THE KING'S SCHOOL

2007
Higher School Certificate
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Calculate, correct to three significant figures, $12\tan^2 1$ **2**

(b) Simplify $\dfrac{x+1}{1+\dfrac{1}{x}}$ **2**

(c) Solve $|3x + 4| \leq 5$ **2**

(d) Solve $3^x = 2$ correct to two decimal places. **2**

(e) Differentiate $1 + \dfrac{1}{x}$ **2**

(f) Find a primitive of $(2x + 9)^3$ **2**

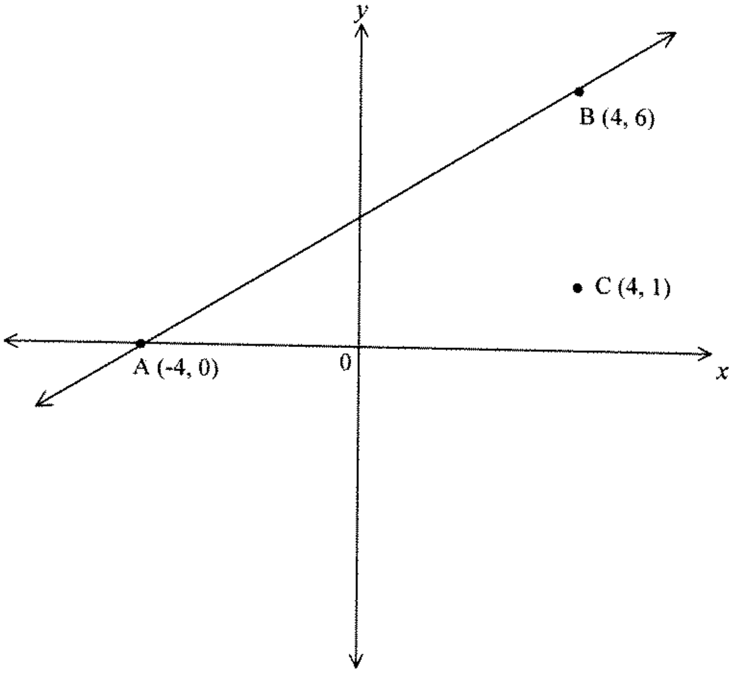
End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve simultaneously

$$y = 4x$$
$$y = 2x^2 - x + 2$$ **3**

(b)



The diagram shows the points $A(-4, 0)$, $B(4, 6)$ and $C(4, 1)$. O is the origin.

(i) Find the gradient of the line AB. **1**

(ii) Deduce that the equation of the line AB is $3x - 4y + 12 = 0$ **2**

(iii) The perpendicular from $C(4, 1)$ meets line AB at D. i.e. $CD \perp AB$ at D.
Find the length of CD. **2**

(iv) Find the length of DB. **1**

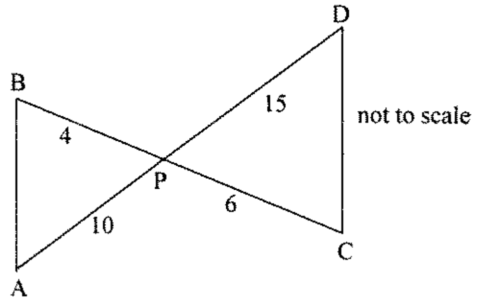
(c) Find the equation of the tangent to the curve $y = x^3 - 2x^2$ at the point $(-1, -3)$. **3**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_0^2 \frac{12x}{x^2+2}\,dx$ expressing your answer in simplest exact form. **3**

(b)



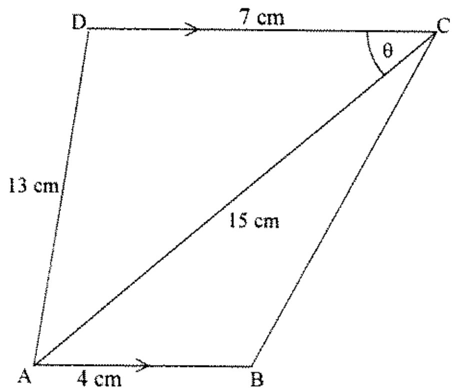
In the diagram, BPC and APD are straight lines.

