**2013**

**TRIAL HSC EXAMINATION**

# Mathematics

**General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11–16.

**Total Marks – 100**

( **Section I** )    Pages 2–3

**10 marks**
- Attempt Questions 1–10
- Allow about 15 minutes for this section

( **Section II** )    Pages 4–10

**90 marks**
- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** What is the supplement of $\dfrac{\pi}{6}$ ?

(A) $\dfrac{\pi}{3}$    (B) $\dfrac{5\pi}{6}$    (C) $\dfrac{5\pi}{3}$    (D) $\dfrac{11\pi}{6}$

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**2** What is the equation of the parabola with vertex (4, 2) and focus (3, 2)?

(A) $(x-4)^2 = 4(y-2)$    (B) $(x-4)^2 = -4(y-2)$

(C) $(y-2)^2 = 4(x-4)$    (D) $(y-2)^2 = -4(x-4)$

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**3** The quadratic equation $2x^2 - 5x + 12 = 0$ has roots $\alpha$ and $\beta$. What is the value of $\alpha^2 + \beta^2$?

(A) $-\dfrac{23}{4}$    (B) $\dfrac{23}{4}$    (C) $\dfrac{25}{4}$    (D) $-\dfrac{25}{4}$

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**4** The number represented by a 1 followed by one hundred zeros is called a googol. Which of the following is equal to a googol?

(A) $10^{10}$    (B) $10^{99}$    (C) $10^{100}$    (D) $10^{101}$

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**5** Which of the following expressions represents $\displaystyle\int \dfrac{x}{2x^2}\, dx$ ?

(A) $\dfrac{x}{2} + C$    (B) $\tfrac{1}{2}\ln x + C$    (C) $\ln\dfrac{x}{2} + C$    (D) $\dfrac{x^2}{4} + C$

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**6** Given that $\sin\theta = \tfrac{5}{13}$ and $\tan\theta < 0$, what is the exact value of $\cos\theta$?

(A) $-\tfrac{12}{5}$    (B) $-\tfrac{12}{13}$    (C) $\tfrac{12}{13}$    (D) $\tfrac{12}{5}$

---

2

**7** Which of the following expressions is the correct simplification of $\dfrac{\operatorname{cosec}\theta \sec\theta}{\tan\theta}$?

    (A) $\sin^2\theta$           (B) $\cos^2\theta$           (C) $\operatorname{cosec}^2\theta$           (D) $\sec^2\theta$

**8** The curve $y = ax^2 - 6x + 3$ has a stationary point at $x = 1$. What is the value of $a$?

    (A) 2           (B) −1           (C) 3           (D) −3

**9** Which of the following correctly represents the sum $1 + x + x^2 + x^3 + \ldots + x^n$?

    (A) $\displaystyle\sum_{k=1}^{n} x^k$           (B) $\displaystyle\sum_{k=1}^{n+1} x^k$           (C) $\displaystyle\sum_{k=1}^{n} x^{k-1}$           (D) $\displaystyle\sum_{k=1}^{n+1} x^{k-1}$

**10** What is the range of the function $y = |x| - x$?

    (A) All real $y$           (B) $y \geq 0$           (C) $y \leq 0$           (D) $y = 0$

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

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**Question 11** (15 marks) Use a new Writing Booklet

(a) Given that $n = 2 - \sqrt{3}$, evaluate $n + \dfrac{1}{n}$, showing all working. Give your answer in exact form. **2**

(b) Differentiate the following with respect to $x$.

    (i) $(x + 1)^2$ **2**

    (ii) $xe^{2x}$ **2**

    (iii) $\ln \dfrac{x}{2x+1}$ **3**

(c) Write $\dfrac{x+1}{x(x-1)} - \dfrac{x-1}{x(x+1)}$ as a single fraction in simplest form. **3**

(d) Find $\displaystyle\int_0^1 1 + e^{2x} \, dx$. Give your answer in exact form. **3**

4

**Question 12** (15 marks) Use a new Writing Booklet

(a) Using the trapezoidal rule, find an approximation to $\int_0^{\frac{\pi}{2}} \cos x \, dx$, using 4 function values. **3**
Give your answer correct to 3 significant figures.

(b) What is the domain and range of the curve $y = \ln(x + 1)$? **2**

(c)
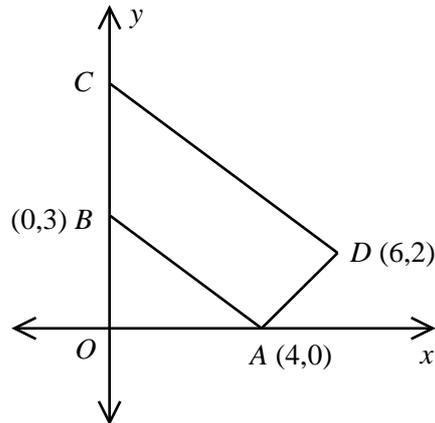


The diagram above shows the points $A(4, 0)$, $B(0, 3)$ and $D(6, 2)$. The point $C$ lies on the $y$-axis, and $CD$ is parallel to $AB$.

(i) Copy of trace the diagram into your writing booklet.

(ii) What type of quadrilateral is $ABCD$? Give a reason to support your answer. **2**

(iii) Determine the gradient of the interval $AB$. **1**

(iv) Show that the equation of $CD$ is $3x + 4y - 26 = 0$. **2**

(v) Find the coordinates of $C$. **1**

(vi) Show that the length of the interval $AB$ is 5 units. **1**

(vii) Show that the length of the interval $CD$ is $7\frac{1}{2}$ units. **1**

(viii) Find the perpendicular distance from $A$ to $CD$. **1**

(ix) Hence, or otherwise, find the area of quadrilateral $ABCD$. **1**

5

**Question 13** (15 marks) Use a new Writing Booklet

(a) For an arithmetic progression, the fifth term is 16 and the eleventh term is 40.

