2 2x = \frac{1}{2}$$

for $0 \leq x \leq 2\pi$.

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Examination continues…

20. Solve $3\tan\theta - \cot\theta = 5$ for $0° \leq \theta \leq 360°$. Give your answers to the nearest degree. 4

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21. Evaluate:

a) **2**

$$\int 1 + e^{7x}\ dx$$

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b) **2**

$$\int_0^3 \frac{8x}{1+x^2}\ dx$$

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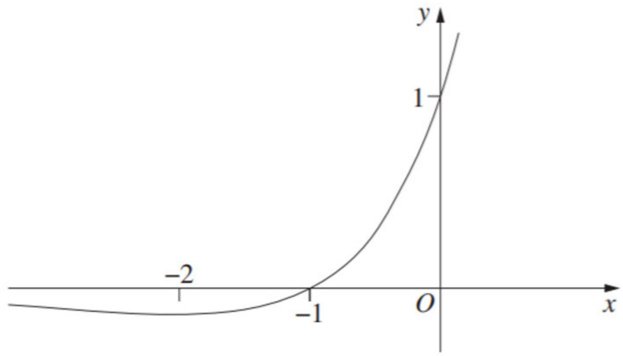
22. The diagram below shows the graph of $y = e^x(x+1)$. **2**



Using a graphical method, determine the number of solutions to the equation

$$2e^x(1+x) = 2 - 3x - 2x^2$$

You may use the diagram above in your solution.

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Examination continues…

23. The probability distribution of a discrete random variable X is given by the table below.

$\boldsymbol{X}$	0	1	2	3	4
$\boldsymbol{P(X=x)}$	0.2	$0.6p^2$	0.1	$1-p$	0.1

a) Show that $p = \frac{2}{3}$ or $p = 1$. **2**

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b) Let $p = \frac{2}{3}$. Calculate, to two decimal places:

i) $E(X)$ **2**

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ii) $P(X \geq E(X))$ **1**

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iii) The variance of X **1**

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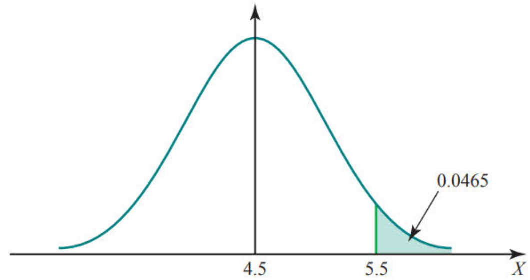
24. A random variable X is normally distributed with mean 4.5. It is given that $P(X > 5.5) = 0.0465$.



a) Show that the standard deviation is 0.595. **2**

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b) Find the probability that a random observation of X lies between 3.8 and 4.8. **2**

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25.

a) Differentiate $e^{3x}(\cos x - 3\sin x)$. **2**

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b) Hence or otherwise, find **2**

$$\int e^{3x} \sin x \, dx$$

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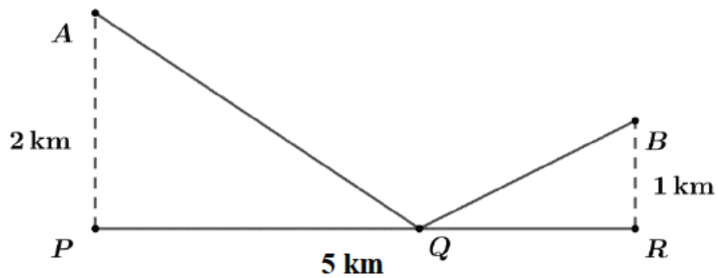
End Section III

Section IV (33 marks) Student Number:__________________

26. A telecommunications box is to be placed at some point Q along a road PQR, as shown in the diagram below. Two cables will connect the box to relay stations at A and B. Let $AP = 2$ km be the perpendicular distance from A to the road, and similarly, let $BR = 1$ km be the perpendicular distance from B to the road. The distance PR is 5 km.



a) By letting $PQ = x$ kilometres, show that the total length L of the cable is given by **1**

$$L = \sqrt{x^2 + 4} + \sqrt{x^2 - 10x + 26}$$

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Question continues…

b) Hence find the location of Q such that the total length of the cable is minimised. **3**

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Examination continues…

27. The equation of a circle is given by

$$x^2 + y^2 + 4x - 2y - 20 = 0$$

a) Find the centre and radius of the circle. **2**

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b) Hence evaluate **2**

$$\int_{-7}^{3} 1 + \sqrt{21 - 4x - x^2}\ dx$$

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28. A boat travelling in a straight line in a still lake has its engine turned off at time $t = 0$ seconds. Its velocity v metres per second at time t seconds thereafter is given by

$$v(t) = \frac{100}{(t+2)^2}$$

a) Discuss $v(t)$ as $t \to \infty$ and hence the motion of the boat. **2**

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b) Show that the acceleration on the boat is always oppositely directed to its motion. **2**

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c) To the nearest metre, how far does the boat travel from $t = 5$ to $t = 10$ seconds? **2**

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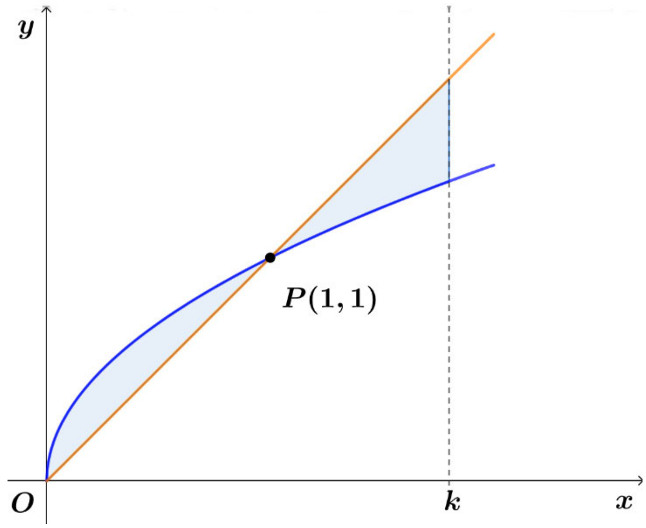
29. The diagram below shows the area bounded between two curves, $y = \sqrt{x}$ and $y = x$. **3**



Consider the region bounded by the curves, the line $x = 0$ and $x = k$, where $k > 1$.

For what value of k will the area to the left of point P be equal to the area to the right of P?

