THE KING'S SCHOOL

2011
Higher School Certificate
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Answer each question in a separate booklet

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

Total marks – 120
Attempt Questions 1-10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Graph the line $y = x + 1$ in the number plane showing the intercepts made with the coordinate axes. **2**

(b) Solve $|2x - 1| < 3$ **2**

(c) Simplify $\dfrac{\sqrt{2}}{\sqrt{2} + 1}$ **2**

(d) Simplify $4^{n} \times 2^{1-n}$ **2**

(e) Evaluate $\displaystyle\sum_{n=1}^{3} \cos\left(\frac{\pi n}{2}\right)$ **2**

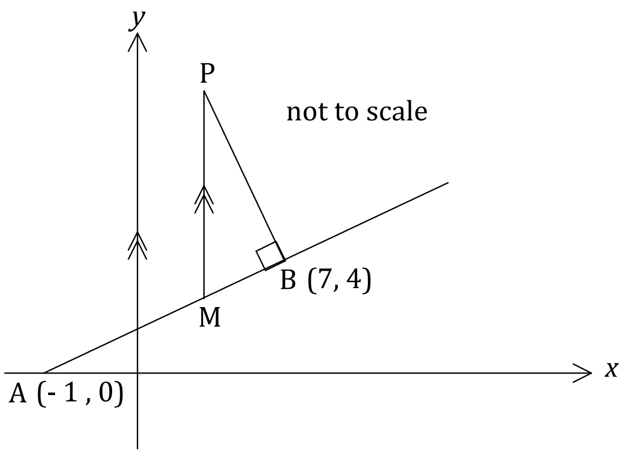
(f) Find the derivative of $1 + x - \dfrac{1}{x}$ **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the shortest distance from the point P $(-1, 2)$ to the line $3x + 4y + 55 = 0$ **2**

(b)



A $(-1, 0)$ and B (7, 4) are two points. M is the mid-point of AB.

P is the point such that PM is parallel to the *y* axis and PB ⊥ AB

(i) Find the gradient of AB **1**

(ii) Hence, or otherwise, show that the equation of PB is $y = -2x + 18$ **2**

(iii) Write down the coordinates of M **1**

(iv) Find the area of Δ MPB **2**

(c) $9 + 20 + 31 + \ldots$ is an arithmetic series.

(i) Find the sum of the first 100 terms. **2**

(ii) Is 2011 a term in the series? Explain. **2**

End of Question 2

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $f(x) = e^{\sin x}$

(i) Find, correct to one decimal place, $f(1)$ **1**

(ii) State the largest value of $f(x)$ **1**

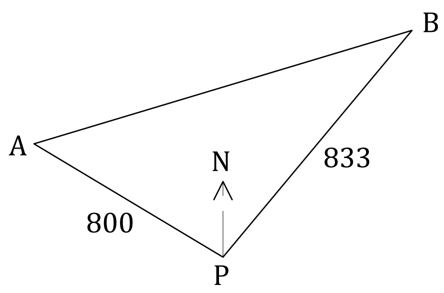
(iii) Find $f'(0)$ **2**

(b) Find the derivative of $x^2(2x + 1)^3$

[YOU MAY LEAVE YOUR ANSWER UNSIMPLIFIED] **2**

(c) Evaluate $\int_0^1 \frac{2x}{2 - x^2}\,dx$ **3**

(d)



Ship A sails from P on a bearing of 310° for 800 km. Ship B sails from P on a bearing of 010° for 833 km.

