THE KING'S SCHOOL

2010
Higher School Certificate Course
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

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Attempt Questions 1-10
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find, correct to the nearest integer, $3420 \sin \frac{\pi}{5}$ **2**

(b) Find the sum of the first 30 terms of the arithmetic series $9 + 13 + 17 + \ldots$ **2**

(c) Write down a primitive function of $\sec^2(2x + 1)$ **2**

(d) Simplify $2\sqrt{3} + \frac{2}{\sqrt{3} - 1}$ **2**

(e) Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - \frac{x^{2010}}{2010}$

Find the value of $f'(-1)$ **2**

(f) Sketch the hyperbola $y - 1 = \frac{1}{x}$, clearly showing its asymptotes and any intercepts made with the coordinate axes. **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) State the domain of the function $f(x) = \ln(1 + x)$ **1**

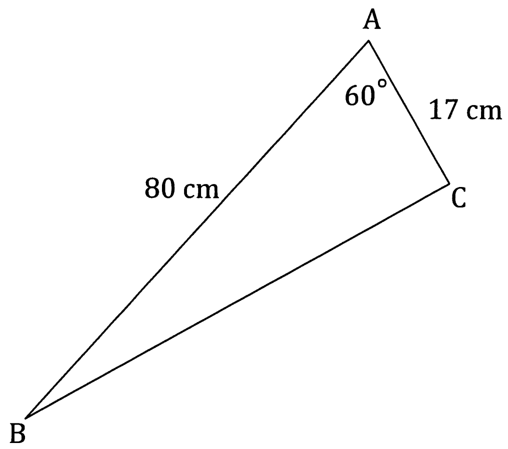
(b) Differentiate

(i) $\sin(x^2)$ **2**

(ii) $x^2 \ln x$ **2**

(c) Evaluate $\displaystyle\int_{-2}^{2} \frac{2}{\sqrt{2x + 5}}\, dx$ **3**

(d)



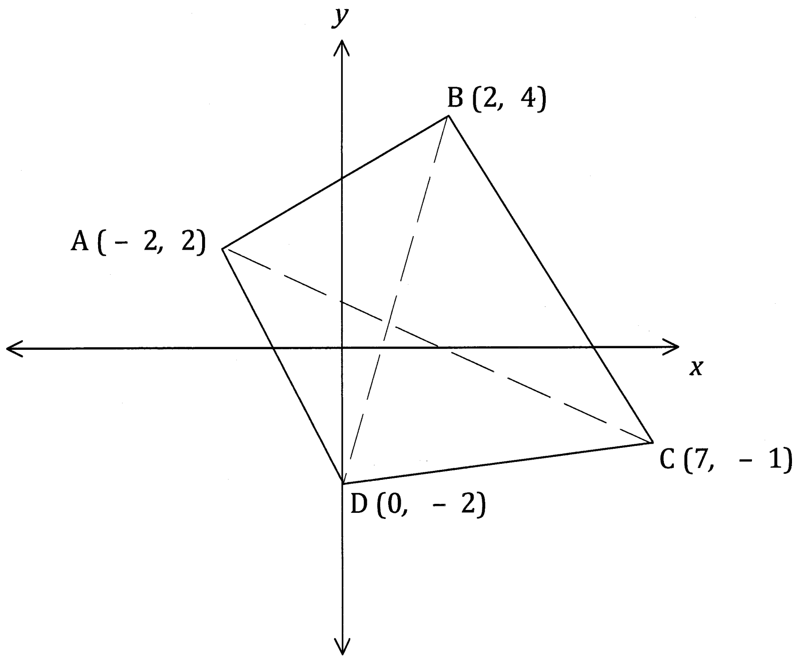
(i) Find the exact area of Δ ABC **2**

(ii) Find the length of BC **2**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the quadrilateral ABCD

(i) Show that the diagonals AC and BD are perpendicular. **2**

(ii) Find the mid-point M of BD. **1**

(iii) Show that M lies on the diagonal AC. **2**

(iv) Hence, or otherwise, find the area of the quadrilateral ABCD. **2**

(b) (i) On the same axes, sketch the parallel lines

$$3x + 4y = 12 \quad \text{and}$$
$$3x + 4y = 24,$$

clearly showing their intercepts made with the coordinate axes. **2**

(ii) Find the perpendicular distance between the lines. **2**

(iii) Hence, or otherwise, find the area of the region enclosed by the two lines in the first quadrant. **1**

End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

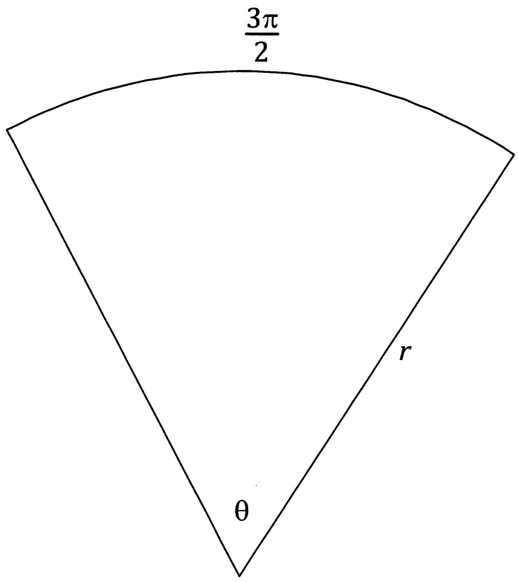
(a) (i) Prove that $\sin x + \cot x \cos x \equiv \operatorname{cosec} x$ **3**

(ii) Find the exact solutions of $\sin x + \cot x \cos x = 2,\ 0 < x < 2\pi$ **2**

(b) $S = \dfrac{\pi^4}{13} + \dfrac{\pi^3}{13} + \dfrac{\pi^2}{13} + \ldots$ is a geometric series.