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30. Consider the probability density function f for a random variable X given by

$$f(x) = \begin{cases} \dfrac{a}{\sqrt{9+2x}} & ; \ 0 \le x \le 8 \\ 0 & ; \ \text{otherwise} \end{cases}$$

where a is a constant.

a) Find the value of a. **2**

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b) Find the cumulative distribution function, $F(x)$. **2**

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c) Hence find the interquartile range, $Q_3 - Q_1$. **3**

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Examination continues…

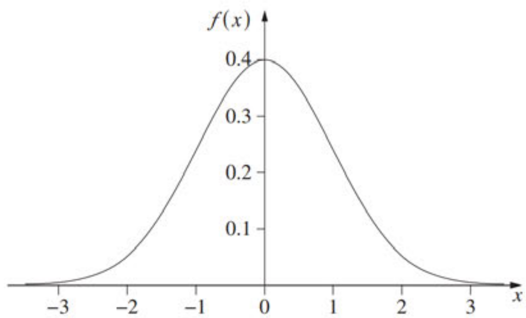
31. Let X denote a normally distributed random variable with mean 0 and standard deviation 1.

The random variable X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

where $-\infty < x < \infty$.

The diagram below shows the graph of $y = f(x)$.



Complete the table of values for the function given. Give your answer correct to four significant figures. **1**

$X = x$	0	1	2
$f(x)$	0.3989		0.05399

b) Using the trapezoidal rule and part (a), determine, to four significant figures, the value of **2**

$$P(-2 \leq X \leq 2) = \int_{-2}^{2} f(x)\, dx$$

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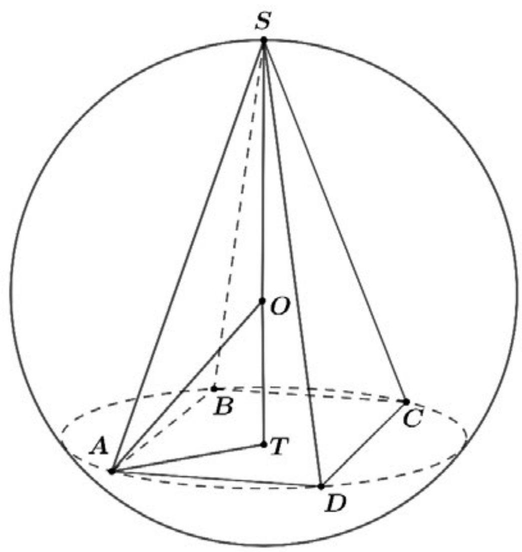
32. The diagram below shows a sphere of radius R and centre O circumscribing a square pyramid. The points A, B, C, D and S all lie on the surface of the sphere and form the vertices of the pyramid with $ABCD$ the base. The point T lies directly beneath O and lies in the base of the pyramid such that $AT \perp OT$.



a) Let $x = OT$. Show that the volume of the pyramid is **3**

$$V = \frac{2}{3}(R+x)^2(R-x)$$

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Question continues…

b) Hence find the maximum volume a square pyramid may take when its vertices are circumscribed by a sphere of radius R. **3**

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END OF EXAMINATION

Section I
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

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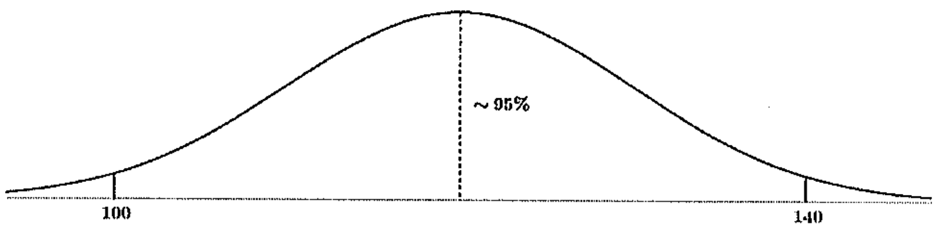
1. For the series,

$$2 - 1 + \frac{1}{2} - \frac{1}{4} + \cdots$$

what is the sum of the first eight terms, correct to two decimal places?

A. 0.68
B. 1.31
C. 1.33
D. 4.00

2. Consider the following graph of a normal distribution, with approximately 95% of the area bounded by the curve, the horizontal axis and $x = 100, x = 140$.



Which of the following parameters best describes the curve?

A. $\mu = 100,\ \sigma^2 = 140$
B. $\mu = 120,\ \sigma^2 = 20$
C. $\mu = 120,\ \sigma^2 = 10$
D. $\mu = 120,\ \sigma^2 = 100$

3. An integer n is chosen at random from the set $\{5, 7, 9, 11\}$. An integer p is also chosen at random from the set $\{2, 6, 10, 14, 18\}$. What is the probability that $n + p = 23$?

A. 0.1
B. 0.2
C. 2.5
D. 0.3

4. If $\ln 3a = \ln b - 2 \ln c$ where $a, b, c > 0$, which of the following is true?

A. $a = \frac{b-c^2}{3}$
B. $a = \frac{b}{3c^2}$
C. $\ln 3a = \frac{b}{c^2}$
D. $\ln 3a = \frac{\ln b}{\ln c^2}$

5. What is the x coordinate of the point on the curve $y = e^{2x}$ where the tangent is parallel to the line $y = 4x - 1$?

A. $x = \frac{1}{2}\ln 2$
B. $x = \ln 2$
C. $x = -\frac{1}{2}\ln 2$
D. $x = 2$

6. A discrete random variable X has the following probability distribution:

x	0	1	2	3
$P(X = x)$	$2q$	$6q$	$3q$	$4q$

The mean of X is which of the following?

A. 1
B. $\frac{8}{5}$
C. $\frac{1}{15}$
D. 2

7. The integral

$$\int_0^6 |x-2|\,dx$$

evaluates to which of the following?