    (i) Find the first term and the common difference. **3**

    (ii) How many terms in the sequence must be added to reach a sum of 312? **2**

(b) Find the equation of the tangent to the curve $y = \log_e x^2$ at the point $(1, 0)$. **3**

(c) Consider the curve $y = 2 + 3x - x^3$.

    (i) Find $\dfrac{dy}{dx}$. **1**

    (ii) Locate any stationary points and determine their nature. **3**

    (iii) For what value(s) of $x$ is the curve concave up? **1**

    (iv) Sketch the curve for $-2 \le x \le 2$. **2**

**Question 14**  (15 marks)  Use a new Writing Booklet

(a)  A bushwalker sets out from base camp (*A*) and walks for 14 km on a bearing of 121° to a lookout (*B*). She then turns and walks on a bearing of 232° until she is directly south of her starting point (*C*).
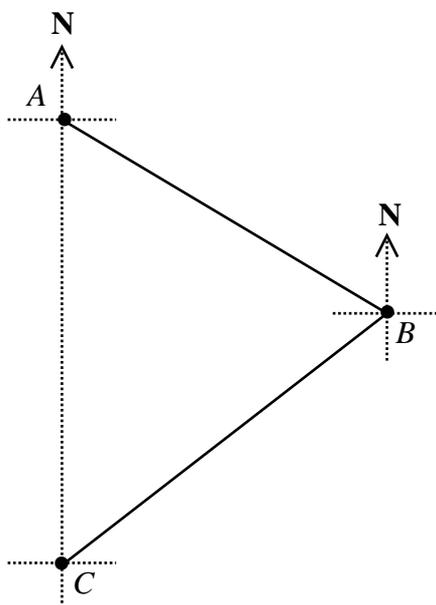


(i)  Copy the diagram above into your writing booklet and complete it by clearly showing the above information.  **1**

(ii)  How far south is the bushwalker from her starting point? Give your answer correct to 1 decimal place.  **3**

(b)  (i)  Write down the amplitude and period of the function $y = 2 \cos 2x - 1$.  **2**

(ii)  Draw a large, neat sketch of the function $y = 2 \cos 2x - 1$ over the domain $-\pi \le x \le \pi$.  **2**

(iii)  On the same diagram, draw a neat sketch of the function $y = x - 1$.  **1**

(iv)  Hence find the number of solutions to the equation $2 \cos 2x - x = 0$.  **1**

**Question 14 (continued)**

(c)
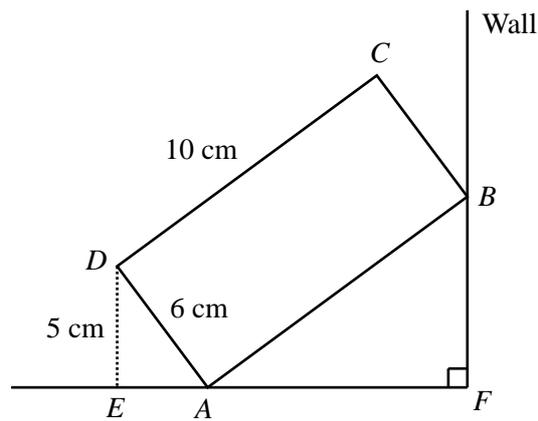


In the diagram above, *ABCD* is the cross section of a rectangular block which has been placed against a wall. The block is 10 cm wide and 6 cm high. The outermost edge (indicated by the point *D* in the diagram*)* of the block is 5 cm above the ground.

(i)  Prove that Δ*EAD* is similar to Δ*FBA*.  **3**

(ii)  At what height does the block touch the wall (indicated by the point *B* in the diagram)? Give your answer in simplest surd form.  **2**

8

**Question 15** (15 marks) Use a new Writing Booklet

(a) The diagram below shows the graph of a gradient function $y = f'(x)$.
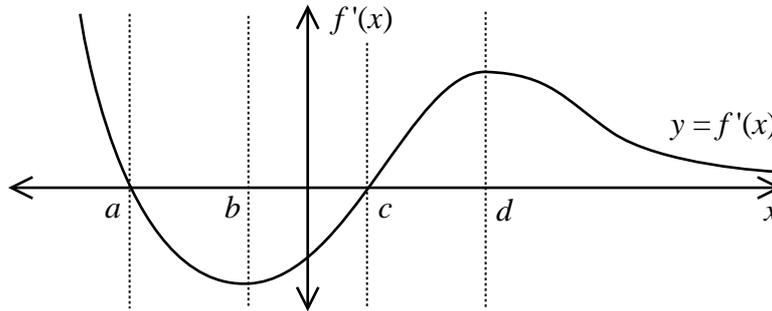


(i) Write down the value(s) of $x$ where the curve is stationary. **2**

(ii) For what value of $x$ will the curve have a maximum turning point? **1**

(iii) Copy or trace the diagram into your writing booklet. Draw a possible curve for $y = f(x)$, clearly showing what is happening to the curve as the values of $x$ increase indefinitely. **2**

(b) Find the exact volume of the solid formed when the area enclosed by the curve $y = \dfrac{1}{\sqrt{x+1}}$ and the lines $x = 0$ to $x = 3$ is rotated about the $x$-axis. **4**