(i) Show that $\angle APB = 60^\circ$ **1**

(ii) Find the distance AB between the ships. **2**

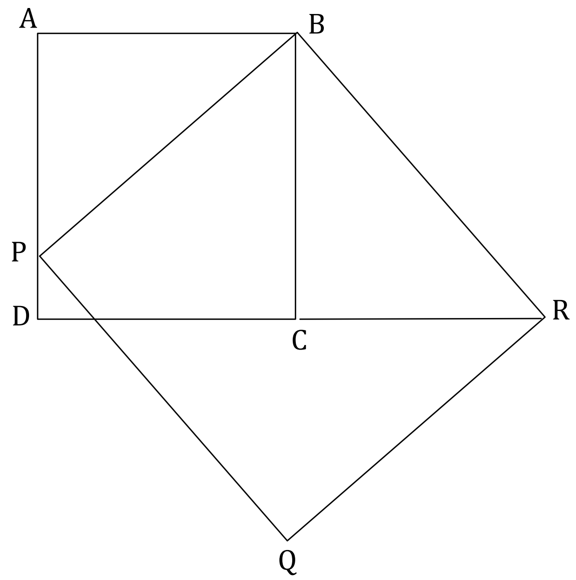
End of Question 3

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Prove that $\dfrac{1}{1 + \sin x} + \dfrac{1}{1 - \sin x} \equiv 2\sec^2 x$ **3**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}\,dx$ **2**

(b)



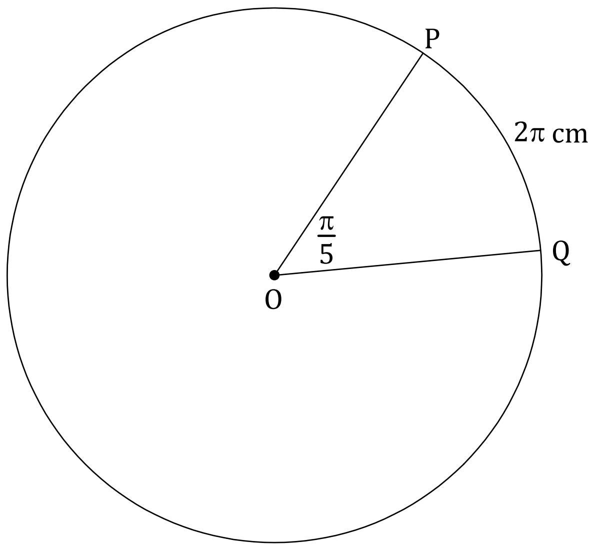
ABCD and BPQR are squares. P is on AD and DCR is a straight line.

(i) Prove that Δ ABP is congruent to Δ BCR **3**

(ii) Deduce that CR + PD = DC **2**

Question 4 continues on the next page

(c)



OPQ is a sector of a circle where $\angle POQ = \frac{\pi}{5}$ and arc $PQ = 2\pi$ cm.

Find the area of the sector. **2**

End of Question 4

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $f(x) = x^3 - 6x^2 + 9x + 2$

(i) Solve the equation $f(x) = 2$ **2**

(ii) Find the stationary points and determine their nature. **3**

(iii) Sketch the graph of $y = f(x)$ **2**

(iv) For what values of x is the curve concave upward? **1**

(b) (i) Show that $\int_{-1}^{1} 4x^3 - 6x^2 - 2x + 4\,dx = 4$ **2**

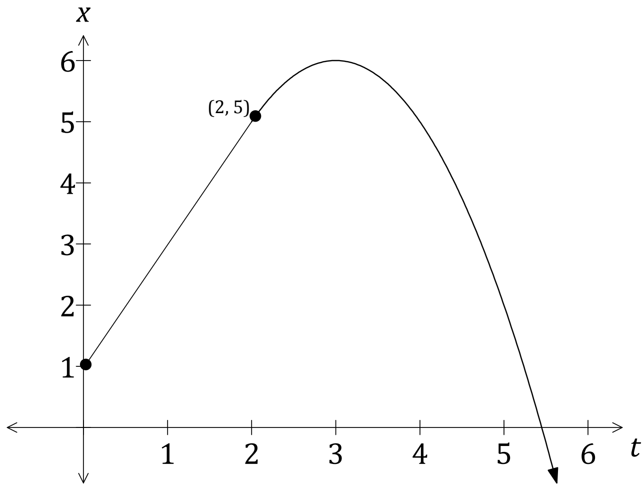
(ii) It is known that Simpson's Rule gives the exact answer for cubic curves.

Verify this for the integral in part (i). **2**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The graph shows the displacement *x* of a particle at time *t*.

For $0 \le t \le 2$ the particle is moving with constant velocity.