AP = 10, PD = 15, BP = 4 and PC = 6

(i) Prove Δ ABP is similar to Δ CDP **2**

(ii) Deduce that AB || CD **2**

(c)



The diagram shows a trapezium ABCD where AB || DC and AB = 4 cm, CD = 7 cm, DA = 13 cm, AC = 15 cm.

Let $\angle DCA = \theta$

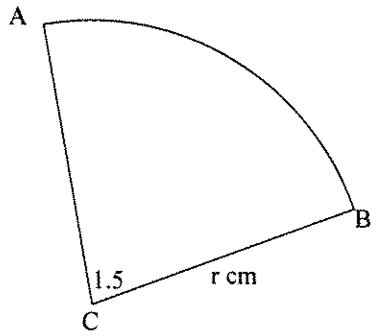
(i) Find θ **2**

(ii) Find the exact area of the trapezium. **3**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



CAB is a sector of a circle with centre C and radius r cm. $\angle ACB = 1.5$ radians.
The perimeter of the sector is 1.4 cm.

(i) Find r. **2**

(ii) Find the area of the sector. **1**

(iii) Find $\angle ACB$ correct to the nearest degree. **1**

(b) 2007 + 2000 + ... is an arithmetic series.

(i) State the common difference. **1**

(ii) Show working to decide whether 12 is a term in the series. **2**

(iii) Find the maximum number of terms for which the sum of the series remains positive. **2**

(c) Solve the equation $79100 \times 1.002^{40} - M\left(1.002^{40} + 1.002^{39} + \ldots + 1.002 + 1\right) = 0$ giving your value for M correct to the nearest integer. **3**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For a particular curve $y = f(x)$, which passes through the point (2, 0), we have $f'(x) = 3x(2-x)$

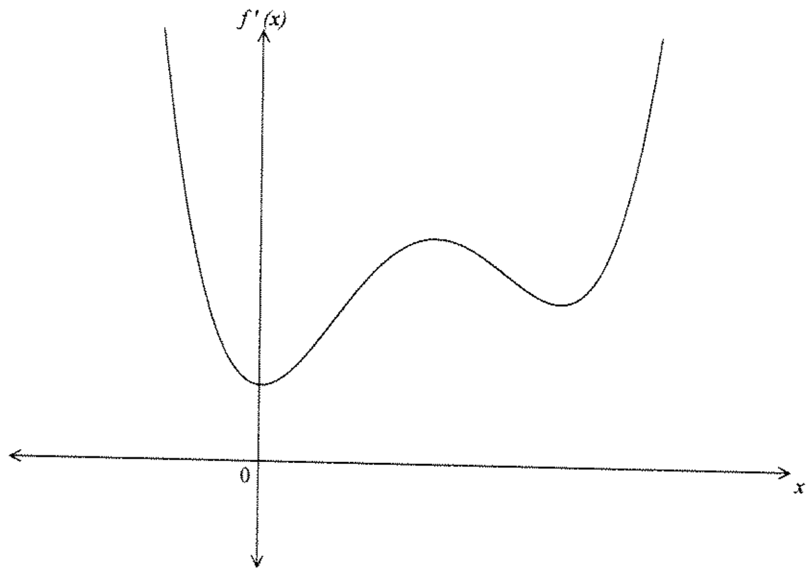
(i) Determine the nature of the stationary points of the curve. **3**

(ii) Prove that $f(-1) = 0$ **3**

(b) (i) Sketch the curve $y = 2\sin \pi x,\ 0 \le x \le 2$ **2**

(ii) Find the area bounded by the curve $y = 2\sin \pi x$ and the x axis from $x = 0$ to $x = 2$ **3**

(c)



The diagram shows a sketch of the gradient function of the curve $y = f(x)$.