(c) The graph below shows the curve $y = e^{-x}$ intersecting with the line $OP$ in the first quadrant. $O$ is the origin and $P$ has coordinates $(1, \dfrac{1}{e})$.
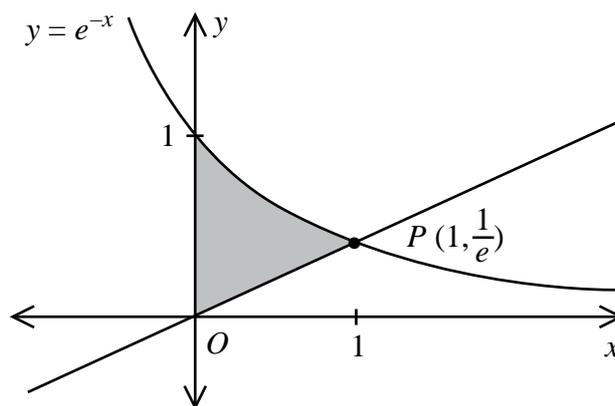


(i) Show that the line $OP$ has equation $y = \dfrac{x}{e}$. **2**

(ii) Hence, or otherwise, calculate the shaded area bound by the curve $y = e^{-x}$, the line $y = \dfrac{x}{e}$ and the $y$-axis. Give your answer in exact form. **4**

9

**Question 16**       (15 marks)              Use a new Writing Booklet

(a)   Solve $2 \sin^2\theta - \cos\theta + 1 = 0$ for the domain $0 \le \theta \le 2\pi$.       **3**

(b)   *ABCDE* is a pentagon with perimeter 30 cm. The pentagon is constructed with an equilateral triangle $\triangle ABE$ joining a rectangle *BCDE*.
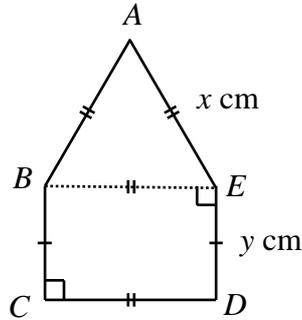


(i)    Show that $y = \dfrac{30 - 3x}{2}$.       **1**

(ii)   Show that the area of $\triangle ABE$ is $\dfrac{\sqrt{3}x^2}{4}$ cm².       **2**

(iii)  Hence show that the area of the pentagon is $15x + \dfrac{(\sqrt{3} - 6)x^2}{4}$ cm².       **2**

(iv)   Find the exact value of $x$ for which the area of the pentagon will be a maximum. Justify your solution.       **3**

(c)   Mr Jones has decided to start saving for an overseas trip when he takes long service leave in 5 years. He would like to have $10 000 to cover all his expenses. He decides to save $150 per month, at the start of each month, in an account which earns interest at 3% per annum, compounded monthly. Will Mr Jones reach his goal of $10 000? By how much will he exceed or fall short of his goal?       **4**

**End of paper**

10

**2 UNIT MATHEMATICS**
**2013 TRIAL HSC EXAMINATION**

## SECTION I

1   $\dfrac{\pi}{6} = 30°$            1   **B**

$\therefore$ Supplement $= 180 - 30°$
$= 150°$
$= \dfrac{5\pi}{6}$

2                                                       2   **D**
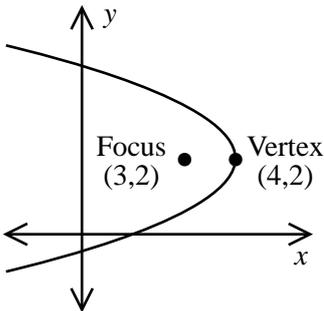


Focal length $= 4 - 3 = 1$ unit
Parabola is sideways, concave left.
$\therefore$ Equation is:

$(y - y_1)^2 = -4a(x - x_1)$
$(y - 2)^2 = -4.1.(x - 4)$
$(y - 2)^2 = -4(x - 4)$

3   $2x^2 - 5x + 12 = 0$                      3   **A**

$\alpha + \beta = \dfrac{-b}{a}$
$= \dfrac{-(-5)}{2}$
$= \tfrac{5}{2}$

$\alpha\beta = \dfrac{c}{a}$
$= \dfrac{12}{2}$
$= 6$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= \left(\tfrac{5}{2}\right)^2 - 2(6)$
$= -\tfrac{23}{4}$

4   $100 = 10^2$                                   4   **C**
$1000 = 10^3$
$\therefore$ 1 followed by 100 zeros $= 10^{100}$

5    $\displaystyle\int \frac{x}{2x^2}\,dx = \int \frac{1}{2x}\,dx$

$$= \tfrac{1}{2}\int \frac{1}{x}\,dx$$

$$= \tfrac{1}{2}\,\ln x + C$$

5   **B**

6



If $\sin\theta > 0$ and $\tan\theta < 0$ then $\theta$ is in the second quadrant.

$\cos\theta = -\frac{12}{13}$

6   **B**

7

$$\frac{\operatorname{cosec}\theta\,\sec\theta}{\tan\theta} = \operatorname{cosec}\theta \times \sec\theta \div \tan\theta$$

$$= \frac{1}{\sin\theta} \times \frac{1}{\cos\theta} \div \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta} \times \frac{1}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{\sin\theta}$$

$$= \frac{1}{\sin^2\theta}$$

$$= \operatorname{cosec}^2\theta$$

7   **C**

8   $y = ax^2 - 6x + 3$

$$\frac{dy}{dx} = 2ax - 6$$

Stationary point at $x = 1$.

$\therefore$ When $x = 1$, $\dfrac{dy}{dx} = 0$.

Therefore:

$$2ax - 6 = 0$$
$$2a(1) - 6 = 0$$
$$2a - 6 = 0$$
$$2a = 6$$
$$a = 3$$

8   **C**

9   $1 + x + x^2 + x^3 + \dots + x^n$ is a geometric progression with $n + 1$ terms.