For $t > 2$ the equation of motion is given by $x = 6t - 3 - t^2$

(i) Find the velocity at $t = 1$ **1**

(ii) At what time is the particle at rest? **1**

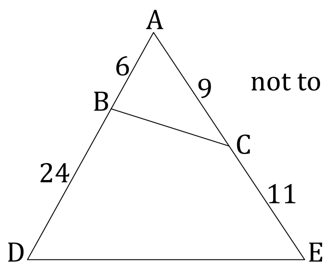
(iii) Find the velocity at $t = 5$ **1**

(iv) Find the distance travelled during the first 6 seconds. **2**

(v) Find the least acceleration. **1**

(vi) At what times is the particle at $x = 4$? **2**

(b)



not to scale

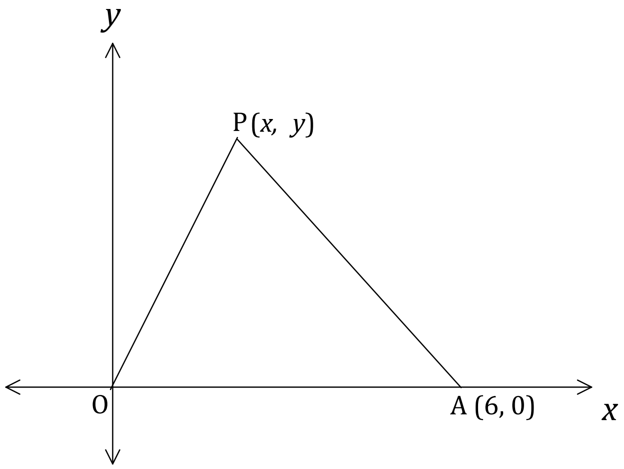
(i) Use the diagram to prove that $\Delta ABC \parallel\!| \Delta ADE$ **2**

(ii) Prove that the ratio area ΔABC : area $\Delta ADE = 9 : 100$ **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



O (0, 0) and A (6, 0) are two fixed points.

P (x, y) is any point in the number plane so that PA = 2PO

(i) Show that the point (2, 0) satisfies the above condition and find the other point in the x axis that also satisfies this condition. **2**

(ii) Prove that the Cartesian locus of P (x, y) is given by $x^2 + y^2 + 4x = 12$ **3**

(iii) Describe the locus of P (x, y) in precise geometrical terms. **2**

(b) (i) Solve $4\cos 2x = 2, \ 0 \le x \le \pi$ **2**

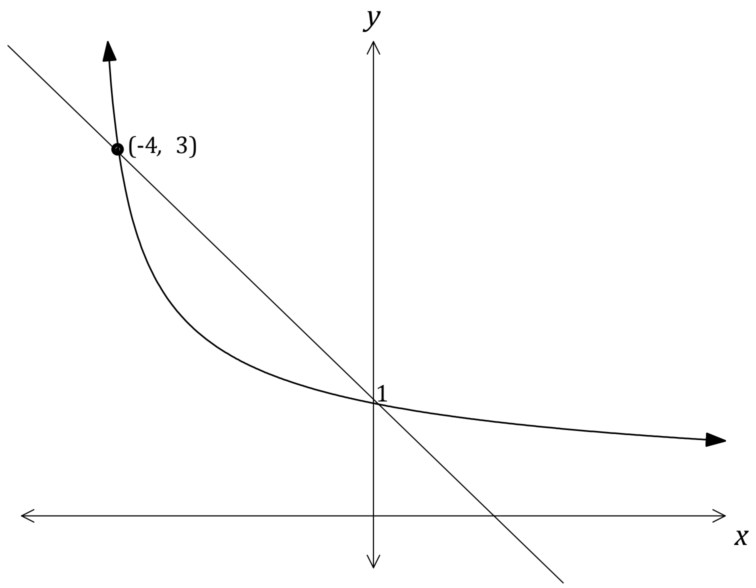
(ii) Sketch the graph of $y = 4\cos 2x, \ 0 \le x \le \pi$ **2**

(iii) Hence, or otherwise, evaluate $\int_0^{\pi} 4\cos 2x \, dx$ **1**

End of Question 7

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The diagram shows the curve $y = \dfrac{3}{\sqrt{2x + 9}}$ and the line $2y + x = 2$ meeting at the points (0, 1) and (-4, 3).

Find the area enclosed between the curve and the line. **4**

(b) A mining boom has seen the town of Digger become bigger.