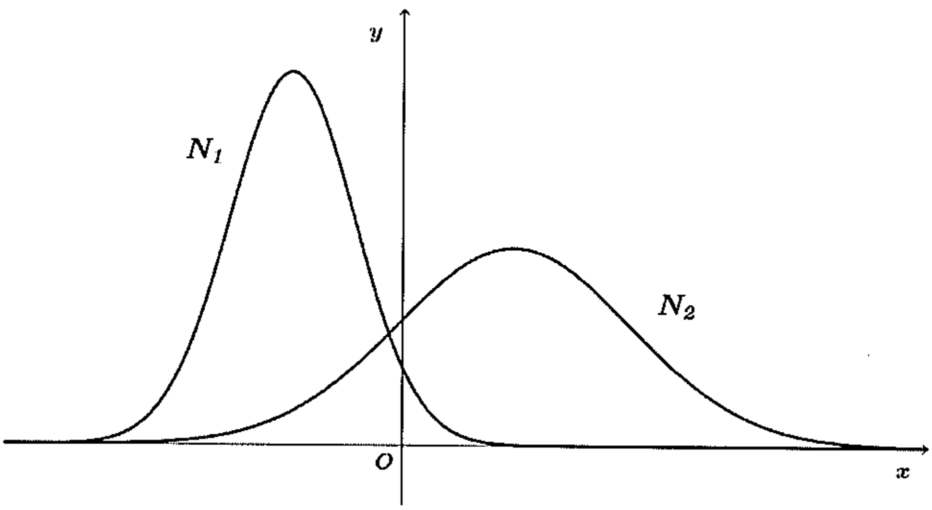
A. 10

B. 20

C. 30

D. None of the above.

8. Consider the following graphs of two normal distributions, N_1 and N_2.



Assuming μ_1, σ_1 are the mean and standard deviation of N_1, and μ_2, σ_2 are the mean and standard deviation of N_2, which of the following statements is true?

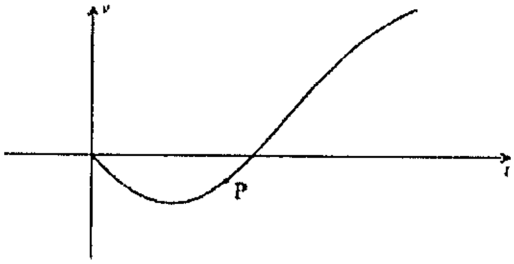
A. $\mu_1 > \mu_2$ and $\sigma_1 > \sigma_2$

B. $\mu_1 > \mu_2$ and $\sigma_1 < \sigma_2$

C. $\mu_1 < \mu_2$ and $\sigma_1 > \sigma_2$

D. $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$

9. The graph shows the velocity v of a particle moving along a straight line as a function of time t.



Taking the rightward direction as positive, which statement describes the motion of the particle at the point P?

A. The particle is moving left at increasing speed.
B. The particle is moving left at decreasing speed.
C. The particle is moving right at decreasing speed.
D. The particle is moving right at increasing speed.

10. A probability density function f is given by

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3) & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

The median m of this function satisfies which of the following equations?

A. $m^4 - 16m^2 = 0$
B. $m^4 - 16m^2 + 24 = 0.5$
C. $m^4 - 16m^2 + 24 = 0$
D. $16m^2 - m^4 - 6 = 0$

Q11,12 /4   Q13,14 /6   Q15,16 /12   /22.

Section II (22 marks) Student Number: TEACHER.

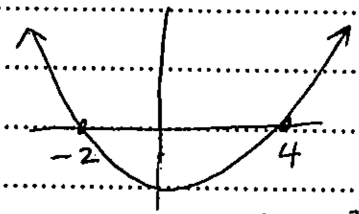
11. Find the largest domain for which $\sqrt{x^2 - 2x - 8}$ is defined. **2**

Undefined when $x^2 - 2x - 8 < 0$

$\therefore x^2 - 2x - 8 \geq 0$

$(x-4)(x+2) \geq 0$

$x \in \mathbb{R},\ x \leq -2$ or $x \geq 4$



1mk → $x^2 - 2x - 8 \geq 0$

Alternatively in Interval form

$(-\infty, -2] \cup [4, \infty)$

1mk → correct answer.

12. For x and y rational numbers and $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$, write $\sqrt{16 + 2\sqrt{55}}$ in the form $\sqrt{x} + \sqrt{y}$. **2**

$(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$

$\therefore \sqrt{x} + \sqrt{y} = \sqrt{x + y + 2\sqrt{xy}}$

$\therefore x + y = 16$

& $xy = 55$ (1)

$\therefore x, y = 11, 5$

$\therefore \sqrt{16 + 2\sqrt{55}} = \sqrt{11} + \sqrt{5}$ (1)

13. Find the values of x and y if the first four terms of a geometric sequence are $3, x, y, 192$. **3**

$\frac{x}{3} = \frac{y}{x} = \frac{192}{y}$ (1 mk)

$\therefore x^2 = 3y$ --- (1)

$y^2 = 192x$ --- (2)

Subst (2) into (1) (1 mk)

$\left(\frac{y^2}{192}\right)^2 = 3y$

$\frac{y^4}{36864} = 3y$

$y = 48$

$\therefore x = 12$ & $y = 48$. (1 mk)

OR Alternatively

$T_4 = ar^{n-1}$

$192 = 3r^3$ (1 mk)

$r^3 = \frac{192}{3}$

$r = 4$ (1 mk)

$\therefore x = 3 \times 4 = 12$

$y = 12 \times 4 = 48$ (1 mk).

14. The patterns below are made using small matchsticks.

#1 #2 #3

Pattern #1 requires 6 matchsticks, pattern #2 requires 11 match sticks and pattern #3 requires 16 matchsticks.

a) Write a formula for the number of sticks, X_n, needed to construct pattern $\#n$. **1**

Sequence is 6, 11, 16 ...

$d = 11 - 6 = 16 - 11 = 5$ $\therefore a = 6$

Using $T_n = a + (n-1)d$ $d = 5$

$X_n = 6 + (n-1)5$ (1)

$= 6 + 5n - 5$

$= 1 + 5n$

Question continues...

b) What is the largest pattern number that can be constructed using 200 matchsticks? **1**

$\therefore \; 1 + 5n = 200$

$5n = 199$

$n = 39 \cdot 8$

$\therefore$ 200 matchsticks could make the 39th pattern. (1)

c) How many matchsticks would be needed to construct all patterns from pattern #1 to pattern #20? **1**

All patterns requires a sum

$S_n = \frac{n}{2}(2a + (n-1)d)$ $\quad a = 6$

$d = 5$

$S_{20} = \frac{20}{2}(2 \times 6 + 19 \times 5)$ $\quad n = 20$

$S_{20} = 1070$ (1)

$\therefore$ 1070 matchsticks required.

