KNOX GRAMMAR SCHOOL
TRIAL H.S.C. EXAMINATION, 1991
MATHEMATICS
2 UNIT

R.J.B.D.

Time allowed: Three hours

STUDENT'S NAME: .... CLASS: ....

DIRECTIONS TO CANDIDATES:

* *All* questions may be attempted.
* *All* questions are of equal value.
* All necessary working should be shown in every question.
* Marks may not be awarded for careless or badly arranged work.
* Approved calculators may be used.

This paper contains 6 parts. Each question should be started on a new page and answers handed in, attached to the appropriate cover sheet, marked A, B, C, D, E, F.

PART A

QUESTION 1.

(a) Find, correct to two decimal places, the value of

$$\frac{\pi}{(3.24)^2 - \sqrt{(32.4)}}$$

(b) Simplify (i) $(27a^{27})^{\frac{1}{3}}$, (ii) $\dfrac{2^{n+1} + 2^n}{2^n}$

(c) Solve for m: $\frac{m}{2} + \frac{m}{3} = \frac{m}{4} + 7$

(d) Factorize $X^3Y - 4XY^3$ fully.

(e) Solve for x: $x^4 - 5x^2 - 36 = 0$

(f) If the price of petrol rises at a constant rate of $11\frac{1}{2}\%$ p.a., how much will a litre of petrol cost in five years time if the current price of petrol is 69.9 cents per litre? (Answer to the nearest cent).

QUESTION 2.

(a) If $\sin A = \cos 40°$, $0° \leq A \leq 360°$. Find A.

(b) Factorize $a^2 + 2ab + b^2 - 16$.

(c) If $2 \log_a 5 + \log_a 4 = 2$, find a.

(d) If $a = 6$, $b = -4$ and $c = -3$, evaluate:

(i) $\dfrac{|a| - |b|}{|c|}$ (ii) $\left|\dfrac{a - b}{c}\right|$

(e) Show that $\dfrac{\cos \theta}{1 - \sin \theta} - \tan \theta = \sec \theta$

(f) Solve for X if $(X - 3)^2 < 4$

PART B

QUESTION 3.

(a) Given the parabola $12y = x^2 - 4x + 52$, express it in the form $(x - x_1)^2 = 4a(y - y_1)$. Find:

i) the coordinates of the vertex,
ii) the focal length,
iii) the coordinates of the focus,
iv) the equation of the directrix,
v) the equation of the axis of the parabola.

(b) On the same set of labelled axes, draw a large diagram of the parallel lines:

$2x - y - 6 = 0$

and $2x - y + 2 = 0$. Find:

i) The shortest distance between these two lines.

ii) The equation of a third line which is parallel to these lines and equidistant from them.

(c) On a number plane, shade in the region which satisfies simultaneously the inequations:

$y \leq \log x$, and $x^2 + y^2 \leq 4$.

QUESTION 4.

(a) Find the derivatives of the following:

i) $\log_e (6x^2 - 3)$

ii) $(x^2 + 5)\, e^x$

iii) $(\text{Cos } x + \text{Sin } x)^3$

(b) Find the equation, in general form, of the normal to the curve $y = x^3 - x + 5$ at the point (2, 11).

(c) Show that the curve $y = x^4 + 2x^2$ is concave upwards for all values of x.

(d) For a certain curve, $\frac{d^2y}{dx^2} = 6x$, find the equation of the curve if it passes through the point (1,-2) with a gradient of -3.

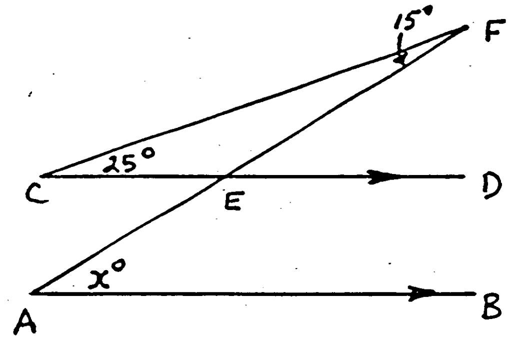
PART C

QUESTION 5.

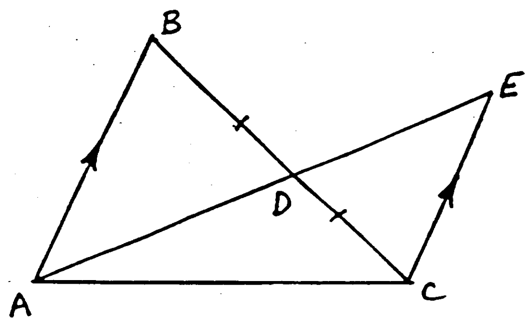
(a) Given CD || AB, $\hat{CFE} = 15°$, $\hat{ECF} = 25°$.