Show that $S < 11$ **3**

(c)



The diagram shows a sector of a circle with radius r and central angle θ radians.

The arc length is $\dfrac{3\pi}{2}$ and the area of the sector is 12π

(i) Show that $r^2\theta = 24\pi$ **1**

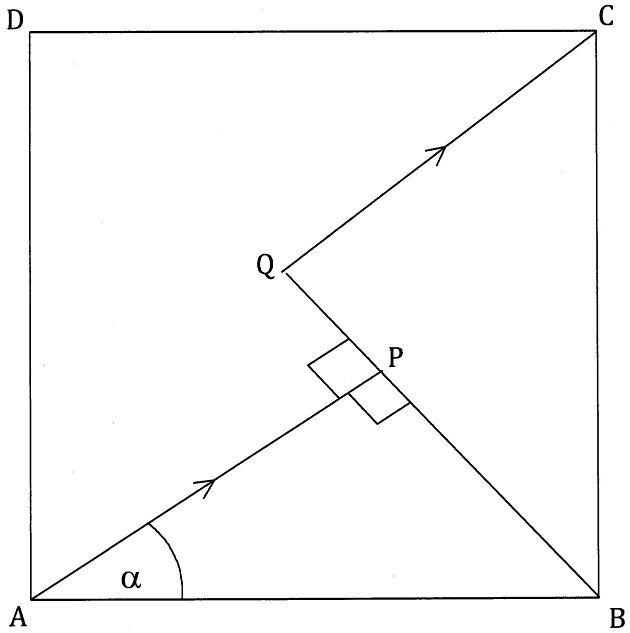
(ii) Find the value of r **2**

(iii) Find the central angle correct to the nearest degree. **1**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



ABCD is a square. P is a point interior to the square so that $\angle APB = 90^\circ$.

BP is produced to Q so that AP || QC

Let $\angle PAB = \alpha$

(i) Explain why $\angle CQB = 90^\circ$ **1**

(ii) Prove that $\Delta APB \equiv \Delta CQB$ **3**

(iii) Deduce that PQ = AP – PB **2**

(b) Let $Q(x) = (k + 1)x^2 - 2(k + 1)x + k$

Prove that there are no values of k for which $Q(x)$ is positive definite. **3**

(c) Solve

(i) $x^2 > 2$ **1**

(ii) $|x^2 - 9| < 7$ **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Find the sum and product of the roots of the equation $3x^2 + 6x - 11 = 0$ **2**

(ii) The line $y = 3x + 6$ meets the hyperbola $y = \dfrac{11}{x}$ at the points P and Q.

Find the coordinates of the mid-point M of the interval PQ. **3**

(b) A particle moves on the x axis with its velocity v at any time $t \geq 0$ given by $v = e^t - 12e^{-t}$

The particle is initially at the origin.

(i) Show that the acceleration is positive at all times. **2**

(ii) The particle stops when $e^t = \sqrt{12}$. **[DO NOT SHOW THIS]**

Find the exact acceleration when the particle stops. **1**

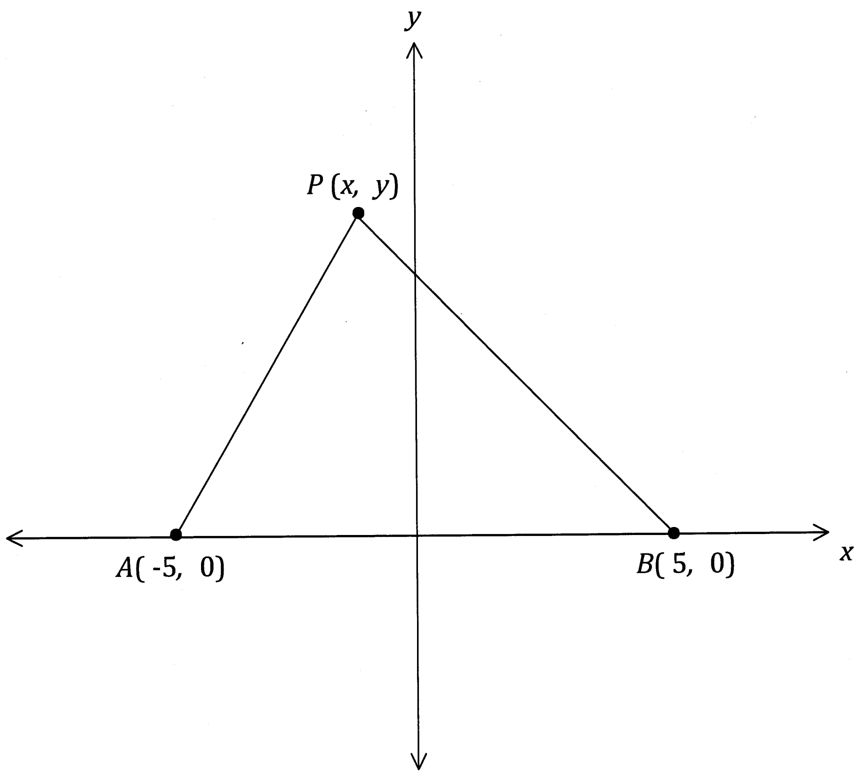
(iii) Find the displacement x at any time t. **2**