15. A curve $y = f(x)$ passes through the point $(0, 7)$. Its gradient function is given by **2**

$$\frac{dy}{dx} = 1 - 6\sin 3x$$

Find the equation of the curve.

$y = \int 1 - 6\sin 3x \, dx$

$y = x + \frac{6\cos 3x}{3} + c$ (1)

When $x = 0 \quad y = 7$

$7 = 0 + \frac{6\cos 0}{3} + c$

$7 = 2 + c \quad \therefore c = 5$

$\therefore \; y = x + 2\cos 3x + 5$ --- (1)

16. Consider the curve $y = 3x^4 - 16x^3 + 24x^2 - 9$.

a) Show that the first and second derivatives are respectively **2**

$$y' = 12x(x-2)^2$$

$$y'' = 12(x-2)(3x-2)$$

$y = 3x^4 - 16x^3 + 24x^2 - 9$

$y' = 12x^3 - 48x^2 + 48x$

$= 12x(x^2 - 4x + 4)$ (1)

$= 12x(x-2)^2$ Note: ans given

$y'' = 36x^2 - 96x + 48$

$= 12(3x^2 - 8x + 4)$ (1)

$= 12(3x-2)(x-2)$

$= 12(x-2)(3x-2)$ Note: ans given.

b) Find and classify all stationary points and points of inflexion.

Stationary points occur when $y' = 0$ ie $12x(x-2)^2 = 0$

∴ $x = 0, 2$. (1mk)

When $x = 0$ $y = -9$

$x = 2$ $y = 7$

∴ Stationary points are found at $(0,-9)$ & $(2,7)$

Testing their nature at $(0,-9)$, $y'' = 12(-2)(-2)$

$= 48 > 0$ ∴ concave up

∴ min at $(0,-9)$ (1mk)

Max 2mks for points only with no reasons

Testing $(2,7)$, $y'' = 12(0)(4) = 0$

∴ possible point of inflexion (horizontal)

Test point either side

x	1	2	3
y''	-12	0	84

Changes sign from concave down to concave up. ∴ $(2,7)$ is a horizontal point of inflexion. (1mk)

Note: you must write values in table, not just say +ve or -ve.

Test to see if other points of inflexion when $y'' = 0$

ie $12(x-2)(3x-2) = 0$

$x = 2$ $x = \frac{2}{3}$

∴ Possible POI at $\left(\frac{2}{3}, -\frac{67}{27}\right)$

x	0	$\frac{2}{3}$	1
y''	$12 \cdot -2 \cdot -2$ = 48	0	-12

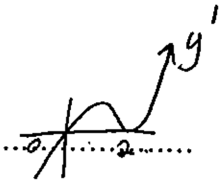
changes concavity ∴ $\left(\frac{2}{3}, -\frac{67}{27}\right)$ PoI. (1mk)

c) Over which interval(s) is the curve decreasing? **1**

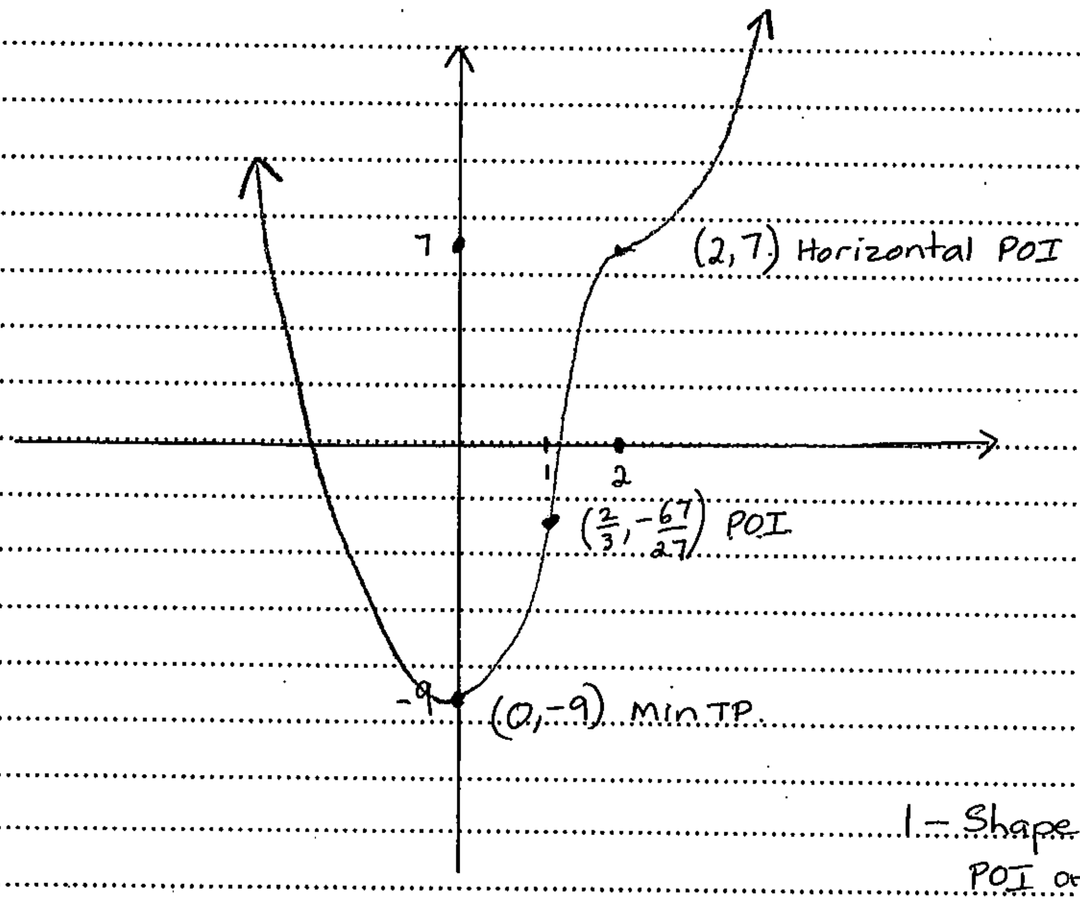
Curve is decreasing when $y' < 0$

ie $12x(x-2)^2 < 0$

Decreasing when $x < 0$ (1)



d) Sketch the curve, ensuring you demonstrate all features found, including the intercept with the y-axis (you may ignore calculating any intercepts with the x-axis). **3**



1 – Shape + label POI at $(\frac{2}{3}, -\frac{67}{27})$

1 – Min at (0, -9)