Find the value of x.

(Give a diagram and all reasons in your answer.)



(b)



The median AD of the triangle ABC is produced to E such that AB || CE.

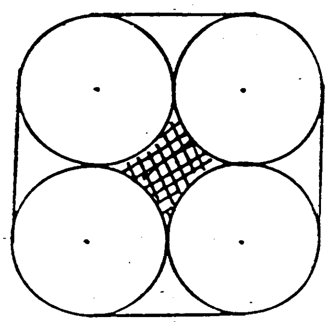
Prove that AB = CE.

(Give a diagram and all reasons in your answer.)

(c) Four pipes, each of diameter 2m, are held together by a strap, as shown in the diagram opposite.

Find, in terms of π ,

i) the length of the strap,
ii) the area of the shaded space between the pipies.



QUESTION 6.

(a) Find k so that 7k, 8k + 4 and 12k - 4 form an arithmetic sequence.

(b) How many terms of a series, 4 + 9 + 14 + ..., must be added to give a sum of 1134?

(c) The limiting sum of a G.P. is 93, and the common ratio is 1/3.

Find: (i) the first term,

(ii) the sum of the first 5 terms.

(d) Two cyclists enter a 24 hour endurance race. Rider "A" intends to cover 40 km in the first hour, 39 km in the second hour, 38 km in the third hour, and so on until the end.

The other rider, "B", intends to cover only 15 km in the first hour then increase by 1 km per hour each hour until a maximum of 30 km per hour is reached, and maintain this until the end.

If the schedules are maintained, which one would win the contest, and by how far?

PART D

QUESTION 7.

(a) Find: (i) $\int (4 - 3x)^5 \, dx$, (ii) $\int \operatorname{Sec}^2 (3x - 1) \, dx$

(iii) $\int_{\pi/6}^{\pi/3} \frac{\cos x}{1 + \sin x} \, dx$. Give your answer correct to 2 d.p.

(b) Evaluate $\int_0^2 e^{x^2} \, dx$, using trapezoidal rule with 4 sub-intervals. (i.e. 5 ordinates). Answer to 3 sig.fig.

(c) Two cars leave a point A at the same time. One car travels at an average speed of 65 km per hour along a straight road in a direction 138°T. The other car averages 80 km per hour along another straight road in a direction 240°T.
How far apart are the cars after 2½ hours? (Draw a neat sketch showing all the information.)

QUESTION 8.

(a) i) Find the coordinates of the points of intersection of the curve $y = x^2 - x - 2$ and the line $y = x + 1$.

ii) Sketch the curves $y = x^2 - x - 2$ and $y = x + 1$ on the same set of axes, and calculate the area enclosed between them.

(b) Find the exact volume of the solid formed when the region bounded by the curve $y = e^x$, the X-axis, the Y-axis and the line $x = 2$, is rotated about the X-axis.

(cont'd.)......

(c) The probability that an electrical component will work is 0.95. Two of these components are wired together in such a way that the machine will operate if either component works.

i) Find the probability that exactly one component will work.

ii) Find the probability that at least one component will work.

iii) If the probability that the machine will not operate is 0.0001, what is the least number of components needed so that the machine will not operate?

PART E

QUESTION 9.

(a) The number of bacteria N in a colony after t minutes is given by

$$N = 10\,000e^{0.05t}.$$ Find:

i) the number of bacteria after 10 minutes.

ii) the time required for the original number to double,

iii) the rate at which the colony increases when

(α) $t = 10$,

(β) $N = 20\,000$.

(b) A particle moves in a straight line and at time t seconds. Its displacement from a fixed origin in the line is x metres and its velocity is v where $v = t^2 - 5t + 6$.

i) Find the acceleration of the particle when it first comes to rest.

ii) Find the position of the particle when the particle comes to rest for the second time, given that $x = -2$ when $t = 0$.

iii) How far does the particle travel in the first four seconds?

iv) During the first four seconds, at what time is the particle travelling its fastest?

QUESTION 10.

(a) Draw a neat sketch of the graph:

$$y = -3 \text{ Sin } 2x, \text{ in } -\pi \leq x \leq \pi.$$

State the period and amplitude.

(b) Solve: $\sqrt{3}$ Tan $2x + 1 = 0$, for $0 \leq x \leq 2\pi$.