(iv) Find the time taken for the particle to return to the origin. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



$A(-5,\ 0)$ and $B(5,\ 0)$ are two fixed points. $P(x,\ y)$ is any point so that $\frac{PB}{PA} = \frac{3}{2}$

(i) Show that $(-1,\ 0)$ is a point on the locus of $P(x,\ y)$ **1**

(ii) Prove that the equation of the locus of $P(x,\ y)$ is

$x^2 + y^2 + 26x + 25 = 0$ **3**

(iii) Describe the locus of P in precise geometrical terms. **2**

(b) Let $f(x) = 1 - \cos \pi x\ ,\ \ -1 \le x \le 1$

(i) State the period of the function. **1**

(ii) Sketch the graph of $y = f(x)$ **2**

(iii) Find the area bounded by the curve and the x axis, $-1 \le x \le 1$ **3**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use Simpson's Rule with three function values to approximate

$$\int_0^1 \sqrt{18x^2 - 3x + 1}\, dx$$ **2**

(b) Let $f(x) = x - 2\sqrt{x} - 3$

(i) By letting $u = \sqrt{x}$, or otherwise, solve $f(x) = 0$ **2**

(ii) Find any stationary points and determine their nature. **3**

(iii) Sketch the graph of $y = f(x)$ indicating the x and y intercepts, if any. **2**

(iv) The line $y = x - 4$ is a normal to the curve $y = f(x)$ at the point P.

Find the coordinates of P. **3**

End of Question 8

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

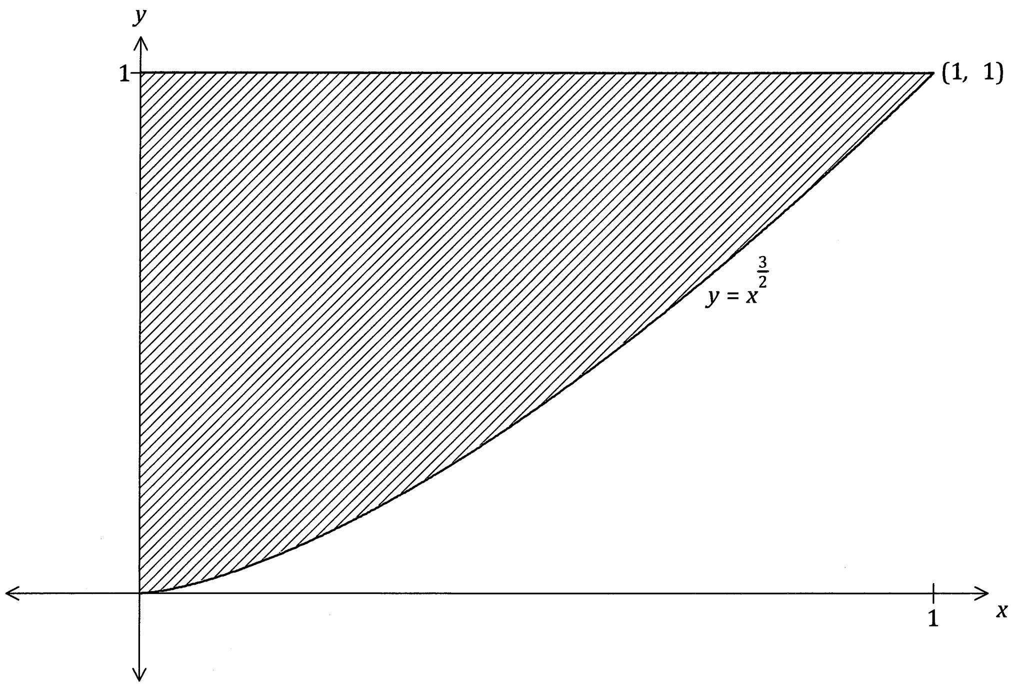
(a) A balloon is being inflated at a rate $R\ cm^3/s$ given by $R = \dfrac{100t}{1 + t^2} + 10$, where $t \geq 0$ is the time in seconds.