How many stationary points are on the curve $y = f(x)$? **1**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) $x^2 - 2Ax + B = 0$ has two different real roots α, β

(i) Show that $A^2 > B$ **2**

(ii) Find the range of values of B if the sum of the roots is equal to the product of the roots. **3**

(b) (i) State the domain of the function $f(x) = \dfrac{1}{1 + \sqrt{x}}$ **1**

(ii) Without using calculus, find the range of the function $f(x) = \dfrac{1}{1 + \sqrt{x}}$ **1**

(iii) Use Simpson's rule with three function values to give a two decimal place approximation to

$$\int_0^1 \frac{1}{1 + \sqrt{x}}\, dx$$

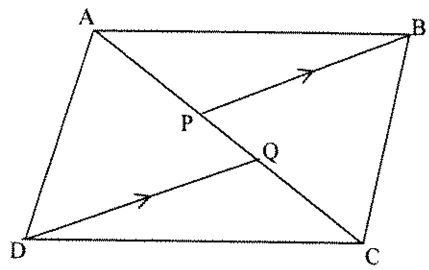
3

(c) Find the focus of the parabola $(x - 1)^2 = 2\left(y + \frac{1}{2}\right)$ **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



In the diagram, ABCD is a parallelogram.

BP and DQ meet the diagonal AC at P and Q, respectively, where $BP \parallel DQ$

(i) Explain why $\angle BPQ = \angle DQP$ **1**

(ii) Prove that ΔABP is congruent to ΔDCQ **3**

(iii) Deduce that $BQ = DP$ **2**

(b) The population P of a town is known to be changing exponentially.
i.e. $P = P_0 e^{kt}$, P_0, k constants, t time ≥ 0

(i) Show that $P = P_0 e^{kt}$ satisfies the equation $\frac{dP}{dt} = kP$ **1**

(ii) At the start of 2001 the population was 25 000 and at the start of 2007 it was 30 000. Prove that the continuous growth rate is approximately 3% p.a. **3**

(c) Simplify $(\sqrt{5} - 2)^4 (\sqrt{5} + 2)^5$ **2**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that $\frac{d}{dx}\left(xe^{-x}\right) = e^{-x} - xe^{-x}$ **1**

(ii) Hence prove that $\int_0^1 xe^{-x}\,dx = 1 - 2e^{-1}$ **2**

(b) The region bounded by the curve $y = x + e^{-x}$ and the x axis from $x = 0$ to $x = 1$ is revolved about the x axis.

Prove that the volume of the solid generated is $\frac{\pi}{6}\left(17 - 24e^{-1} - 3e^{-2}\right)$ **4**

(c) $A(-3, 0)$ and $B(6,0)$ are two points in the number plane. $P(x, y)$ is any point in the plane such that P is twice as far from B as it is from A. i.e. $PB = 2PA$.

(i) Prove that the cartesian equation of the locus of $P(x, y)$ is $x^2 + 12x + y^2 = 0$ **3**

(ii) Describe the locus of P in precise geometrical terms. **2**

End of Question 8

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particle is moving on the x axis. Its position at time t seconds is given by

$$x = t^2(t-6), \quad t \geq 0$$

(i) At what times is the particle at the origin? **1**

(ii) Find expressions for the velocity $\dot{x}$ and the acceleration $\ddot{x}$ **2**

(iii) In what direction is the particle moving at $t = 2$? **1**

(iv) For what values of t is the velocity increasing? **1**

(v) Find the total distance travelled during the first six seconds of the motion. **2**

(b) A rainwater tank is initially empty. The rate, R *L/s*, at which water is entering the tank is given by

$$R = 1 - \frac{1}{\sqrt{2t+1}}, \quad t \geq 0 \text{ is the time in seconds}$$

(i) Find the rate at which the tank is filling after one minute. **1**

(ii) The tank is full to its capacity after 66 minutes. Explain why the capacity of the tank is less than 3960 L. **1**

(iii) Determine the capacity of the tank. **3**

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Sketch the curve $y = \ln(1 - 2x)$ **2**