(c) During an airline discount season, Sápmoc Airlines offer a special student holiday deal. Provided 80 students participate, Sapmoc provide a return flight and accommodation for two weeks on Nightmare Island for \$830 per person. However, for every student in excess of the 80 minimum, the fare is reduced by \$5 per person.

i) Let x be the number of students in excess of 80. Determine an expression for R, the revenue (the money) obtained by Sapmoc Airlines under this arrangement.

ii) Determine the number of students needed to enable Sapmoc to maximize its revenue.

iii) What will be the cost per student if Sapmoc Airlines maximizes its revenue?

END OF PAPER

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0.$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0.$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0.$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0.$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0.$$

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Q1. a) 0.65 2dp.

b) (i) $3a^9$ (ii) $\frac{2^n(2+1)}{2^n} = 3$

c) $6m+4m = 3m+7\times 12 \quad 7m = 84 \quad m=12$

d) $xy(x+2y)(x-2y)$

e) $(x^2-9)(x^2+4)=0.$

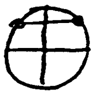
$(x-3)(x+3)(x^2+4)=0.$

$\therefore x=3$ or $x=-3$ (x^2+4 Has No Sol.)

f). $A = 69.9(1.115)^5 = 120$ cent ℓ^{-1}

All [2] each

Q2 a). $A=50°$ or $A=130°$



b) $(a+b)^2-4^2 = (a+b+4)(a+b-4)$.

c) $\log_a 100 = 2 \quad \therefore a=10.$

d) (i) $\frac{6-4}{3} = \frac{2}{3}$ (ii) $\left|\frac{6--4}{-3}\right| = \frac{10}{3}$

e) $\frac{\cos\theta}{1-\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\cos^2\theta+\sin^2\theta-\sin\theta}{\cos\theta(1-\sin\theta)}$

$= \frac{1-\sin\theta}{\cos\theta(1-\sin\theta)}$

$= \sec\theta.$

f) $|x-3|<2 \quad \therefore -2<x-3<2.$

$1<x<5$

All [2] each

Q3. a) $(x-2)^2 = 12(y-4)$

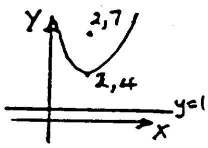
(i) Vertex $(2,4)$

(ii) focal length 3

(iii) focus $(2,7)$

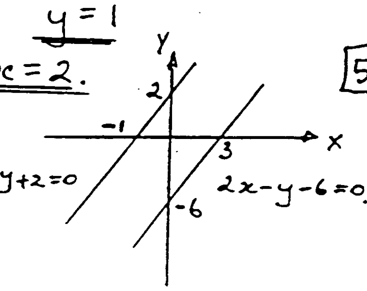
(iv) Directrix $y=1$

(v) Axis $x=2.$



[5]

b)



Take (0,2).

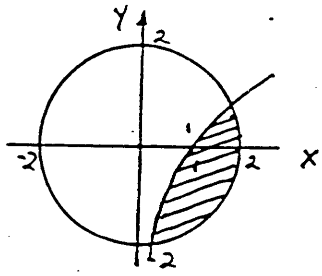
(i) $d = \left|\frac{0-2-6}{\sqrt{2^2+1^2}}\right| = \frac{8}{\sqrt5}$ units

(ii) mid-point on y axis $(0,-2)$ $m=2$

$y=2x-2$. or $2x-y-2=0.$

[5]

c)



[2]

4 a) (i) $\frac{dy}{dx} = \frac{12x}{6x^2-3}$ or $\frac{4x}{2x^2-1}$

(ii) $\frac{dy}{dx} = e^x(2x)+e^x(x^2+5)$

$= e^x(x^2+2x+5)$

(iii) $\frac{dy}{dx} = 3(-\sin x+\cos x)(\cos x+\sin x)^2$

[4]

b) $\frac{dy}{dx} = 3x^2-1 \quad m_T = 11 \,;\, m_N = -\frac{1}{11}$

$y-11 = -\frac{1}{11}(x-2)$

$11y-121 = -x+2 \quad \therefore x+11y-123=0.$

[3]

c) $\frac{dy}{dx} = 4x^3+4x \quad \frac{d^2y}{dx^2} = 12x^2+4.$

$12x^2+4>0$ because $x^2\geq 0$ $\therefore$ c.c up.