When $k = 1$, $x^k = x^1$
$$= x$$

When $k = 1$, $x^{k-1} = x^{1-1}$
$$= x^0$$
$$= 1$$

$\therefore\ 1 + x + x^2 + x^3 + \dots + x^n = \displaystyle\sum_{k=1}^{n+1} x^{k-1}$

9   **D**

$y = |x| - x$
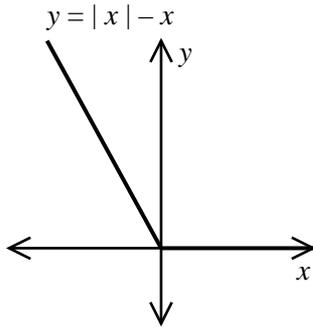


When $x \geq 0$, $y = x - x$
$\qquad = 0$
The range of this section is $y = 0$.
When $x < 0$, $y = -x - x$
$\qquad = -2x$
The range of this section is $y > 0$.
$\therefore$ Range of function is $y \geq 0$.

## SECTION II

## QUESTION 11

(a)

$$n + \frac{1}{n} = \frac{n^2 + 1}{n}$$
$$= \frac{(2 - \sqrt{3})^2 + 1}{2 - \sqrt{3}}$$
$$= \frac{4 - 4\sqrt{3} + 3 + 1}{2 - \sqrt{3}}$$
$$= \frac{8 - 4\sqrt{3}}{2 - \sqrt{3}}$$
$$= \frac{4(2 - \sqrt{3})}{2 - \sqrt{3}}$$
$$= 4$$

(b)  (i)  $y = (x + 1)^2$

$\qquad \dfrac{dy}{dx} = 2.1.(x + 1)^1$

$\qquad\qquad = 2(x + 1)$
$\qquad\qquad = 2x + 2$

(ii)  $y = xe^{2x}$

$\qquad \dfrac{dy}{dx} = x.2e^{2x} + e^{2x}.1$

$\qquad\qquad = e^{2x}(2x + 1)$

(iii) $y = \ln \dfrac{x}{2x+1}$

$= \ln x - \ln (2x + 1)$

$\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{2}{2x+1}$

$= \dfrac{2x+1}{x(2x+1)} - \dfrac{2x}{x(2x+1)}$

$= \dfrac{2x+1-2x}{x(2x+1)}$

$= \dfrac{1}{x(2x+1)}$

**or**

$y = \ln \dfrac{x}{2x+1}$

Let $f(x) = \dfrac{x}{2x+1}$

$f'(x) = \dfrac{(2x+1).1 - x(2)}{(2x+1)^2}$

$= \dfrac{2x+1-2x}{(2x+1)^2}$

$= \dfrac{1}{(2x+1)^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

$= \dfrac{1}{(2x+1)^2} \div \dfrac{x}{2x+1}$

$= \dfrac{1}{(2x+1)^2} \times \dfrac{2x+1}{x}$

$= \dfrac{1}{x(2x+1)}$

(c)

$\dfrac{x+1}{x(x-1)} - \dfrac{x-1}{x(x+1)} = \dfrac{(x+1)(x+1)}{x(x-1)(x+1)} - \dfrac{(x-1)(x-1)}{x(x+1)(x-1)}$

$= \dfrac{(x+1)(x+1) - (x-1)(x-1)}{x(x-1)(x+1)}$

$= \dfrac{(x^2 + 2x+1) - (x^2 - 2x+1)}{x(x-1)(x+1)}$

$= \dfrac{x^2 + 2x+1 - x^2 + 2x-1}{x(x-1)(x+1)}$

$= \dfrac{4x}{x(x-1)(x+1)}$

$= \dfrac{4}{(x-1)(x+1)}$

4

(d)

$$\int_0^1 1 + e^{2x} = [x + \tfrac{1}{2}e^{2x}]_0^1$$
$$= [1 + \tfrac{1}{2}e^2] - [0 + \tfrac{1}{2}e^0]$$
$$= 1 + \tfrac{1}{2}e^2 - \tfrac{1}{2}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}e^2$$
$$= \tfrac{1}{2}(1 + e^2)$$
$$= \frac{1 + e^2}{2}$$

## QUESTION 12

(a)

| $x$ | $f(x)$ | $W$ | $P$ |
|-----|--------|-----|-----|
| 0 | 1.0000 | 1 | 1.0000 |
| $\frac{\pi}{6}$ | 0.8660 | 2 | 1.7321 |
| $\frac{\pi}{3}$ | 0.5000 | 2 | 1.0000 |
| $\frac{\pi}{2}$ | 0.0000 | 1 | 0.0000 |
| | | | 3.7321 |

$$\int_0^{\frac{\pi}{2}} \cos x \; dx \approx \tfrac{1}{2} \times h \times \text{Sum}$$

$$\approx \tfrac{1}{2} \times \tfrac{\pi}{6} \times 3.7321$$

$$\approx 0.9771$$

(b)   Domain: $\{x: x > -1\}$
Range: $\{y: y \in \mathbb{R}\}$

(c)   (i)
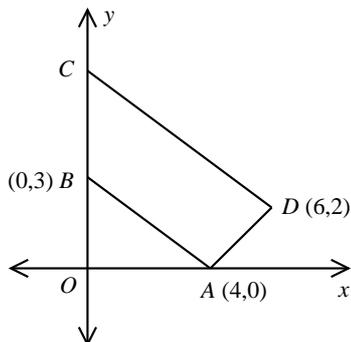


(ii)   $AB$ is parallel to $CD$           (given)
    $\therefore$ $ABCD$ is a trapezium     (one pair of opposite sides parallel)