The population P of Digger is increasing exponentially so that after t years from January 2001 we have $P = Ae^{kt}$ where A and k are positive constants.

(i) Show that the rate of increase of the population at any time t is given by kP. **1**

(ii) In January 2006 the population was 2 840 and in January 2009 it was 3 500.

Taking $t = 0$ for January 2001, find the population of Digger in January 2001.

Give your answer correct to the nearest 10 people. **4**

(c) The velocity of a particle at any time t is given by

$$v = 7\ln(3t + 1) - 9\ln(t + 1), \quad t \geq 0$$

Find at what time the acceleration is 0. **3**

End of Question 8

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Holly invests $400 000 into a fund paying interest of 8% p.a. compounded annually. She intends to withdraw $40 000 at the end of each year immediately after the interest has been paid into the fund. Holly hopes to be able to do this for at least 20 years.

(i) What amount is in the fund after the first withdrawal? **1**

Let $\$A_n$ be the amount in the fund after n years.

It can be shown that $A_n = 400000\,(1.08)^n - 40000\left(1 + 1.08 + \ldots + 1.08^{n-1}\right)$

[DO NOT PROVE THIS]

(ii) Show that $A_n = 100000\left(5 - 1.08^n\right)$ **3**

(iii) Will Holly's hopes be realised? **2**

(b) Water is flowing into a tank of capacity 1 000 L. The rate of flow, R L/h, is given by

$R = 8t^2(6 - t)$, where t is the time in hours, $t \geq 0$

(i) Explain why $t \leq 6$ **1**

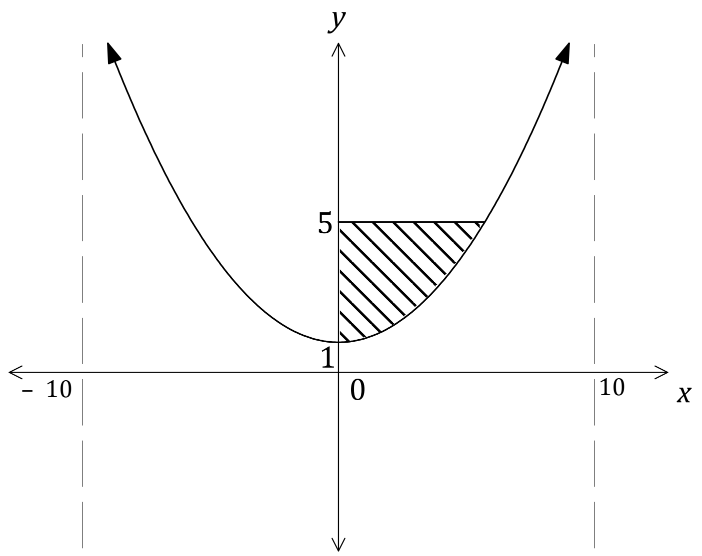
(ii) At what time does the rate of flow begin to decrease? **2**

(iii) Initially the tank held 100L of water. Will the tank overflow after the 6 hours? **3**

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The sketch shows the shaded region bounded by the curve $y = \dfrac{10}{\sqrt{100 - x^2}}$ and the *y* axis between $y = 1$ and $y = 5$

This region is revolved about the *y* axis.

Find the volume of the solid of revolution. **4**

(b) (i) By differentiation alone, use the trigonometric identity $\cos 2\theta = 1 - 2\sin^2\theta$ to prove that $\sin 2\theta = 2\sin\theta\cos\theta$ **2**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin\theta\cos\theta\, d\theta$ **2**