(i) At what rate is the balloon being inflated after 2 seconds? **1**

(ii) Determine the greatest rate of inflation. **3**

(iii) The balloon held $69\ cm^3$ of air initially. Find, correct to 3 significant figures, the volume of air held by the balloon after 10 seconds. **3**

(b) (i)

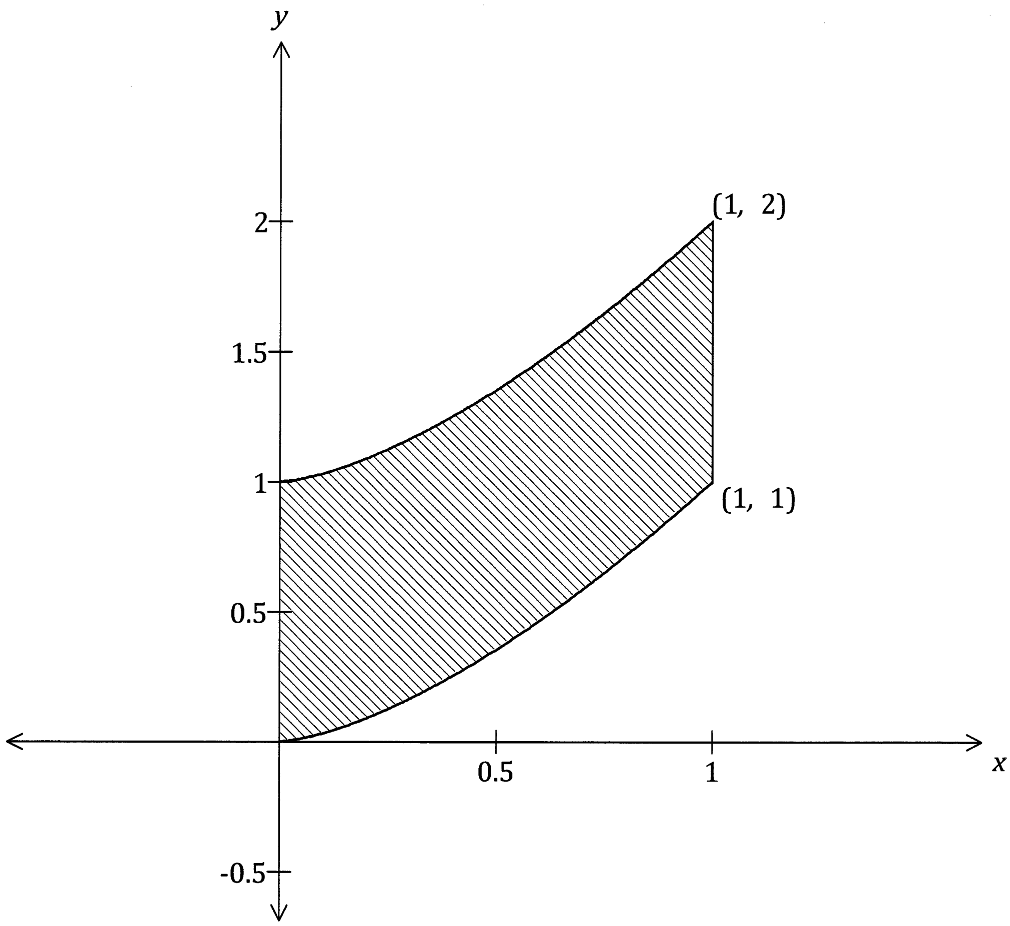


The diagram shows the shaded region between the curve $y = x^{\frac{3}{2}}$, the y axis and the line $y = 1$. This region is revolved about the y axis.

Find the volume of the solid generated. **3**

Question 9 continues on the next page

(b) (ii)



The diagram shows the shaded region between the curves $y = x^{\frac{3}{2}}$ and $y = x^{\frac{3}{2}} + 1$, the y axis and the line $x = 1$.

This region is revolved about the y axis.

Find the volume of the solid generated. **2**

End of Question 9

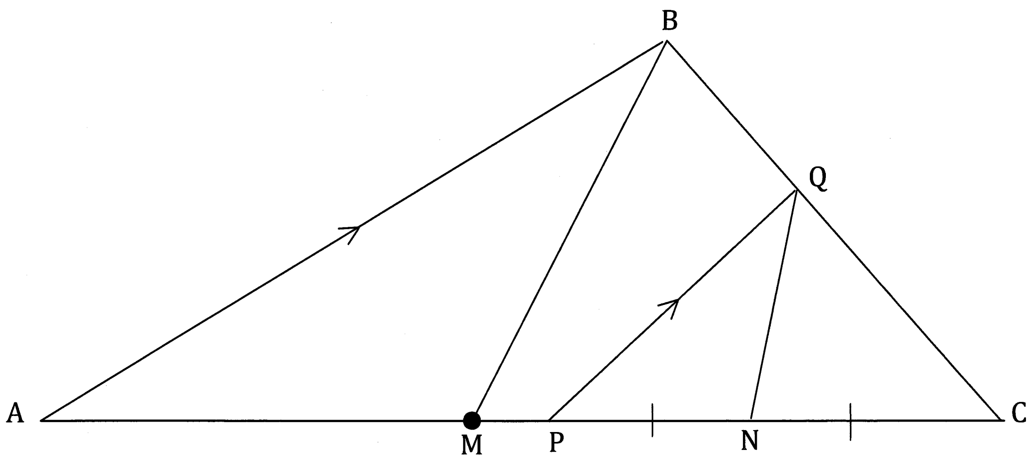
Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the smallest integer n for which $0.5^n < 12 \times 10^{-2010}$ **2**

(b) (i) Sketch the graph of the function $y = 2|x| - x + 1$ **2**

(ii) Evaluate $\int_{-1}^{1} 2|x| - x + 1 \, dx$ **2**

(c)



In the diagram, M is the mid-point of side AC of ΔABC.