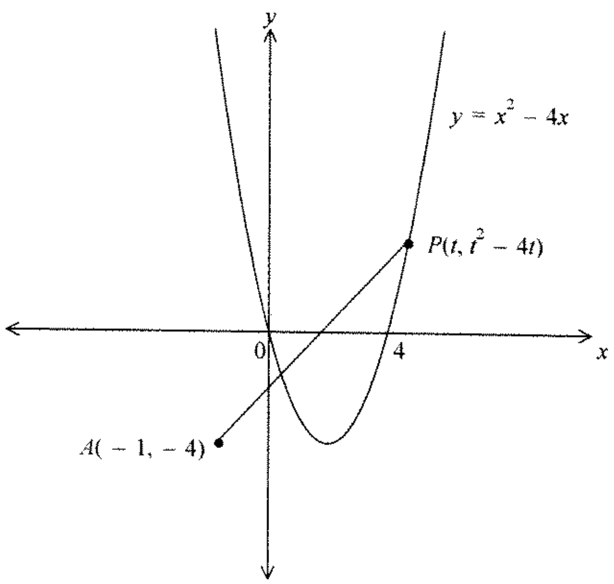
(b) $m \sin x = n \cos x$ where $m,\ n > 0$ and $0 < x < \frac{\pi}{2}$

Prove that $\sin x \cos x = \dfrac{mn}{m^2 + n^2}$ **3**

(c) Let $f(t) = 2(t-2)^3 + t + 1$ where $f(1) = 0$

By considering $f'(t)$ or otherwise deduce that $f(t) = 0$ only if $t = 1$ **2**

(d)



The sketch shows the parabola $y = x^2 - 4x$ and the point $A(-1, -4)$

Let $P(t, t^2 - 4t)$ be any point on the parabola and let $AP^2 = l$

(i) Show that $l = (t+1)^2 + (t-2)^4$ **2**

(ii) Hence or otherwise find the minimum length of AP. **3**

End of Examination

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

TKS MATHEMATICS TRIAL 2007 SOLUTIONS

Qn 1

(a) 29.1 [PAY 1 for 0.00366]

(b) $= \dfrac{x(x+1)}{x+1} = x$

(c) $-5 \le 3x+4 \le 5$

$\therefore \quad -9 \le 3x \le 1 \Rightarrow -3 \le x \le \frac{1}{3}$

(d) $\ln 3^x = \ln 2$

$\therefore \; x \ln 3 = \ln 2 \Longrightarrow x = \dfrac{\ln 2}{\ln 3} = 0.63$, 2 d.p.

(e) $y = 1 + x^{-1}$

$\therefore \; \dfrac{dy}{dx} = -x^{-2} = -\dfrac{1}{x^2}$

(f) $\dfrac{(2x+9)^4}{4 \times 2} = \dfrac{(2x+9)^4}{8}$

Question 2

(a) $\therefore \quad 0 = 2x^2 - 5x + 2$

$\Rightarrow (2x - 1)(x - 2) = 0$

$$\begin{cases} x = \frac{1}{2} \text{ or } 2 \\ y = 2 \text{ or } 8 \end{cases} \qquad \text{i.e. } \left(\tfrac{1}{2}, 2\right) \text{ and } (2, 8)$$