(c) (i) Solve $x - 9 \leq 3 - 2x \leq 2x - 1$ **3**

(ii) Solve $x^2 - 9 \leq 3 - 2x^2 \leq 2x^2 - 1$ **1**

End of Examination Paper

Standard Integrals

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

Student Number

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Mathematics

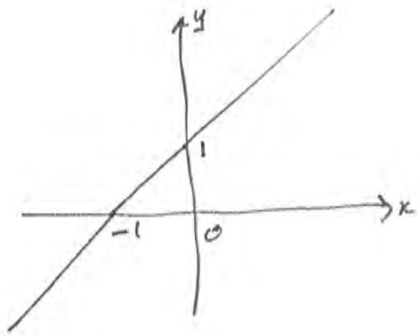
Question	*Algebra and Number*	*Differential Calculus*	*Functions*	*Geometry*	*Integral Calculus*	*Trigonometry*	*Total*
1	*b-e* **/ 8**	*f* **/ 2**	*a* **/ 2**				**/ 12**
2	*c* **/ 4**		*a, b* **/ 8**				**/ 12**
3		*a, b* **/ 6**			*c* **/ 3**	*d* **/ 3**	**/ 12**
4				*b* **/ 5**	*a-ii* **/ 2**	*a-i, c* **/ 5**	**/ 12**
5		*a* **/ 8**			*b* **/ 4**		**/ 12**
6		*a* **/ 8**		*b* **/ 4**			**/ 12**
7			*a* **/ 7**			*b* **/ 5**	**/ 12**
8		*b, c* **/ 8**			*a* **/ 4**		**/ 12**
9	*a* **/ 6**	*b* **/ 6**					**/ 12**
10	*c* **/ 4**	*b-i* **/ 2**			*a, b-ii* **/ 6**		**/ 12**
Total	**/22**	**/40**	**/17**	**/ 9**	**/19**	**/13**	**/120**

TKS MATHEMATICS TRIAL H.S.C. 2011 SOLUTIONS

Question 1

(a) $x=0, y=1$

$y=0, x=-1$



(b) $\therefore \quad -3 < 2x-1 < 3$

$-2 < 2x < 4$

$\Rightarrow \quad -1 < x < 2$

(c) $\dfrac{\sqrt{2}(\sqrt{2}-1)}{2-1} = 2-\sqrt{2}$

(d) $2^{2n} \times 2^{1-n} = 2^{n+1}$

(e) $\cos\frac{\pi}{2} + \cos\pi + \cos\frac{3\pi}{2} = 0 - 1 + 0 = -1$

(f) $y = 1 + x - x^{-1}$

$\therefore \dfrac{dy}{dx} = 1 + x^{-2} = 1 + \dfrac{1}{x^2}$

Question 2

(a) $d = \dfrac{-3+8+55}{\sqrt{3^2+4^2}} = \dfrac{60}{5} = 12$

(b) (i) grd $AB = \dfrac{4}{8} = \dfrac{1}{2}$

(ii) grd $PB = -2$ since $PB \perp AB$

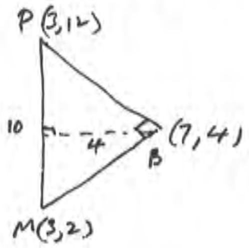
$\therefore PB: \quad y-4 = -2(x-7)$

ie $y = -2x+18$

(iii) $M = \left(\frac{6}{2}, \frac{4}{2}\right) = (3, 2)$

(iv) at P, $x = 3$ $\therefore y = -6+18 = 12$

$\Rightarrow$



ie area $= \frac{1}{2} \cdot 10 \cdot 4 = 20$

(c) (i) $a = 9$, $d = 11$

$\therefore S_{100} = 50(18 + 99 \times 11) = 55350$

(ii) Put $a + (n-1)d = 9 + 11(n-1) = 2011$

$\therefore 11(n-1) = 2002$

$n - 1 = 182$

$\Rightarrow$ 2011 is the 183rd term.

Question 3

(a) (i) $f(1) = e^{\sin 1} = 2 \cdot 3$, 1 d.p.

(ii) $e^1 = e$

(iii) $f'(x) = e^{\sin x} \cos x = e^0 \times 1 = 1$

(b) $x^2\, 3(2x+1)^2 2 + (2x+1)^3 2x$

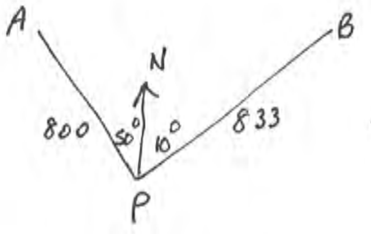
$\left[= 6x^2(2x+1)^2 + 2x(2x+1)^3\right] = \left[2x(2x+1)^2(5x+1)\right]$