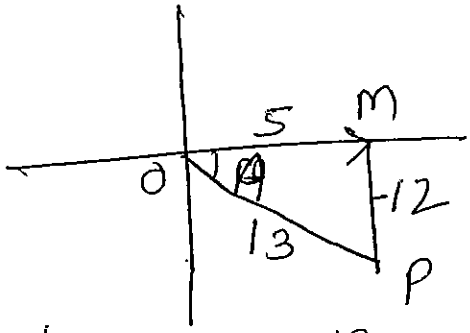
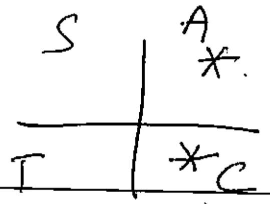
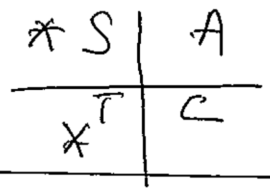
1 – Horizontal POI labelled at (2, 7)

End Section II

MATHEMATICS ~~Extension~~ Advanced 1 : Question 17

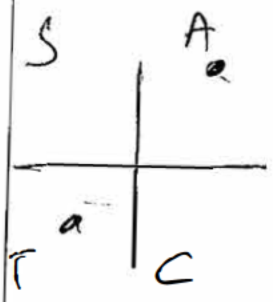
Suggested Solutions	Marks	Marker's Comments
a) common ratio = 0·9		
First term a = 2000kg		
Nth term $T_n = a r^{n-1}$		This question was answered correctly by the majority of students.
$T_n = 200 (0.9)^{n-1}$	1	
10th term $T_{10} = 200 (0.9)^9 = 77.48\text{kg}$	1	
b) limiting solution		majority of students has answered it correctly.
$S_\infty = \frac{200}{1-0.9} = 2000\text{kg}$	1	
<u>question 18</u>		
a) over 120 g		
Determine $z = \frac{120-100}{10} = 2$	1	most students have answered it correctly.
$P(X > 120) = \Phi(z > 2)$		
$= 1 - 0.9772$		
$= 0.0228$	1	

MATHEMATICS ~~Extension~~ Advanced 1 : Question 19

Suggested Solutions	Marks	Marker's Comments
a) $\operatorname{cosec} A = -\frac{13}{12}$ $\quad \sec A > 0$		
$\sin A = -\frac{12}{13}$ $\quad \cos A > 0$		
The above information indicates $\angle$ in the IV quadrant	1	Majority of the student have answered it correctly.
		
$\tan A = -\frac{12}{5}$	1	
b) $\cos^2 2x = \frac{1}{2}$		
$\cos 2x = \frac{1}{\sqrt{2}}$ $\quad \vert \quad$ $\cos 2x = -\frac{1}{\sqrt{2}}$	1	Some students did not realise that $0 \le 2x \le 4\pi$ is the domain and hence were penalized.
$2x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$ $\quad \vert \quad$ $2x = \frac{3}{4}\pi, \frac{5\pi}{4}, \frac{11}{4}\pi, \frac{13\pi}{4}$		
$x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$ $\quad \vert \quad$ $x = \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$		
$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$	1	
 		

MATHEMATICS ~~Extension~~ Advanced **1 : Question** 20

Suggested Solutions	Marks	Marker's Comments
$3\tan\theta - \cot\theta = 5$		
$\cot\theta = \frac{1}{\tan\theta}$		
$3\tan^2\theta - 1 = 5\tan\theta$		
$3\tan^2\theta - 5\tan\theta - 1 = 0$	1	
~~3 tanθ~~		
$\tan\theta = \frac{5 \pm \sqrt{25+12}}{6}$		
$\tan\theta = \frac{5 \pm \sqrt{37}}{6}$	1	Some students lost marks as they did not account for $\theta = 350^\circ$ and 170° values.
$\tan\theta = \frac{5+\sqrt{37}}{6}$ | $\tan\theta = \frac{5-\sqrt{37}}{6}$		
$\theta = 62^\circ, 180^\circ + 60^\circ$ | $\theta = -10.22^\circ + 360^\circ = 350^\circ$	1	
$\theta = 62^\circ, 242^\circ$		
$\theta = -10.22^\circ + 180^\circ = 170^\circ$	1	

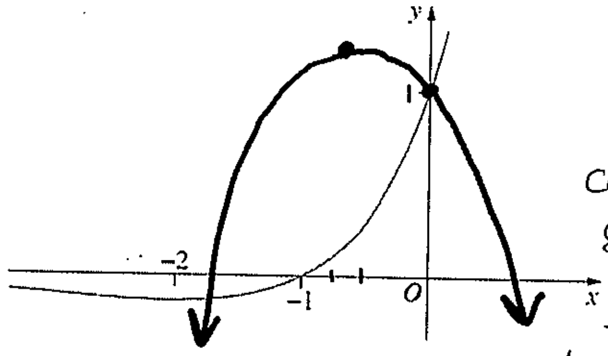


MATHEMATICS ~~Extension 1~~ Advanced : Question 21

Suggested Solutions	Marks	Marker's Comments
a) $\int 1+e^{7x}\,dx$		
$= x + \dfrac{e^{7x}}{7} + c$	1 + 1	Students were penalised if they did not write down the constant of integration.
b) $\int_0^3 \dfrac{8x}{1+x^2}\,dx$		
$= 4\int_0^3 \dfrac{2x}{1+x^2}\,dx$	1	
$= 4 \ln(1+x^2)\Big\vert_0^3$		Some students did not use $\int \frac{f'(x)}{f(x)}\,dx = \ln x + c$ formula correctly. So they were penalised.
$= 4\ln 10$	1	

Mathematics Advanced

22. The diagram below shows the graph of $y = e^x(x+1)$. **2**



1 mark for concave downward shape and 1 mark for two intersection points.

Using a graphical method, determine the number of solutions to the equation

$$2e^x(1+x) = 2 - 3x - 2x^2$$

You may use the diagram above in your solution.

$2e^x(1+x) = 2 - 3x - 2x^2 \Leftrightarrow e^x(1+x) = 1 - 1.5x - x^2$

Where $1 - 1.5x - x^2$ meets $e^x(1+x)$, we have solutions.

Sketch: $-(x^2 + 1.5x) + 1 = -((x+0.75)^2 - 0.75^2) + 1$

$= -(x+0.75)^2 + 1.5625$

Note $x=0$ is a solution. Concave-down, max at $x = -0.75$, cuts through $(0,2)$ & again on negative side.