P is on AC and Q is on BC so that PQ||AB.

N is the mid-point of PC.

(i) Explain why $\frac{CB}{CQ} = \frac{CA}{CP}$ **1**

(ii) Deduce that $\frac{CB}{CQ} = \frac{CM}{CN}$ **1**

(iii) Explain why ΔCBM is similar to ΔCQN **2**

(iv) Prove that MB||NQ **2**

End of Examination Paper

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

MATHEMATICS SOLUTIONS TKS TRIAL 2010

Question 1

(a) 2010

(b) $S_{30} = \frac{30}{2}(18 + 29 \times 4) = 2010$

(c) $\frac{1}{2}\tan(2x+1)$

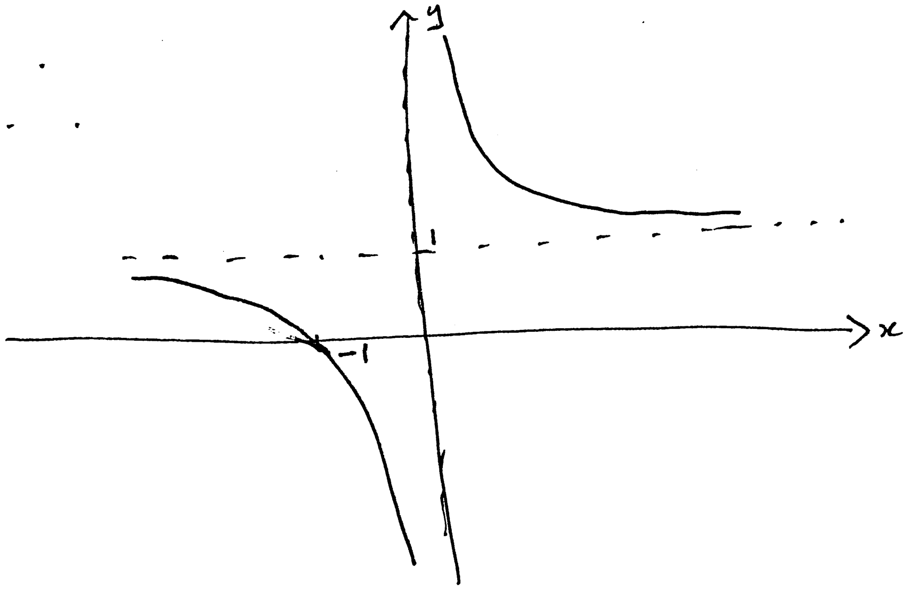
(d) $2\sqrt{3} + \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = 2\sqrt{3} + \frac{2(\sqrt{3}+1)}{2} = 3\sqrt{3} + 1$

(e) $f'(x) = 1 - x + x^2 - x^3 + \cdots - x^{2009}$

$\therefore f'(-1) = 1 + 1 + 1 + \cdots + 1$, 2010 terms

$= 2010$

(f) $x \neq 0,\ y \neq 1$; $y = 0,\ x = -1$



Question 2

(a) $1 + x > 0 \Rightarrow$ domain is $x > -1$

(b) (i) $2x \cos(x^2)$

(ii) $x^2 \cdot \frac{1}{x} + \ln x \cdot 2x$ (will do)

$= x + 2x \ln x$ or $x(1 + 2\ln x)$

(c) $I = \int_{-2}^{2} 2(2x+5)^{-\frac{1}{2}}\, dx$

$= 2\, \frac{2\left[(2x+5)^{\frac{1}{2}}\right]_{-2}^{2}}{2} = 2\left[\sqrt{2x+5}\right]_{-2}^{2}$