(b) (i) grd $AB = \dfrac{6}{8} = \dfrac{3}{4}$

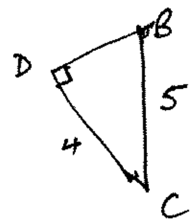
(ii) AB is $y - 0 = \frac{3}{4}(x + 4)$

$\therefore \quad 4y = 3x + 12$

i.e. $3x - 4y + 12 = 0$

(iii) $CD = \dfrac{12 - 4 + 12}{\sqrt{3^2 + 4^2}} = \dfrac{20}{5} = 4$

(iv) we have



$\therefore DB = 3$

(c) $y' = 3x^2 - 4x = 3 + 4$ at $x = -1$

$= 7$

$\therefore$ tangent is $y + 3 = 7(x + 1)$ will do

i.e. $y = 7x + 4$

Question 3

(a) $I = 6\int_0^2 \frac{2x}{x^2+2}\,dx$

$= 6\left[\ln(x^2+2)\right]_0^2 = 6(\ln 6 - \ln 2) = 6\ln 3$

(b) (i) $\angle BPA = \angle DPC$, vertically opposite

$\frac{PB}{PC} = \frac{PA}{PD} = \frac{2}{3}$

$\therefore \triangle ABP \ ||| \ \triangle CDP$, sim. $\triangle$ test with included angle

(ii) $\angle A = \angle D$, $\triangle$s similar

But these are alternate angles

$\therefore AB \parallel CD$

(c) (i) $\cos\theta = \frac{7^2+15^2-13^2}{2\times 7\times 15} = \frac{105}{210} = \frac{1}{2}$

$\therefore \theta = 60^\circ$

(ii) Area $= \frac{1}{2}\cdot 7.15\sin 60^\circ + \frac{1}{2}\cdot 4.15.\sin 60^\circ$, $\angle CAB = \theta$

$= \frac{1}{2}\cdot 11.\ 15\sin 60^\circ$

$= \frac{1}{2}.\ 11.\ 15.\ \frac{\sqrt{3}}{2} = \frac{165\sqrt{3}}{4}$ cm^2

Question 4

(a) (i) arc $AB = 1.5r$

$\therefore \quad 1.5r + 2r = 1.4$

$3.5r = 1.4 \Rightarrow r = \frac{1.4}{3.5} = 0.4$

(ii) Area $= \frac{1}{2}(0.4)^2(1.5)\ cm^2 = 0.12\ cm^2$

(iii) $1.5^c = 1.5 \times \frac{180}{\pi}^{\circ} = 86^{\circ}$, nearest degree

(b) (i) $d = 2000 - 2007 = -7$

(ii) $T_n = 2007 - 7(n-1) = 12$

$\therefore 7(n-1) = 1995$

$n - 1 = 285 \quad$ or $n = 286$

Yes. 12 is the 286^{th} term

(iii) We need $\frac{n}{2}\left(4014 - 7(n-1)\right) > 0$

$\Rightarrow \quad 4014 - 7(n-1) > 0$

$\therefore \quad 7(n-1) < 4014$

$n - 1 < \frac{4014}{7} = 573.4\ldots\ldots$

ie $\quad n < 574.4\ldots\ldots$

$\Rightarrow$ max. no. of terms is 574

(c) $\therefore \quad 79100 \times 1.002^{40} - \frac{M(1.002^{41} - 1)}{1.002 - 1} = 0$

$\therefore \quad M = \frac{79100 \times 1.002^{40} \times 0.002}{1.002^{41} - 1} = 2007$, nearest integer

Question 5

(a) (i) $f'(x) = 0 \Rightarrow 3x(2-x) = 0$

$\therefore x = 0,\ 2$

$f'(x) = 6x - 3x^2$

$f''(x) = 6 - 6x$ & $f''(0) > 0 \Rightarrow \cup$

& $f''(2) < 0 \Rightarrow \cap$

$\therefore$ at $x=0$ there's a minimum turning point

& at $x = 2$ there's a maximum turning point

(ii) Since $f'(x) = 6x - 3x^2$

then $f(x) = \frac{6x^2}{2} - \frac{3x^3}{3} + c = 3x^2 - x^3 + c$

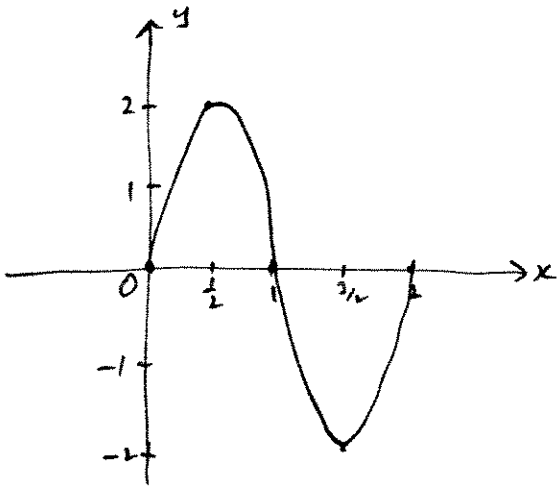
$\therefore f(2) = 12 - 8 + c = 0 \quad \therefore c = -4$