So → <u>two</u> solutions.

Examination continues...

MATHEMATICS ~~Extension 1~~ Advanced : Question 22		
Suggested Solutions	**Marks**	**Marker's Comments**
a) we know $\sum_{x \in X} P(X=x) = 1$	1	
$0.2 + 0.6P^2 + 0.1 + 1 - P + 0.1 = 1$ $0.6P^2 - P + 0.4 = 0$ $3P^2 - 5P + 2 = 0$ $3P^2 - 3P - 2P + 2 = 0$ $3P(P-1) - 2(P-1) = 0$ $(3P-2)(P-1) = 0$		Some students came out with only one value of P, so were penalised.
$P = 2/3$ | $P = 1$	1	
b) $P = 2/3$ i) $E(X) = 0.2 \times 0 + 0.6 \times \left(\frac{2}{3}\right)^2 + 0.1 \times 2 + \left(1 - \frac{2}{3}\right) \times 3 + 0.1 \times 4$ $\doteq 1.87$	1	This is a wrong or right answer type question. So students were awarded one mark if right or zero mark if wrong.

MATHEMATICS ~~Extension 1~~ Advanced : Question 23

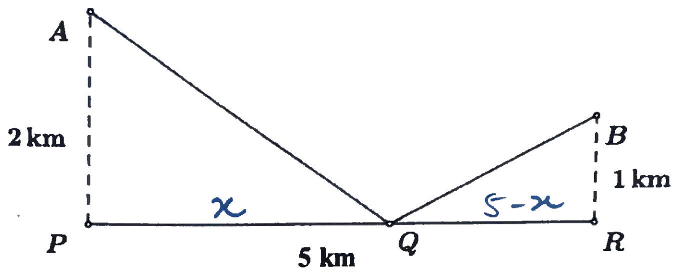
Suggested Solutions	Marks	Marker's Comments
(ii) $P(X \geq E(X))$ $P(X \geq 1.87) = P(X = 2, 3, 4)$ $= 0.1 + 1 - 2/3 + 0.1$ $= 0.5\dot{3} \doteqdot 0.5$	1	Most students got it correct.
(iii) $Var(X) = E(X^2) - \mu^2$ $= 0 + 0 + 0.6 \times \left(\frac{2}{3}\right)^2 + 0.1 \times 4$ $+ \frac{1}{3} \times 9 + 16 \times 0.1 - 1.87^2$ $= 5.267 - 3.497$ $= 1.77$ (2 d.p)	1	Right or wrong question. Most students got it correct.

MATHEMATICS ~~Extension 1~~ Advanced : Question 24		
Suggested Solutions	**Marks**	**Marker's Comments**
a) $\mu = 4.5$ $z = \frac{5.5-\mu}{\sigma}$ Given $P(X > 5.5) = 0.0465$ $\frac{X-\mu}{\sigma} > \frac{5.5-\mu}{\sigma}$ $\phi\left(z > \frac{5.5-\mu}{\sigma}\right) = 0.0465$ $1 - 0.0465 = 0.9535$ z for this 1.68 from the table	1	majority of the students got it right
$\frac{5.5-\mu}{1.68} = \sigma$ $\sigma = \frac{5.5-4.5}{1.68} = 0.5952$	1	
b) $P(3.8 < X < 4.8)$ $= \phi\left(\frac{3.8-4.5}{0.595} < z < \frac{4.8-4.5}{0.595}\right)$ $= \phi(-1.180 < z < 0.500)$	1	I rounded up z-score to 2 d.p. Hence variations in answers expected and were given full credit.
~~= 0.384~~ $= \phi(0.50) + \phi(1.18) - 1$ $= 0.6915 + 0.8810 - 1$ $= 0.5725$	1	

MATHEMATICS ~~Extension 1~~ Advanced : Question. 25		
Suggested Solutions	**Marks**	**Marker's Comments**
a) $\frac{d}{dx} e^{3x}(\cos x - 3\sin x)$ use product rule $u = e^{3x}$ $u' = 3e^{3x}$ $v = \cos x - 3\sin x$ $v' = -\sin x - 3\cos x$ $\frac{d}{dx} e^{3x}(\cos x - 3\sin x)$ $= 3e^{3x}(\cos x - 3\sin x) + e^{3x}(-\sin x - 3\cos x)$	1	This question was fairly done by most students.
$= e^{3x}[3\cos x - 9\sin x - \sin x - 3\cos x)$ $= e^{3x}[-10\sin x)$		
$= -10e^{3x}\sin x$	1	
b) use result from the previous question $\int \frac{d}{dx} e^{3x}(\cos x - 3\sin x) = -10\int e^{3x}\sin x\, dx$	1	Some students did not effectively use the result from the part 'a' and got it wrong. Hence they were penalised.
$e^{3x}\sin x = -\frac{1}{10} e^{3x}(\cos x - 3\sin x) + C$	1	

Section IV (33 marks) Student Number:____________________

26. A telecommunications box is to be placed at some point Q along a road PQR, as shown in the diagram below. Two cables will connect the box to relay stations at A and B. Let $AP = 2$ km be the perpendicular distance from A to the road, and similarly, let $BR =$ km be the perpendicular distance from B to the road. The distance PR is 5 km.



a) By letting ~~X~~ P $Q = x$ kilometres, show that the total length L of the cable is given by **1**

$$L = \sqrt{x^2 + 4} + \sqrt{x^2 - 10x + 26}$$

$AQ = \sqrt{x^2+2^2}$ $QB = \sqrt{(5-x)^2+1^2}$

By Pythagoras Theorem

$\therefore L = \sqrt{x^2+4} + \sqrt{x^2-10x+26}$ — (1)

Question continues...

b) Hence find the location of Q such that the total length of the cable is minimised. **3**

$$\frac{dL}{dx} = \frac{x}{\sqrt{x^2+4}} + \frac{x-5}{\sqrt{x^2-10x+26}}$$ (1)

Stationary point when $\frac{dL}{dx} = 0$

$$\Rightarrow \frac{x}{\sqrt{x^2+4}} = \frac{5-x}{\sqrt{x^2-10x+26}} \quad (*)$$

Squaring both sides (note: this introduces an extra solution).

$$\frac{x^2}{x^2+4} = \frac{(5-x)^2}{x^2-10x+26}$$

$$x^4 - 10x^3 + 26x^2 = 25x^2 - 10x^3 + x^4 + 100 - 40x + 4x^2$$

$$0 = 3x^2 - 40x + 100$$

$$0 = (3x-10)(x-10)$$

(1) $x = 10$ or $\frac{10}{3}$

But 10 exceeds construction, and does not satisfy (*). (it is the extra solution introduced by squaring).