$= 2(3-1) = 4$

(d) (i) $A = \frac{1}{2} \cdot 80 \cdot 17 \sin 60^\circ$ cm^2

$= 40 \cdot 17 \cdot \frac{\sqrt{3}}{2}$ cm^2 $= 340\sqrt{3}$ cm^2

(ii) $BC^2 = 80^2 + 17^2 - 2 \cdot 80 \cdot 17 \cos 60^\circ$

$= 80^2 + 17^2 - 80 \times 17 = 5329$

$\therefore BC = \sqrt{5329}$ cm $= 73$ cm

Question 3

(a) (i) grd $AC = \dfrac{-1-2}{7+2} = -\dfrac{3}{9} = -\dfrac{1}{3}$

grd $BD = \dfrac{-2-4}{0-2} = 3$

+ since $-\dfrac{1}{3} \times 3 = -1$ then $AC \perp BD$

(ii) $M = (1, 1)$

(iii) grd $MA = \dfrac{2-1}{-2-1} = -\dfrac{1}{3}$ = grd AC

$\Rightarrow$ A, M, C collinear ie M is on AC

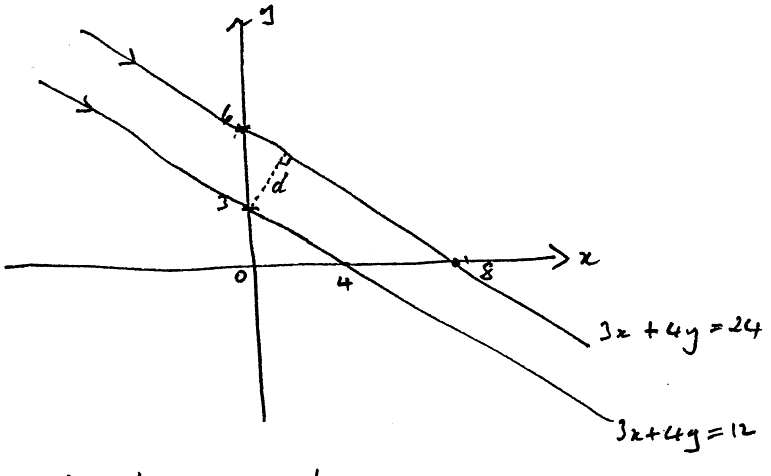
(iv) Area $= \frac{1}{2}.AC.BM + \frac{1}{2}.AC.DM$

$= \frac{1}{2} AC (BM + DM) = \frac{1}{2}.AC.BD$

$= \frac{1}{2}.\sqrt{9^2+3^2}\sqrt{2^2+6^2}$

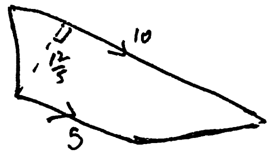
$= 30$

(b) (i)



(ii) $d = \dfrac{|0+12-24|}{\sqrt{3^2+4^2}} = \dfrac{12}{5}$

(iii)



TRAPEZIUM

$A = \frac{1}{2}(10+5)\frac{12}{5} = 18$

OR $A = \frac{1}{2}.8.6 - \frac{1}{2}.3.4 = 18$

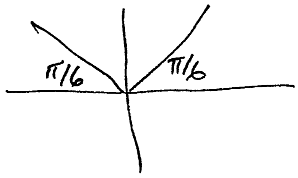
Question 4

(a) (i) $LS = \sin x + \frac{\cos x}{\sin x} \cdot \cos x$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x}$$

$$= \frac{1}{\sin x} = \operatorname{cosec} x$$

(ii) From (i), $\operatorname{cosec} x = 2$



$$\Rightarrow \sin x = \frac{1}{2}$$

$$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$$

(b) G. S., $a = \frac{\pi^4}{13}$, $r = \frac{1}{\pi}$ (& $-1 < r < 1$)

$$\therefore S = \frac{\frac{\pi^4}{13}}{1 - \frac{1}{\pi}} = \frac{\pi^5}{13(\pi - 1)} = 10\cdot 99\ldots < 11$$

$\Rightarrow S < 11$ since it's a series of positive terms

(c) (i) $A = \frac{1}{2} r^2 \theta = 12\pi \Rightarrow r^2 \theta = 24\pi$

(ii) $r^2\theta = 24\pi$

$r\theta = \frac{3\pi}{2}$

$$\therefore r = \frac{24\pi}{\frac{3\pi}{2}} = 16$$

(iii) $\theta = \frac{3\pi}{2 \times 16} = \frac{3 \times 180}{32}^{\circ} = 17^{\circ}$, nearest degree

Question 5

(a) (i) $\angle CQB = \angle QPA = 90°$, alternate $\angle s$ in // lines AP, QC

(ii) $\angle PBA = 90° - \alpha°$, $\angle$ sum ΔAPB

$\therefore \angle CBQ = \alpha°$, $\angle ABC = 90°$, ABCD a square

$\therefore$ In Δs APB, CQB

$\angle P = \angle Q = 90°$, (i)

$\angle PAB = \angle CBQ = \alpha°$

$AB = BC$, sides of square ABCD

$\therefore$ congruent, AAS

(iii) $PQ = QB - PB$

$= AP - PB$ since $QB = AP$, corr. sides in cong Δs (ii)

(b) Firstly, $k+1 > 0 \Rightarrow k > -1$

Next, $\Delta = 4(k+1)^2 - 4k(k+1)$

$= 4(k+1)(k+1-k)$

$= 4(k+1) < 0$ for P.D

$\Rightarrow k < -1$

$\therefore$ no value of k makes $Q(x)$ P.D.