$\therefore f(x) = 3x^2 - x^3 - 4$

$\therefore f(-1) = 3 + 1 - 4 = 0$

(b)

(i)



(ii) $A = 2\int_0^1 2\sin \pi x \, dx = \frac{4}{\pi}\left[-\cos \pi x\right]_0^1$

$= \frac{4}{\pi}(1+1) = \frac{8}{\pi}\ u^2$

(c) 0 since $f'(x) > 0$ for all x

Question 6

(a) (i) $\therefore \Delta > 0 \Rightarrow 4A^2 - 4B > 0$

or $A^2 > B$

(ii) We have $2A = B$

$\therefore 4A^2 = B^2$

$\therefore$ from (i), $B^2 > 4B$

$\Rightarrow B(B-4) > 0 \Rightarrow B < 0$ or $B > 4$

(b) (i) $x \geqslant 0$

(ii) greatest value when $x = 0$ giving $y = 1$

$\Rightarrow$ range is $0 < y \leq 1$

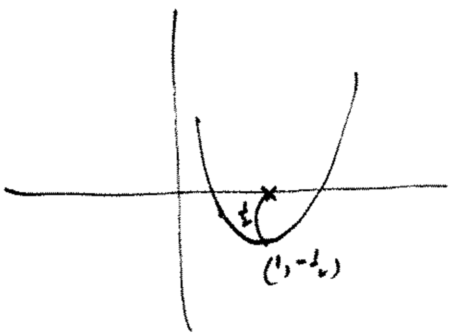
(iii) $I \simeq \frac{1}{6} \cdot 1 \left[1 + \frac{1}{2} + 4 \times \frac{1}{1 + \sqrt{0.5}} \right]$

$= 0.64$, 2 d.p.

(c) Vertex $= (1, -\frac{1}{2})$, $4a = 2$

$a = \frac{1}{2}$

$\Rightarrow$

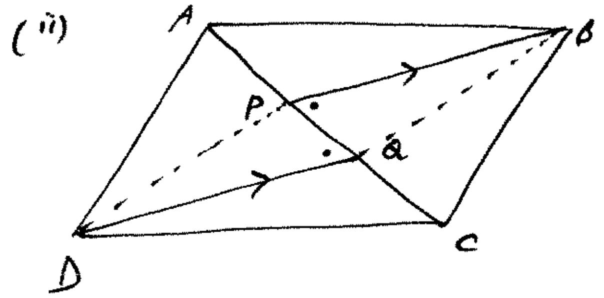


$\Rightarrow$ focus is $(1, 0)$

Question 7

(a) (i) alternate angles in $\parallel$ lines

(ii)



In Δs ABP, DCQ

$\angle APB = \angle CQD$, both supplements of equal angles in (i)

$\angle BAP = \angle DCQ$, alt. angles in $\parallel$ lines AB, DC

$AB = DC$, opp. sides of $\parallel$ogram

$\therefore \Delta ABP \equiv \Delta DCQ$, AAS

(iii) Join BQ, DP

From (ii), $BP = DQ$

and $BP \parallel DQ$, given

$\therefore$ $BPDQ$ is a $\parallel$ogram [one pair of opp sides equal and parallel]

$\therefore BQ = DP$, opp. sides of $\parallel$ogram

(b) (i) $\dfrac{dP}{dt} = P_0 e^{kt} \times k = k\left(P_0 e^{kt}\right) = kP$

(ii) 2001, $t=0$, $P = 25\,000$

2007, $t=6$, $P = 30\,000$

$\therefore P = 25000\, e^{kt}$

and $30\,000 = 25000\, e^{6k}$

$\therefore e^{6k} = \dfrac{30}{25} \Rightarrow 6k = \ln\left(\frac{6}{5}\right)$ i.e. $k = \frac{1}{6}\ln\left(\frac{6}{5}\right)$

$\simeq 0.0303\ldots$

i.e. growth rate $\simeq 3\%$ p.a.