Hence, $x = \frac{10}{3}$ only.

x	0	10/3	5
$\frac{dL}{dx}$	$-\frac{5}{\sqrt{26}}$	0	$\frac{5}{\sqrt{29}}$

(1)

$\therefore$ $x = \frac{10}{3}$ is a local minimum.

$\therefore$ Q should be located $\frac{10}{3}$ km from P to minimise the length of cable.

Examination continues...

27. The equation of a circle is given by

$$x^2 + y^2 + 4x - 2y - 20 = 0$$

a) Find the centre and radius of the circle. **2**

$x^2 + 4x + 4 + y^2 - 2y + 1 = 25$

$(x+2)^2 + (y-1)^2 = 25$

centre : $(-2, 1)$

radius = 5 units.

b) Hence evaluate **2**

$$\int_{-7}^{3} 1 + \sqrt{21 - 4x - x^2}\, dx$$

$21 - (x^2 + 4x) = 21 - ((x+2)^2 - 4)$

$= 25 - (x+2)^2$

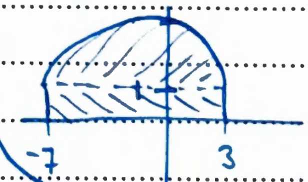
OR $(x+2)^2 + (y-1)^2 = 25$

$(y-1)^2 = 25 - (x+2)^2$

$y - 1 = \sqrt{25 - (x+2)^2}$

$y = 1 + \sqrt{21 - 4x - x^2}$

Hence $\int_{-7}^{3} 1 + \sqrt{21 - 4x - x^2}\, dx$



= Area of rectangle + Area of semicircle — ①

$= (3 - -7) \times 1 + \frac{1}{2}\pi(5)^2$

$= 10 + \frac{25\pi}{2}\ u^2$ — ①

OR $= 49.27\ u^2$ (2dp).

28. A boat travelling in a straight line in a still lake has its engine turned off at time $t = 0$ seconds. Its velocity v metres per second at time t seconds thereafter is given by

$$v(t) = \frac{100}{(t+2)^2}$$

a) Discuss $v(t)$ as $t \to \infty$ and hence the motion of the boat. **2**

① — $\lim_{t \to \infty} v(t) = 0^+$ $v(t) > 0$ as $(t+2)^2 > 0$, $t \geq 0$

① — So the boat forever slows approaching zero velocity but never reaching a complete stop.

b) Show that the acceleration on the boat is always oppositely directed to its motion. **2**

$v(t) > 0$ as $t \geq 0$, $\frac{100}{(t+2)^2} > 0$

$a(t) = v'(t) = \frac{-200}{(t+2)^3}$ — ①

Since $t \geq 0$, $(t+2)^3 \geq 0$ so $\frac{-200}{(t+2)^3} < 0$ ①

∴ velocity is always positively directed and acceleration negatively directed. Hence acceleration opposes motion.

c) To the nearest metre, how far does the boat travel from $t = 5$ to $t = 10$ seconds? **2**

$x = \int_5^{10} 100(t+2)^{-2}\,dt$

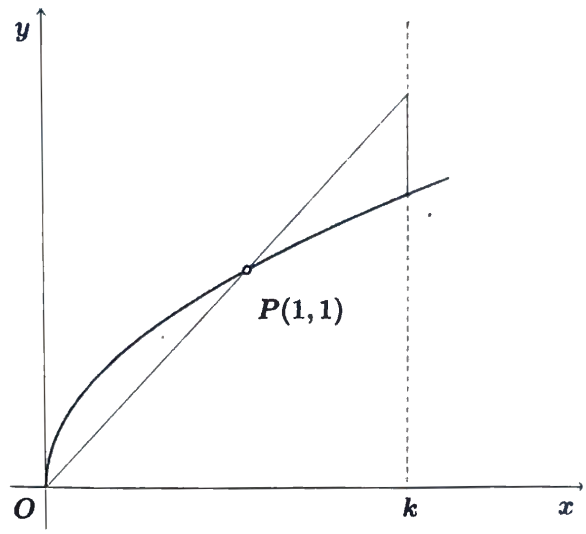
$= -100\left[(t+2)^{-1}\right]_5^{10}$ — ①

$= -100\left(\frac{1}{12} - \frac{1}{7}\right)$

$= 5.95$

≈ 6m (nearest metre) — ①

29. The diagram below shows the area bounded between two curves, $y = \sqrt{x}$ and $y = x$. **3**



Consider the region bounded by the curves, the line $x = 0$ and $x = k$, where $k > 1$.

For what value of k will the area to the left of point P be equal to the area to the right of P?

$$\int_0^1 x^{\frac{1}{2}} - x \, dx = \int_1^k x - x^{\frac{1}{2}} \, dx \quad \text{— (1)}$$

$$\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{2}\right]_0^1 = \left[\frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3}\right]_1^k$$

$$\frac{2}{3} - \frac{1}{2} = \frac{k^2}{2} - \frac{2k^{\frac{3}{2}}}{3} - \frac{1}{2} + \frac{2}{3}$$

$$\frac{k^2}{2} - \frac{2k^{\frac{3}{2}}}{3} = 0 \quad \text{— (1)}$$

$$k^{\frac{3}{2}}\left(\frac{k^{\frac{1}{2}}}{2} - \frac{2}{3}\right) = 0$$

$$\frac{k^{\frac{1}{2}}}{2} = \frac{2}{3} \quad (\text{since } k \neq 0,\ k > 1)$$

$$\sqrt{k} = \frac{4}{3}$$

$$k = \frac{16}{9} \quad \text{— (1)}$$

30. Consider the probability density function f for a random variable X given by

$$f(x) = \begin{cases} \dfrac{a}{\sqrt{9+2x}} & ; \ 0 \le x \le 8 \\ 0 & ; \ \text{otherwise} \end{cases}$$

where a is a constant.