(c) (i) $x > \sqrt{2}$ or $x < -\sqrt{2}$

(ii) $\therefore -7 < x^2 - 9 < 7$

or $2 < x^2 < 16 \Rightarrow x > \sqrt{2}$ or $x < -\sqrt{2}$

and $-4 < x < 4$

$\therefore -4 < x < -\sqrt{2}$ or $\sqrt{2} < x < 4$

Question 6

(a) (i) $\Sigma = -\frac{6}{3} = -2, \quad \Pi = -\frac{11}{3}$

(ii) at P, Q $\quad 3x+6 = \frac{11}{x}$

$\Rightarrow 3x^2+6x=11$

or $3x^2+6x-11=0$ has roots x_1, x_2 say

$\Rightarrow x_1+x_2=-2 \quad \Rightarrow \frac{x_1+x_2}{2} = -1$

ie at M, $x=-1$, $y=-3+6$ ie $M=(-1,3)$

(b) (i) $\ddot{x} = e^t + 12e^{-t} > 0 \ \forall t$ since $e^t, e^{-t} > 0 \ \forall t$

$\therefore \ddot{x} > 0 \ \forall t$

(ii) $\ddot{x} = \sqrt{12} + \frac{12}{\sqrt{12}} = \sqrt{12}+\sqrt{12} = 2\sqrt{12}$ or $4\sqrt{3}$

(iii) $x = e^t + 12e^{-t} + c$

: $0 = 1 + 12 + c, \quad c = -13$

$\therefore x = e^t + 12e^{-t} - 13$

(iv) $x=0 \Rightarrow e^{2t} + 12 - 13e^t = 0$

$e^{2t} - 13e^t + 12 = 0$

$(e^t - 1)(e^t - 12) = 0$

$\Rightarrow e^t = 1$ or 12 $\Rightarrow$ returns to origin when $e^t = 12$

ie $t = \ln 12$

Question 7

(a) (i) If $P = (-1, 0)$, $\frac{PB}{PA} = \frac{6}{4} = \frac{3}{2}$ $\therefore$ result

(ii) $3PA = 2PB$ or $9PA^2 = 4PB^2$

$\therefore 9\left((x+5)^2 + y^2\right) = 4\left((x-5)^2 + y^2\right)$

ie $9(x^2 + 10x + 25 + y^2) = 4(x^2 - 10x + 25 + y^2)$

ie. $5x^2 + 5y^2 + 130x + 125 = 0$

ie $x^2 + y^2 + 26x + 25 = 0$

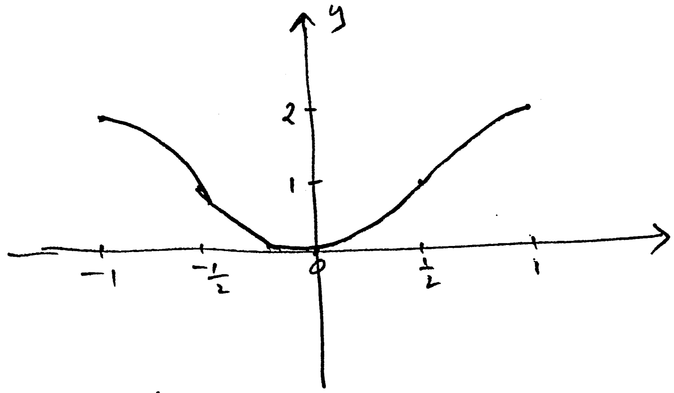
(iii) $\therefore (x+13)^2 - 169 + y^2 = -25$

ie $(x+13)^2 + y^2 = 144$

$\Rightarrow$ circle, centre $(-13, 0)$, radius 12

(b) (i) $p = \frac{2\pi}{\pi} = 2$

(ii)



(iii) $A = 2\int_0^1 1 - \cos \pi x \, dx$

$= 2\left[x - \frac{\sin \pi x}{\pi}\right]_0^1$

$= 2(1 - 0 - (0)) = 2$

Question 8

(a) $I \approx \frac{1}{6} \cdot 1 \left[1 + 4 + 4 \times 2 \right]$

$= \frac{13}{6}$

(b) (i) $f(x) = 0 \Rightarrow (\sqrt{x} + 1)(\sqrt{x} - 3) = 0$

$\Rightarrow \sqrt{x} = 3$ only as $\sqrt{x} + 1 \neq 0$

$\therefore x = 9$

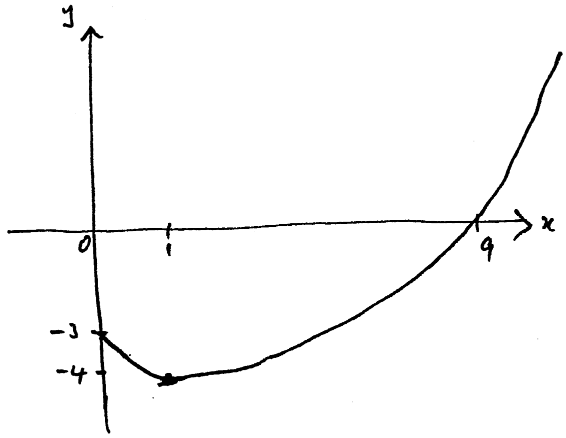
(ii) $f'(x) = 1 - 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} = 1 - \frac{1}{\sqrt{x}}$

$= 0$ if $x = 1$

$f''(x) = \frac{1}{2} x^{-3/2} > 0$ if $x = 1 \Rightarrow \cup$

ie minimum T.P $(1, -4)$

(iii) $x = 0, y = -3$; $x = 9, y = 0$, (i)