(iii)   $m_{AB} = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{3 - 0}{0 - 4}$$

$$= -\tfrac{3}{4}$$

5

(iv)  Since *CD* is parallel to *AB*, $m_{CD} = -\frac{3}{4}$

Equation of *CD* is:
$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\tfrac{3}{4}(x - 6)$$
$$4(y - 2) = -3(x - 6)$$
$$4y - 8 = -3x + 18$$
$$3x + 4y - 26 = 0$$

(v)   When $x = 0$:
$$3(0) + 4y - 26 = 0$$
$$4y - 26 = 0$$
$$4y = 26$$
$$y = 6\tfrac{1}{2}$$

$\therefore\ C \equiv (0, 6\tfrac{1}{2})$

(vi)

$$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (4 - 0)^2 + (0 - 3)^2$$
$$= 16 + 9$$
$$= 25$$
$$AB = 5 \text{ units}$$

(vii)

$$CD^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (6 - 0)^2 + (2 - 6\tfrac{1}{2})^2$$
$$= 36 + 20\tfrac{1}{4}$$
$$= 56\tfrac{1}{4}$$
$$= \tfrac{225}{4}$$
$$CD = \tfrac{15}{2}$$
$$= 7\tfrac{1}{2} \text{ units}$$

(viii)

$$\text{Distance} = \left| \frac{ah + bk + c}{\sqrt{a^2 + b^2}} \right|$$
$$= \left| \frac{3.4 + 4.0 - 26}{\sqrt{3^2 + 4^2}} \right|$$
$$= \left| \frac{-14}{\sqrt{25}} \right|$$
$$= \frac{14}{5}$$
$$= 2\tfrac{4}{5} \text{ units}$$

(ix)

$$\text{Area } ABCD = \tfrac{1}{2}h(a + b)$$
$$= \tfrac{1}{2} \times 2\tfrac{4}{5} \times (5 + 7\tfrac{1}{2})$$
$$= 17\tfrac{1}{2} \text{ square units}$$

6

**QUESTION 13**

(a) (i) Solving simultaneously:
$$\begin{cases} T_5 = 16 \\ T_{11} = 40 \end{cases}$$
$$\begin{cases} a + (5-1)d = 16 \\ a + (11-1)d = 40 \end{cases}$$
$$\begin{cases} a + 4d = 16 \\ a + 10d = 40 \end{cases}$$
$$6d = 24$$
$$d = 4$$
Substituting:
$$a + 4(4) = 16$$
$$a = 0$$
$\therefore$ First term = 0, common difference = 4

(ii) We need $S_n = 312$. Therefore:
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$312 = \frac{n}{2}[2.0 + (n-1).4]$$
$$312 = \frac{n}{2}[4n - 4]$$
$$624 = n[4n - 4]$$
$$624 = 4n^2 - 4n$$
$$4n^2 - 4n - 624 = 0$$
$$n^2 - n - 156 = 0$$
$$(n-13)(n+12) = 0$$
$\therefore n = 13$ or $-12$
Since the term number must be positive we have $n = 13$.
$\therefore$ 13 terms in the sequence will add to a sum of 312.

(b) $y = \ln x^2$
$$\frac{dy}{dx} = \frac{2x}{x^2}$$
$$= \frac{2}{x}$$
When $x = 1$, $y = \ln 1^2$
$$= \ln 1$$
$$= 0$$
When $x = 1$, $\frac{dy}{dx} = \frac{2}{1}$
$$= 2$$
$\therefore$ Gradient of tangent to curve at (1,0) is 2.
$\therefore$ Equation of tangent is:
$$y - y_1 = m(x - x_1)$$
$$y - 0 = 2(x - 1)$$
$$y = 2x - 2$$
$$2x - y - 2 = 0$$

7

(c) (i) $y = 2 + 3x - x^3$

$$\frac{dy}{dx} = 3 - 3x^2$$

$$\frac{d^2y}{dx^2} = -6x$$

(ii) For turning points, $\frac{dy}{dx} = 0$.

$$3 - 3x^2 = 0$$
$$3(1 - x^2) = 0$$
$$3(1 + x)(1 - x) = 0$$

$1 + x = 0$ or $1 - x = 0$

$x = -1$ or $1$

When $x = -1$, $y = 2 + 3(-1) - (-1)^3$
$$= 2 - 3 + 1$$
$$= 0$$

When $x = -1$, $\frac{d^2y}{dx^2} = -6(-1)$

$$= 6$$
$$> 0$$

$\therefore$ Minimum turning point at $(-1,0)$

When $x = 1$, $y = 2 + 3(1) - (1)^3$
$$= 2 + 3 - 1$$
$$= 4$$

When $x = -1$, $\frac{d^2y}{dx^2} = -6(1)$

$$= -6$$
$$< 0$$

$\therefore$ Maximum turning point at $(1,4)$

For points of inflection, $\frac{d^2y}{dx^2} = 0$.

$$-6x = 0$$
$$x = 0$$

When $x = 0$, $y = 2 + 3(0) - (0)^3$
$$= 2$$

When $x = -0.1$, $\frac{d^2y}{dx^2} = -6(-0.1) > 0$

When $x = 0.1$, $\frac{d^2y}{dx^2} = -6(0.1) < 0$

$\therefore$ Point of inflection at $(0,2)$

(iii) Curve is concave up when $\frac{d^2y}{dx^2} > 0$.

$$-6x > 0$$
$$x < 0$$

$\therefore$ Curve is concave up when $x < 0$.