(c) $I = -\int_0^1 \frac{-2x}{2-x^2}\, dx$

$= -\left[\ln(2-x^2)\right]_0^1 = -(\ln 1 - \ln 2) = \ln 2$

(d)

(i)



$\Rightarrow \angle APB = 50° + 10° = 60°$

(ii) $AB^2 = 800^2 + 833^2 - 2 \times 800 \times 833 \cos 60°$

$= 667489$

$\therefore AB = 817$ km

Question 4

(a) (i) $LS = \dfrac{1-\sin x + 1 + \sin x}{1-\sin^2 x}$

$= \dfrac{2}{\cos^2 x} = 2\sec^2 x$

(ii) From (i), $I = 2\int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

$= 2\left[\tan x\right]_0^{\frac{\pi}{4}} = 2(1-0) = 2$

(b) (i) In Δs ABP, BCR

$AB = BC$, ABCD a square

$BP = BR$, BPQR a square

$\angle BAP = \angle BCR = 90^\circ$, ABCD a square, DCR a st. line

$\therefore \Delta ABP \equiv \Delta BCR$, RHS

(ii) From (i), $CR = AP$

$\therefore CR + PD = AP + PD = AD$

$= DC$, ABCD a square

(c) $l = r\theta \Rightarrow 2\pi = \frac{\pi}{5} \cdot r$ ie $r = 10$

$\therefore$ Area $= \frac{1}{2} \cdot 10^2 \cdot \frac{\pi}{5}$ cm^2

$= 10\pi$ cm^2

Question 5

(a) (i) $f(x) = 2 \Rightarrow x^3 - 6x^2 + 9x = 0$

$\therefore x(x^2 - 6x + 9) = 0$

$x(x-3)^2 = 0 \quad \therefore x = 0, 3$

(ii) $f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

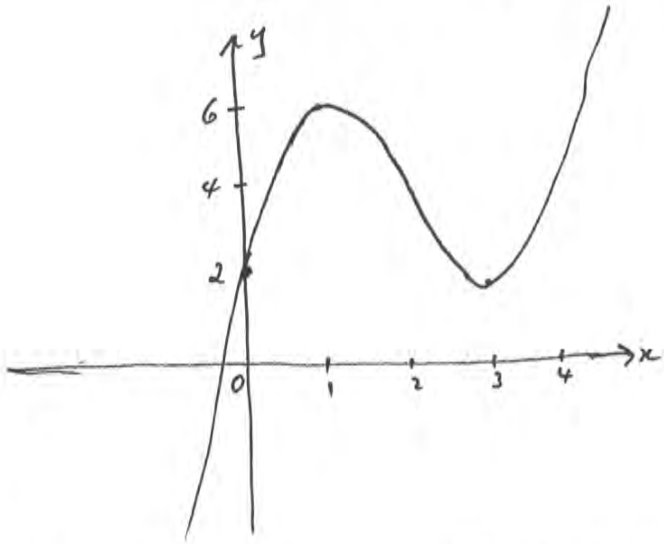
$= 0$ if $\begin{cases} x = 1 \\ y = 6 \end{cases}$ or $\begin{cases} x = 3 \\ y = 2 \end{cases}$

$f''(x) = 6x - 12$

Now $f''(1) < 0 \Rightarrow (1, 6)$ max. turning pt

$f''(3) > 0 \Rightarrow (3, 2)$ min turning pt

(iii)



(iv) Need $f''(x) > 0 \Rightarrow 6x - 12 > 0$

i.e. for $x > 2$

(b) (i) $I = \left[x^4 - 2x^3 - x^2 + 4x\right]_{-1}^{1} = 1 - 2 - 1 + 4 - (1 + 2 - 1 - 4) = 4$