(c) $= \left((\sqrt{5}-2)(\sqrt{5}+2)\right)^4(\sqrt{5}+2) = (5-4)^4(\sqrt{5}+2) = \sqrt{5}+2$

Question 8

(a) (i) $\frac{d(xe^{-x})}{dx} = e^{-x}(1) + x(-e^{-x})$

$= e^{-x} - xe^{-x}$

(ii) From (i), $xe^{-x} = e^{-x} - \frac{d(xe^{-x})}{dx}$

$\therefore \int_0^1 xe^{-x}\,dx = \left[-e^{-x} - xe^{-x}\right]_0^1$

$= -e^{-1} - e^{-1} - (-1-0)$

$= 1 - 2e^{-1}$

(b) $V = \pi\int_0^1 (x+e^{-x})^2\,dx$

$= \pi\int_0^1 x^2 + 2xe^{-x} + e^{-2x}\,dx$

$= \pi\int_0^1 x^2 + e^{-2x}\,dx + 2\pi(1-2e^{-1})$ from (a)(ii)

$= \pi\left[\frac{x^3}{3} - \frac{1}{2}e^{-2x}\right]_0^1 + 2\pi(1-2e^{-1})$

$= \pi\left(\frac{1}{3} - \frac{1}{2}e^{-2} - (0 - \frac{1}{2})\right) + 2\pi(1-2e^{-1})$

$= \pi\left(\frac{5}{6} - \frac{1}{2}e^{-2}\right) + 2\pi(1-2e^{-1})$

$= \frac{\pi}{6}\left(5 - 3e^{-2} + 12 - 24e^{-1}\right)$

$= \frac{\pi}{6}\left(17 - 24e^{-1} - 3e^{-2}\right)$

(c) (i) $PB = 2PA \quad \therefore \; PB^2 = 4 \times PA^2$

$$\Rightarrow \quad (x-6)^2 + y^2 = 4\left((x+3)^2 + y^2\right)$$

$$x^2 - 12x + 36 + y^2 = 4x^2 + 24x + 36 + 4y^2$$

or $\quad 3x^2 + 3y^2 + 36x = 0$

i.e. $\quad x^2 + 12x + y^2 = 0$

(ii) Locus is $\quad (x+6)^2 + y^2 = 36$

i.e. a circle, centre $(-6, 0)$, radius 6

Question 9

(a) (i) $x = 0 \Rightarrow t^2(t-6) = 0$ i.e. at $t = 0, 6$

(ii) $x = t^3 - 6t^2$

$\therefore \dot{x} = 3t^2 - 12t$ and $\ddot{x} = 6t - 12$

(iii) $t = 2$, $\dot{x} = 12 - 24 < 0$

$\therefore$ moving from right to left

i.e. negative direction

(iv) v is increasing when $\ddot{x} > 0$

$\Rightarrow 6t - 12 > 0$ or $\underline{t > 2}$

(v) $v = 0$, $3t(t-4) = 0$, $t = 0, 4$

when $t = 4$, $x = 16(-2) = -32$

$[t = 0, x = 0 \quad ; \quad t = 6, x = 0]$

$\therefore$ distance travelled $= 32 \times 2 = 64$

(b) (i) $t = 60$, $R = 1 - \dfrac{1}{\sqrt{121}}$ L/s $= \dfrac{10}{11}$ L/s

(ii) Clearly $R < 1$ for all t

$\therefore$ Capacity $< 66 \times 60$ L $= 3960$ L

(iii) $\dfrac{dV}{dt} = 1 - (2t+1)^{-\frac{1}{2}}$

$\therefore V = t - 2\dfrac{(2t+1)^{\frac{1}{2}}}{2} + c$ i.e. $V = t - \sqrt{2t+1} + c$ $[t=0, V=0]$