8

(iv)
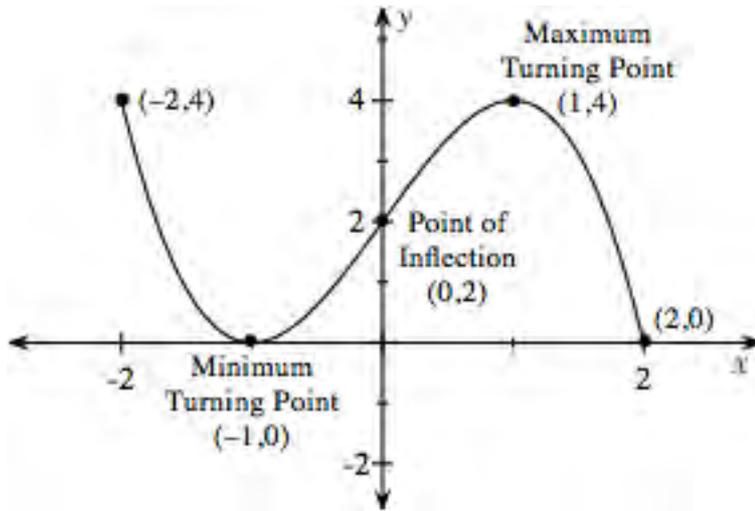


**QUESTION 14**

(a) (i)
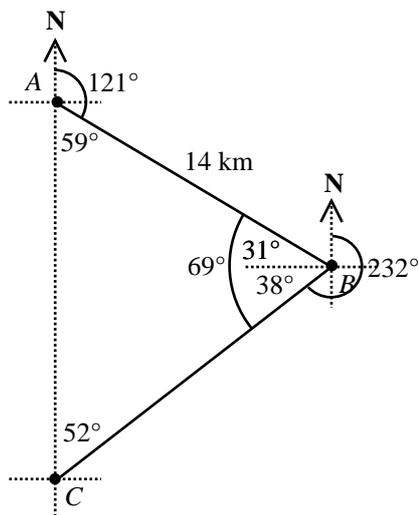


(ii) Using the sine rule:

$$\frac{AC}{\sin 69°} = \frac{14}{\sin 52°}$$

$$AC = \frac{14 \sin 69°}{\sin 52°}$$

$$= 16.58622793$$

$$= 16.6 \text{ km}$$

∴ The bushwalker is 16.6 km south from her starting point.

9

(b)  (i)  Amplitude = 2

Period $= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{2}$

$= \pi$

(ii), (iii)
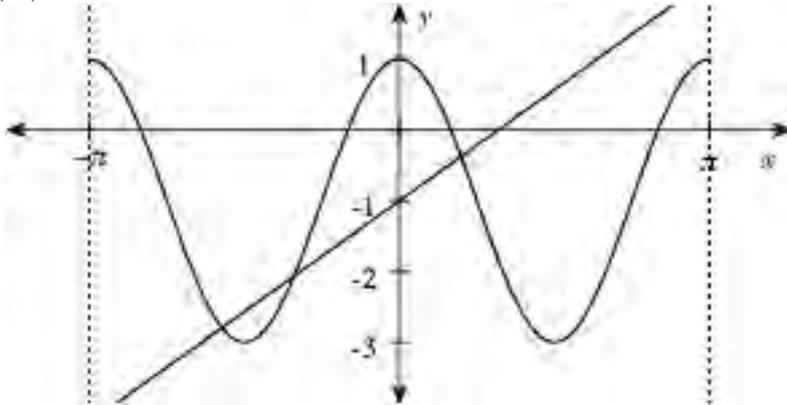


(iv)  Solving simultaneously:

$2\cos 2x - 1 = x - 1$

$2\cos 2x = x$

$2\cos 2x - x = 0$

$\therefore$ There are 3 solutions to the equation $2\cos 2x - x = 0$.

(c)
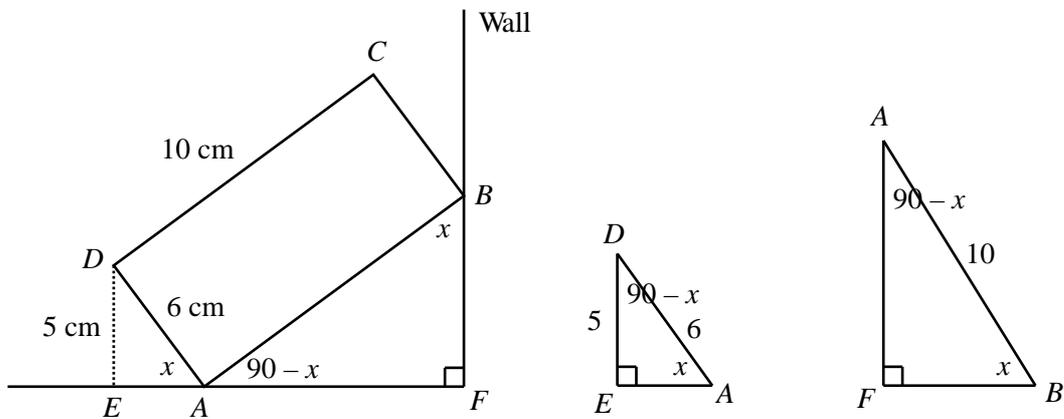


(i)  Let $x = \angle DAE$

| | | |
|---|---|---|
| $\angle BAF = 180 - 90 - x$ | | (supplementary angles) |
| $= 90 - x$ | | |
| $\angle ABF = 180 - 90 - (90 - x)$ | | (angle sum of triangle) |
| $= 180 - 90 - 90 + x$ | | |
| $= x$ | | |
| $\therefore \angle DAE = \angle ABF = x$ | | (as shown) |
| $\angle DEA = \angle AFB$ | | (given) |
| $\therefore \triangle EAD$ is similar to $\triangle FBA$ | | (two pairs of corresponding angles equal) |