a) Find the value of a. **2**

Require $1 = \int_{\mathbb{R}} f(x) = 0 + \int_0^8 \frac{a}{\sqrt{9+2x}}\,dx$

so $1 = a \int_0^8 (9+2x)^{-\frac{1}{2}}\,dx$

$1 = a\left[(9+2x)^{\frac{1}{2}}\right]_0^8$ —— ①

$1 = a[5-3]$

$1 = 2a$

$a = \frac{1}{2}$ —— ①

b) Find the cumulative distribution function, $F(x)$. **2**

for $0 \le x \le 8$: $F(x) = \frac{1}{2}\int_0^x (9+2x)^{-\frac{1}{2}}\,dx$

$= \frac{1}{2}\left[\sqrt{9+2x}\right]_0^x$

$= \frac{1}{2}\left(\sqrt{9+2x} - 3\right)$ —— ①

so $F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{\sqrt{9+2x}-3}{2} & \text{for } 0 \le x \le 8 \\ 1 & \text{for } x > 8 \end{cases}$ —— ①

c) Hence find the interquartile range, $Q_3 - Q_1$. **3**

$IQR = Q_3 - Q_1$

$F(x) = \frac{1}{2}\sqrt{9+2x} - \frac{3}{2} \implies x = \frac{\left[2\left(\frac{3}{2} + F(x)\right)\right]^2 - 9}{2}$

Q_3 when $F(x) = 3/4$,

$Q_3 = \frac{\left(2\left(\frac{3}{2} + 3/4\right)\right)^2 - 9}{2} = \frac{45}{8}$ or 5.625

Q_1 when $F(x) = 1/4$

$Q_1 = \frac{\left(2\left(\frac{3}{2} + 1/4\right)\right)^2 - 9}{2} = \frac{13}{8}$ or 1.625

$\therefore IQR = Q_3 - Q_1 = 4$

Examination continues...

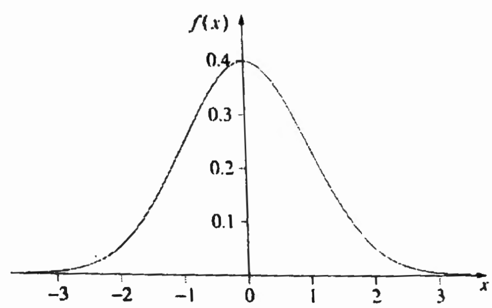
31. Let X denote a normally distributed random variable with mean 0 and standard deviation 1.

The random variable X has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

where $-\infty < x < \infty$.

The diagram below shows the graph of $y = f(x)$.



Complete the table of values for the function given. Give your answer correct to four significant figures. **1**

$X = x$	0	1	2
$f(x)$	0.3989	0.2420	0.05399

(1)

b) Using the trapezoidal rule and part (a), determine, to four significant figures, the value of **2**

$$P(-2 \leq X \leq 2) = \int_{-2}^{2} f(x)\, dx$$

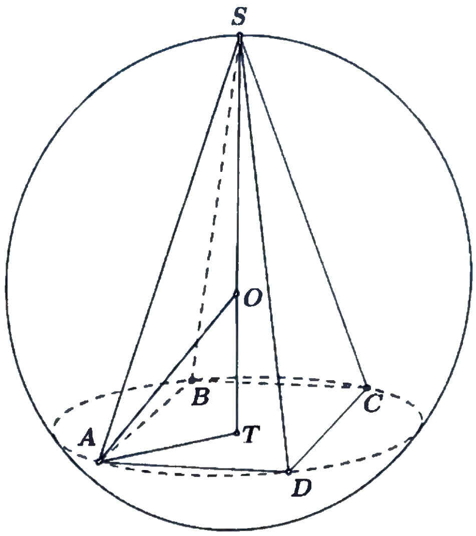
Since $f(-x) = f(x)$ $\int_{-2}^{2} f(x)\,dx = 2\int_{0}^{2} f(x)\,dx$.

So $P(-2 \leq X \leq 2) = 2\int_{0}^{2} f(x)\,dx$

$\approx 2 \times \frac{1}{2}\left[0.3989 + 2 \times 0.2420 + 0.05399\right]$

≈ 0.9369 (4 significant figures).

32. The diagram below shows a sphere of radius R and centre O circumscribing a square pyramid. The points A, B, C, D and S all lie on the surface of the sphere and form the vertices of the pyramid with $ABCD$ the base. The point T lies directly beneath O and lies in the base of the pyramid such that $AT \perp OT$.



a) Let $x = OT$. Show that the volume of the pyramid is **3**

$$V = \frac{2}{3}(R+x)^2(R-x)$$

Base $= 2\sqrt{R^2-x^2}\sqrt{R^2-x^2}$ ①

$= 2(R^2-x^2)$

Height $= x + R$ ①

$\therefore V = \frac{1}{3} \times 2 \times (R^2-x^2) \times (x+R)$

$= \frac{2}{3}(R+x)(R-x)(R+x)$ ①

$V = \frac{2}{3}(R+x)^2(R-x)$.

Question continues...

b) Hence find the maximum volume a square pyramid may take when its vertices are circumscribed by a sphere of radius R. **3**

$V = \frac{2}{3}(R+x)^2(R-x)$

$\frac{dV}{dx} = \frac{2}{3}\left[2(R+x)(R-x) - (R+x)^2\right]$

Stationary Pt. when $\frac{dV}{dx} = 0$

$0 = \frac{2}{3}\left[2(R+x)(R-x) - (R+x)^2\right]$

$0 = (R+x)\left[2(R-x) - (R+x)\right]$

$0 = 2R - 2x - R - x$ $(R \neq -x;\ R, x > 0$ as lengths$)$.

$0 = R - 3x$

$x = \frac{R}{3}$ —— (1)

$\frac{d^2V}{dx^2} = \frac{2}{3}\left[2(-2x) - 2(R+x)\right]$

$= -\frac{4}{3}(3x + R)$ —— (1) test

when $x = \frac{R}{3}$ $\frac{d^2V}{dx^2} = -\frac{4}{3}(R+R) < 0$ as $R > 0$

So local max at $x = \frac{R}{3}$

Hence max volume : $V\left(\frac{R}{3}\right) = \frac{2}{3}\left(R+\frac{R}{3}\right)^2\left(R-\frac{R}{3}\right)$

$= \frac{2}{3} \times \frac{16R^2}{9} \times \frac{2R}{3} = \frac{64}{81}R^3$ —— (1)

END OF EXAMINATION