(iv) $y = x - 4$ has gradient 1 $\Rightarrow$ gradient of tangent $= -1$

$f'(x) = 1 - \frac{1}{\sqrt{x}} = -1 \Rightarrow \frac{1}{\sqrt{x}} = 2$ or $\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$

$\therefore P = \left(\frac{1}{4}, -3\frac{3}{4}\right)$

Question 9

(a) (i) $t = 2$, $R = \frac{200}{5} + 10 \text{ cm}^3/\text{s} = 50 \text{ cm}^3/\text{s}$

(ii) $\frac{dR}{dt} = 100 \left(\frac{1+t^2 - t.2t}{(1+t^2)^2} \right)$

$= 100 \frac{(1-t^2)}{(1+t^2)^2} = 0$ if $t^2 = 1 \Rightarrow t = 1$ since $t \geq 0$

$t = 0$, $\frac{dR}{dt} > 0$; $t = 2$, $\frac{dR}{dt} < 0 \Rightarrow$ max

$\therefore$ max $R = \frac{100}{2} + 10 \text{ cm}^3/\text{s} = 60 \text{ cm}^3/\text{s}$

(iii) $R = \frac{dV}{dt} \Rightarrow V = \int \frac{100t}{1+t^2} + 10 \, dt$

$= 50 \ln(1+t^2) + 10t + c$

$\therefore 69 = 0 + 0 + c$, $c = 69$

$\therefore V = 50 \ln(1+t^2) + 10t + 69$

$t = 10$, $V = 50 \ln 101 + 100 + 69$

$= 399.75\ldots\ldots \text{ cm}^3$

$= 400 \text{ cm}^3$, 3 sig figs (ie nearest cm^3)

(b) (i) $V = \pi \int_0^1 x^2 \, dy$: $y = x^{\frac{3}{2}}$

$\therefore (x^{\frac{3}{2}})^{\frac{4}{3}} = x^2 = y^{\frac{4}{3}}$

$\therefore V = \pi \int_0^1 y^{\frac{4}{3}} dy = \pi \frac{3}{7} [y^{\frac{7}{3}}]_0^1 = \frac{3\pi}{7} \text{ u}^3$

(ii) (LOTS OF WAYS)

e.g. using (i), $V = \frac{3\pi}{7} + \pi . 1^2 . 1 - \frac{3\pi}{7} = \pi \text{ u}^3$

Question 10

(a) Choose base 10 for obvious reasons (base e okay, of course)

$\therefore \log_{10} 0{\cdot}5^n < \log_{10}(12 \times 10^{-2010})$

or $n \log_{10} 0{\cdot}5 < \log_{10} 12 - 2010$

or $n > \dfrac{\log_{10} 12 - 2010}{\log_{10} 0{\cdot}5}$ since $\log_{10} 0{\cdot}5 < 0$

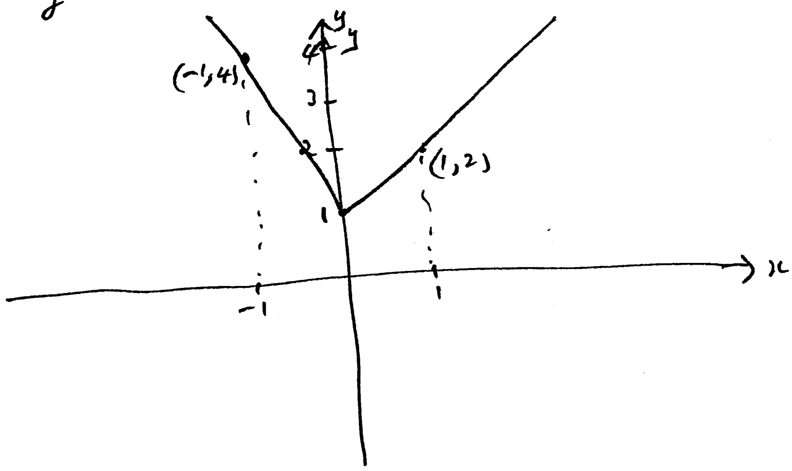
$= 6673{\cdot}49\ldots$

$\therefore$ least $n = 6674$

(b) (i) $x \geq 0$, $y = 2x - x + 1 = x + 1$

$x \leq 0$, $y = -2x - x + 1 = -3x + 1$

$\therefore$



(ii) $\therefore I = \frac{1}{2} \cdot (1+4)1 + \frac{1}{2}(1+2)1 = 4$

10 (c)

(i) ratio intercept theorem in parallel lines (AB, PQ)

(ii) $\frac{CB}{CQ} = \frac{2CM}{2CN}$, (i) and data

$= \frac{CM}{CN}$

(iii) $\angle C$ is common

(ii) $\Rightarrow$ two pairs of corresponding sides are in proportion with $\angle C$ the included angle

$\therefore$ similar

(iv) From (iii), $\angle BMC = \angle QNC$, {matching / corr. $\angle$s in similar $\triangle$s,

and these angles are corresponding

$\therefore MB \parallel NQ$