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(ii)   Since the triangles are similar, we have:

$$\frac{AF}{DE} = \frac{AB}{DA}$$

$$\frac{AF}{5} = \frac{10}{6}$$

$$AF = \tfrac{25}{3}$$

Using Pythagoras' theorem:

$$BF^2 = AB^2 - AF^2$$

$$= 10^2 - \left(\tfrac{25}{3}\right)^2$$

$$= 100 - \tfrac{625}{9}$$

$$= \tfrac{275}{9}$$

$$BF = \sqrt{\tfrac{275}{9}}$$

$$= \tfrac{\sqrt{275}}{3}$$

$$= \tfrac{5\sqrt{11}}{3}$$

∴ The block touches the wall at a height of $\frac{5\sqrt{11}}{3}$ cm above the floor.

## QUESTION 15

(a)  (i)   The curve is stationary when $x = b$ and $x = d$.
     (ii)   The curve has a maximum turning point when $x = d$.
     (iii)  Using the graph of $y = f'(x)$, we can see that $f'(x) = 0$ when $x = a$ and $x = c$.
            Therefore turning points exist at $x = a$ and $x = c$.
            Since $f'(x)$ changes from +ve to –ve at $x = a$, the turning point at $x = a$ is maximum.
            Since $f'(x)$ changes from –ve to +ve at $x = c$, the turning point at $x = c$ is minimum.
            Turning points of the graph $y = f'(x)$ indicate points of inflection on the graph of $y = f(x)$.
            Therefore points of inflection exists at $x = b$ and $x = d$.
            As $x \to \infty$, the gradient of $f(x)$ remains positive, but approaches zero.
            This means the curve for $f(x)$ is increasing but flattens out.

A possible graph of $y = f(x)$ is shown.



(b)   $y = \dfrac{1}{\sqrt{x+1}}$

$$y^2 = \left(\dfrac{1}{\sqrt{x+1}}\right)^2$$

$$= \dfrac{1}{x+1}$$
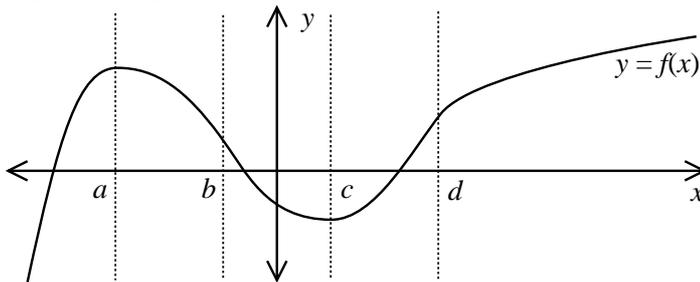
Therefore:

$$\text{Volume} = \pi \int_0^3 y^2 \, dx$$

$$= \pi \int_0^3 \frac{1}{x+1} \, dx$$

$$= \pi \left[ \ln (x+1) \right]_0^3$$

$$= \pi[\ln (3+1)] - \pi[\ln (0+1)]$$

$$= \pi \ln 4 - \pi \ln 1$$

$$= \pi \ln 4 - \pi \times 0$$

$$= \pi \ln 4 \text{ cubic units}$$

(c) (i) Equation of $OP$:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 0}{x - 0} = \frac{\frac{1}{e} - 0}{1 - 0}$$

$$\frac{y}{x} = \frac{\frac{1}{e}}{1}$$

$$\frac{y}{x} = \frac{1}{e}$$

$$ey = x$$

$$y = \frac{x}{e}$$

**or**

Gradient of $OP = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{\frac{1}{e} - 0}{1 - 0}$$

$$= \frac{1}{e}$$

Equation of $OP$:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{1}{e}(x - 0)$$

$$y = \frac{x}{e}$$

(ii) Let $A$ be the point $(1,0)$.
Now in $\Delta PAO$:

$$\text{Area} = \tfrac{1}{2}bh$$

$$= \tfrac{1}{2} \times 1 \times \frac{1}{e}$$

$$= \frac{1}{2e}$$

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**or**

$$\text{Area} = \int_0^1 \frac{x}{e}\, dx$$

$$= \frac{1}{e} \int_0^1 x\, dx$$

$$= \frac{1}{e} [\tfrac{1}{2} x^2]_0^1$$

$$= \frac{1}{e} [\tfrac{1}{2}(1)^2] - \frac{1}{e} [\tfrac{1}{2}(0)^2]$$

$$= \frac{1}{e} [\tfrac{1}{2}]$$

$$= \frac{1}{2e}$$

And under the curve $y = e^{-x}$:

$$\text{Area} = \int_0^1 e^{-x}\, dx$$

$$= \left[-e^{-x}\right]_0^1$$

$$= [-e^{-1}] - [-e^0]$$

$$= -\frac{1}{e} + 1$$

$$= 1 - \frac{1}{e}$$

Therefore:

$$\text{Shaded area} = \left(1 - \frac{1}{e}\right) - \frac{1}{2e}$$

$$= \frac{2e}{2e} - \frac{2}{2e} - \frac{1}{2e}$$

$$= \frac{2e - 3}{2e} \text{ square units}$$

## QUESTION 16

(a)

$$2\sin^2\theta - \cos\theta + 1 = 0$$
$$2(1 - \cos^2\theta) - \cos\theta + 1 = 0$$
$$2 - 2\cos^2\theta - \cos\theta + 1 = 0$$
$$-2\cos^2\theta - \cos\theta + 3 = 0$$
$$2\cos^2\theta + \cos\theta - 3 = 0$$
$$(2\cos\theta + 3)(\cos\theta - 1) = 0$$

Therefore:

$$2\cos\theta + 3 = 0$$
$$2\cos\theta = -3$$
$$\cos\theta = -\tfrac{3}{2}$$

which has no solution.

or:

$$\cos\theta - 1 = 0$$
$$\cos\theta = 1$$
$$\theta = 0° \text{ or } 360°$$
$$= 0 \text{ or } 2\pi$$

$\therefore$ Solution to the equation is $\theta = 0$ or $2\pi$.