(ii) $I = \frac{1}{6} \cdot 2(-4 + 0 + 4 \times 4)$

$= \frac{1}{3}(12) = 4$

Question 6

(a) (i) vel $= \frac{5-1}{2-0} = 2$

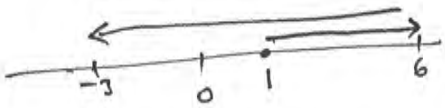
(ii) $t = 3$

(iii) For $t > 2$, $v = 6 - 2t$

$\therefore$ at $t = 5$, $v = 6 - 10 = -4$

(iv) $t = 6$, $x = 36 - 3 - 36 = -3$

$\Rightarrow$



ie. dist trav. $= 5 + 9 = 14$

(v) For $0 \le t \le 2$, $\ddot{x} = 0$

For $t > 2$, $\ddot{x} = \frac{dv}{dt} = -2$

$\Rightarrow$ least acceln $= -2$

(vi) $t = 1\frac{1}{2}$ and $6t - 3 - t^2 = 4$, $t > 2$

$\therefore t^2 - 6t + 7 = 0$

$\therefore t = \frac{6 \pm \sqrt{36-28}}{2} = \frac{6 \pm 2\sqrt{2}}{2}$

$\Rightarrow t = 3 + \sqrt{2}$ since $t > 2$

(b) (i) Now $\frac{AB}{AE} = \frac{6}{20} = \frac{3}{10}$ and $\frac{AC}{AD} = \frac{9}{30} = \frac{3}{10}$

and in Δs ABC, ADE, $\angle A$ is common

$\therefore$ similar, 2 pairs of corr. sides in proportion and included angles equal

(ii) Ratio $= \frac{\frac{1}{2} . 6 . 9 \sin A}{\frac{1}{2} . 20 . 30 \sin A} = \frac{9}{20 \times 5} = \frac{9}{100}$

Question 7

(a) (i) $P = (2,0) \Rightarrow PA = 6-2 = 4 = 2 \times PO$

Clearly, $P = (-6,0)$ since then $PA = 12 = 2\times 6 = 2\,PO$

(ii) $PA = 2PO \Rightarrow PA^2 = 4 \cdot PO^2$

$\therefore (x-6)^2 + y^2 = 4(x^2+y^2)$

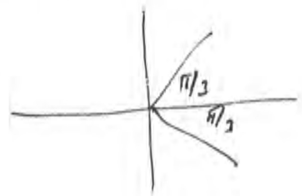
$\Rightarrow 3x^2 + 3y^2 + 12x = 36$

or $x^2 + y^2 + 4x = 12$

(iii) locus is $(x+2)^2 + y^2 = 12 + 4 = 16$

$\Rightarrow$ circle, centre $(-2,0)$, radius 4

(b) (i) $\cos 2x = \frac{1}{2}$, $0 \le 2x \le 2\pi$ $\Rightarrow$

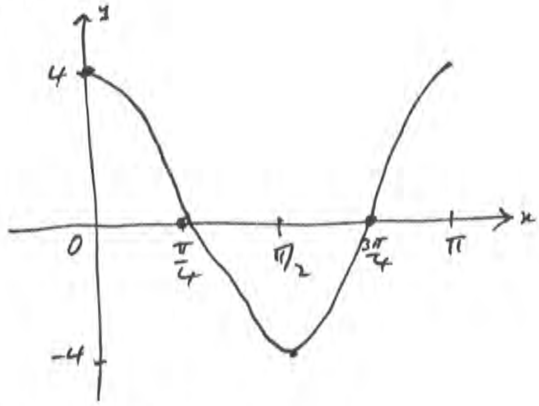


$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

(ii) period $= \frac{2\pi}{2} = \pi$

$\therefore$



(iii) From sketch, $I = 0$

Question 8

(a) For line, $y = \frac{2-x}{2} = 1 - \frac{x}{2}$ [or could use trapezium]

$$\therefore A = \int_{-4}^{0} 1 - \frac{x}{2} - \frac{3}{\sqrt{2x+9}}\,dx$$

$$= \int_{-4}^{0} 1 - \frac{x}{2} - 3(2x+9)^{-\frac{1}{2}}\,dx$$

$$= \left[x - \frac{x^2}{4} - 3\frac{(2x+9)^{1/2}}{\frac{1}{2}\cdot 2}\right]_{-4}^{0}$$

$$= \left[x - \frac{x^2}{4} - 3\sqrt{2x+9}\right]_{-4}^{0}$$

$$= 0 - 3\times 3 - (-4 - 4 - 3) = 2$$

(b) (i) Since $P = Ae^{kt}$

then $\frac{dP}{dt} = Ae^{kt}.k = k(Ae^{kt}) = kP$

(ii) $\therefore t=5,\ P = 2840$ and $t = 8,\ P = 3500$

$$\Rightarrow \quad 2840 = Ae^{5k}$$
$$3500 = Ae^{8k}$$
$$\therefore \frac{e^{8k}}{e^{5k}} = \frac{350}{284} = e^{3k}$$
$$\therefore 3k = \ln\left(\frac{350}{284}\right) \Rightarrow k = \frac{1}{3}\ln\left(\frac{350}{284}\right) \approx 0{\cdot}07$$