$: 0 = 0 - 1 + c, c = 1$

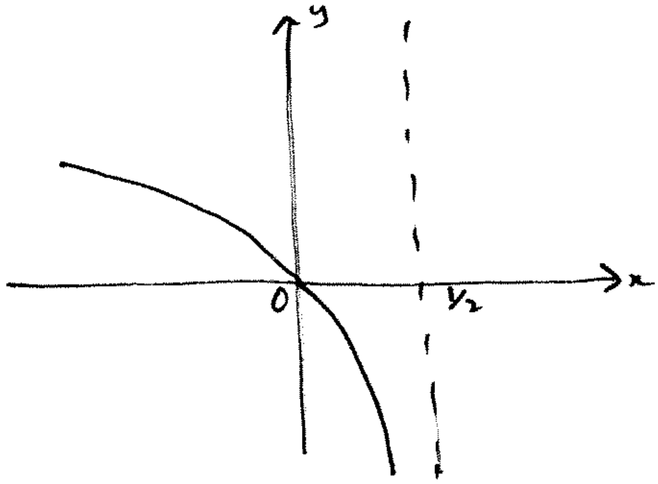
$\therefore V = t - \sqrt{2t+1} + 1$ [$t = 66$ min, it's full]

$\therefore$ capacity $= 66 \times 60 - \sqrt{7921} + 1 = 3872$ L

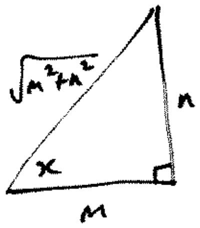
Question 10

(a) $1 - 2x > 0 \Rightarrow 2x < 1$ or $x < \frac{1}{2}$

$x = 0, \ y = 0$



(b) $\therefore \ m \tan x = n$ or $\tan x = \frac{n}{m}$

$\Rightarrow$ 

$\therefore \ \sin x \cos x = \dfrac{n}{\sqrt{m^2+n^2}} \cdot \dfrac{m}{\sqrt{m^2+n^2}} = \dfrac{mn}{m^2+n^2}$

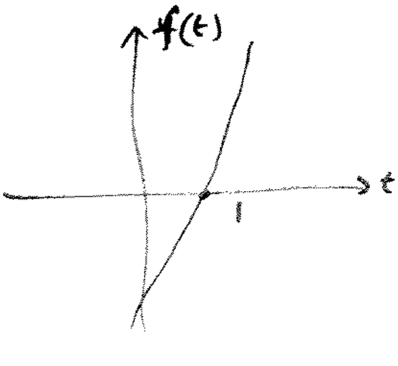
(c) $f'(t) = 6(t-2)^2 + 1 \geq 1 > 0$ for all t

$\therefore \ f(t)$ increases for all t

$\Rightarrow$ curve could only cut t axis $\underline{\underline{\text{once}}}$

$\Rightarrow f(t) = 0$ only if $t = 1$

ie



(d) (i) $l = AP^2 = (t+1)^2 + (t^2-4t+4)^2$

$$= (t+1)^2 + ((t-2)^2)^2$$

$$= (t+1)^2 + (t-2)^4$$

(ii) $\dfrac{dl}{dt} = 2(t+1) + 4(t-2)^3$

$$= 2\left(2(t-2)^3 + t + 1\right)$$

$= 0$ only if $t = 1$ from (c)

OR USE (c) * $\dfrac{d^2l}{dt^2} = 2(6(t-2)^2 + 1) > 0$ if $t = 1$

[in fact > 0 for all t]

$\Rightarrow$ min l when $t = 1$

$\Rightarrow$ min AP when $t = 1$ since $l = AP^2$

$\therefore$ min length $AB = \sqrt{2^2 + 1} = \sqrt{5}$