(b) (i) Perimeter = 30 cm. Therefore:

$$x + y + x + y + x = 30$$
$$3x + 2y = 30$$
$$2y = 30 - 3x$$
$$y = \frac{30 - 3x}{2}$$
$$= 15 - \frac{3x}{2}$$

(ii) Using the area rule:

$$\text{Area of } \Delta ABE = \tfrac{1}{2} \times x \times x \times \sin 60°$$
$$= \frac{x^2}{2} \times \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}x^2}{4}$$

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**or**

Using Pythagoras' theorem:

Let $h$ be the perpendicular height of $\Delta ABE$. Therefore:

$$x^2 = h^2 + \left(\frac{x}{2}\right)^2$$

$$x^2 = h^2 + \frac{x^2}{4}$$

$$4x^2 = 4h^2 + x^2$$

$$3x^2 = 4h^2$$

$$4h^2 = 3x^2$$

$$h^2 = \frac{3x^2}{4}$$

$$h = \frac{\sqrt{3}x}{2}$$

Therefore:

$$\text{Area of } \Delta ABE = \tfrac{1}{2}bh$$

$$= \frac{1}{2} \times x \times \frac{\sqrt{3}x}{2}$$

$$= \frac{\sqrt{3}x^2}{4}$$

(iii)

$$\text{Area } BEDC = lb$$

$$= xy$$

$$= x \times \left(15 - \frac{3x}{2}\right)$$

$$= 15x - \frac{3x^2}{2}$$

$$\text{Area } AEDCB = \text{Area } \Delta AEB + \text{Area } BEDC$$

$$= \frac{\sqrt{3}x^2}{4} + 15x - \frac{3x^2}{2}$$

$$= \frac{\sqrt{3}x^2}{4} + \frac{60x}{4} - \frac{6x^2}{4}$$

$$= \frac{60x}{4} + \frac{\sqrt{3}x^2}{4} - \frac{6x^2}{4}$$

$$= 15x + \frac{(\sqrt{3}-6)x^2}{4}$$

(iv) $A = 15x + \dfrac{(\sqrt{3}-6)x^2}{4}$

$$\frac{dA}{dx} = 15 + \frac{(\sqrt{3}-6)x}{2}$$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{3}-6}{2} < 0$$

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For maximum area, we need $\dfrac{dA}{dx} = 0$ and $\dfrac{d^2A}{dx^2} < 0$. Therefore:

$$15 + \frac{(\sqrt{3}-6)x}{2} = 0$$
$$\frac{(\sqrt{3}-6)x}{2} = -15$$
$$(\sqrt{3}-6)x = -30$$
$$x = \frac{-30}{\sqrt{3}-6}$$
$$= \frac{30}{6-\sqrt{3}}$$
$$= \frac{30}{6-\sqrt{3}} \times \frac{6+\sqrt{3}}{6+\sqrt{3}}$$
$$= \frac{30(6+\sqrt{3})}{6^2-(\sqrt{3})^2}$$
$$= \frac{30(6+\sqrt{3})}{36-3}$$
$$= \frac{30(6+\sqrt{3})}{33}$$
$$= \frac{10(6+\sqrt{3})}{11}$$

Since we know that $\dfrac{d^2A}{dx^2} < 0$, we have a maximum area when $x = \dfrac{10(6+\sqrt{3})}{11}$.

(c)   Interest rate $= 3\%$ p.a.
$= 0.25\%$ per month
$= 0.0025$ per month

Let $A_n =$ the final value of each amount invested.

The first \$150 is invested at 0.25% p.a. for 60 months.

The second \$150 is invested at 0.25% p.a. for 59 months.

The third \$150 is invested at 0.25% p.a. for 58 months.

The final \$150 is invested at 0.25% p.a. for 1 month.

Now:

$A_1 = 150(1 + 0.0025)^{60} = 150(1.0025)^{60}$

$A_2 = 150(1 + 0.0025)^{59} = 150(1.0025)^{59}$

$A_3 = 150(1 + 0.0025)^{58} = 150(1.0025)^{58}$

$A_{60} = 150(1 + 0.0025)^{1} = 150(1.0025)^{1}$

Therefore:

$\text{Total} = A_1 + A_2 + A_3 + ... + A_{60}$

$= 150(1.0025)^{15} + 150(1.0025)^{14} + 150(1.0025)^{13} + ... + 150(1.0025)^{1}$

$= 150(1.0025)^{1} + 150(1.0025)^{2} + 150(1.0025)^{3} + ... + 150(1.0025)^{15}$

$= 150[\underbrace{1.0025^{1} + 1.0025^{2} + 1.0025^{3} + ... + 1.0025^{60}}_{\text{GP: } a=1.0025, r=1.0025, n=60}]$

$= 150\left[\dfrac{a(r^{n}-1)}{r-1}\right]$

$= 150\left[\dfrac{1.0025(1.0025^{60}-1)}{1.0025-1}\right]$

$= 150 \times 64.8083294$

$= \$9721.249411$

$\approx \$9721.25$

$\therefore$ Mr Jones will not reach his goal and will fall short by \$278.75.