[0·06965.---]

$$\therefore A = 2840e^{-5k} = 2001{\cdot}3 - - -$$

[2004·78.--]

$\Rightarrow$ popn on Jan 2001 = 2000 [nearest 10]

(c) $\frac{dv}{dt} = 7.\frac{3}{3t+1} - 9.\frac{1}{t+1} = 0$

$$\Rightarrow 7(t+1) - 3(3t+1) = 0$$
$$-2t + 4 = 0$$

i.e. at $t = 2$

Question 9

(a) (i) Amount $= \$400000(1.08) - 40000 = \$392000$

(ii) $A_n = 400000(1.08)^n - 40000\left(\dfrac{1.08^n - 1}{1.08 - 1}\right)$

$= 400000(1.08)^n - 500000(1.08^n - 1)$

$= 100000(4 \times 1.08^n - 5 \times 1.08^n + 5)$

$= 100000(5 - 1.08^n)$

(iii) $A_{20} = 100000(5 - 1.08^{20})$

$= 100000 \times 0.339\ldots$

> 0

$\therefore$ Yes. There's still money in the fund after 20 years.

(b) (i) We need $R \geq 0$ for water flow

$\Rightarrow 6 - t \geq 0$ i.e. $t \leq 6$

(ii) Now $\dfrac{dR}{dt} = \dfrac{d}{dt} 8(6t^2 - t^3)$

$= 8(12t - 3t^2)$

$= 24t(4 - t) < 0 \Rightarrow R$ decreasing

$\Rightarrow 4 - t < 0$ i.e. $t > 4$

i.e. Begins to decrease at $t = 4$

(iii) $\dfrac{dV}{dt} = 48t^2 - 8t^3$

$\therefore V = \dfrac{48t^3}{3} - \dfrac{8t^4}{4} + c$

$\therefore 100 = 0 - 0 + c, \quad c = 100$

i.e. $V = 16t^3 - 2t^4 + 100$

When $t = 6$, $V = 16 \times 6^3 - 2 \times 6^4 + 100 = 964 < 1000$

$\therefore$ tank won't overflow

Question 10

(a) $V = \pi \int_1^5 x^2 \, dy$ where $y^2 = \dfrac{100}{100-x^2}$

$\therefore \; 100 - x^2 = \dfrac{100}{y^2}$

or $x^2 = 100 - \dfrac{100}{y^2} = 100(1 - y^{-2})$

$\therefore V = 100\pi \int_1^5 1 - y^{-2} \, dy$

$= 100\pi \left[y + \frac{1}{y} \right]_1^5$

$= 100\pi \left(5 + \frac{1}{5} - (1+1) \right) = 320\pi$

(b) (i) If $\cos 2\theta = 1 - 2(\sin\theta)^2$

then $-2 \sin 2\theta = -4 \sin\theta \cos\theta$

$\Rightarrow \sin 2\theta = 2 \sin\theta \cos\theta$

(ii) From (i), $I = \frac{1}{2} \int_0^{\pi/4} \sin 2\theta \, d\theta$

$= \frac{1}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/4} = -\frac{1}{4}(0 - 1) = \frac{1}{4}$

(c) (i) $\therefore \; 3 - 2x \geq x - 9$ and $3 - 2x \leq 2x - 1$

$\Rightarrow \; 12 \geq 3x$ and $4 \leq 4x$

i.e. $x \leq 4$ and $x \geq 1$

i.e. $1 \leq x \leq 4$

(ii) From (i), we have $1 \leq x^2 \leq 4$

$\Rightarrow 1 \leq x \leq 2$ or $-2 \leq x \leq -1$