[2]

d) $\frac{dy}{dx} = 3x^2+c \quad (-3=3+c \;\therefore c=-6)$

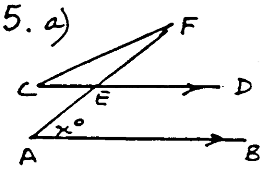
$=3x^2-6.$

$y = x^3-6x+c \quad (-2=1-6+c \;\therefore c=3)$

$\therefore y = x^3-6x+3.$

[3]

5. a)



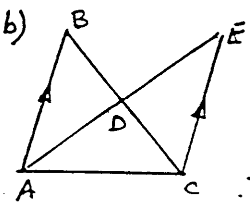
$D\hat{E}F = x°$ alt.ang. AB||CD

$D\hat{E}F = 15°+25°$ Ext.ang. of Δ = int opp.

$\therefore x = 40$

[3]

b)



in ΔABD, ΔECD.

BD = CD AD is med. ∴ D is mid point

$B\hat{A}D = C\hat{E}D$ alt.ang. AB||EC

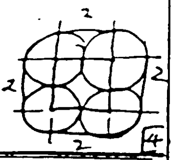
$A\hat{D}B = E\hat{D}C$ Vertical opposite.

∴ ΔABD ≡ ΔECD AAS.

[5]

Q5 c) i) $\ell = 8 + 2\pi$ m

(ii) $A = 4 - \pi$ m^2



[4]

Q6. a). $2(8K+4) = 7K + 12K - 4$

$16K + 8 = 19K - 4$

$3K = 12$ $\therefore K = 4.$ [2]

b) $a = 4$, $d = 5$, $S_n = 1134$

$1134 = \frac{n}{2}(8 + \overline{n-1}\,5)$

$2268 = n(3 + 5n)$

$5n^2 + 3n - 2268 = 0$

$n = -\frac{3}{10} \pm \frac{\sqrt{9 + 45360}}{10}$ $\therefore n = -\frac{3}{10} \pm \frac{213}{10}.$

$\therefore n = 21$ ($n = -21.6$ is not a sol $n > 0$) [4]

c) $93 = \frac{a}{1 - \frac{1}{3}}$ $\therefore$ (i) $a = 62.$

(ii) $S_5 = \frac{62(1 - \frac{1}{3}^5)}{1 - \frac{1}{3}} = 92.617$ (3dp.) [2]

d). "A" $40 + 39 + 38 + \ldots\ldots + 17$
$u_1 + u_2 + \ldots\ldots + u_{24}$

$S_{24} = 12(40 + 17) = 684.$

"B" $15 + 16 + 17 \ldots\ldots + 30$
$u_1 + u_2 + u_3 + \ldots u_{16}.$ $+ 8 \times 30$

$S_{24} = 8(30 + 15) + 240 = 600.$

A wins by 84 km. [4]

Q7. a) (i) $\int (4 - 3x)^5 dx = -\frac{1}{18}(4 - 3x)^6 + c.$

(ii) $\int \sec^2(3x - 1)dx = \frac{1}{3}\tan(3x - 1) + c$

(iii) $\int_{\pi/6}^{\pi/3} \frac{\cos x}{1 + \sin x} dx = \log(1 + \sin x)\Big]$

$= \log(1 + \frac{\sqrt{3}}{2}) - \log(1 + \frac{1}{2})$

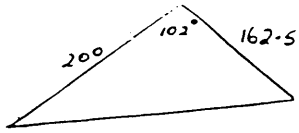
$= 0.22.$ (2dp.) [4]

b) $x_0 = 0$ $x_1 = \frac{1}{2}$ $x_2 = 1$ $x_3 = 1\frac{1}{2}$ $x_4 = 2$
$y_0 = 1$ $y_1 = 1.284$ $y_2 = 2.7183$ $y_3 = 9.4877$ 54.598

$\int_0^2 e^{x^2} dx \doteqdot \frac{\frac{1}{2}}{2}\left[1 + 54.598 + 2(1.284 + 2.7183 + 9.4877)\right]$

$\doteqdot \frac{1}{4}(82.5782362) = 20.6$ (3S.F.)

c).



$d^2 = 200^2 + (162.5)^2 - 2(200)(162.5)\cos 102.$

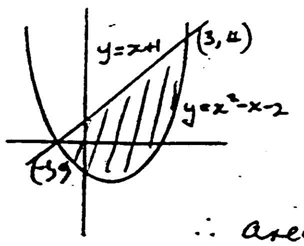
$= 40000 + 26406.25 + 13514.26.$

$= 79920.51$ $\therefore d = 282.7$ km (1dp) [6]

Q8. a) i) $x^2 - x - 2 = x + 1$ $\therefore x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$ $\therefore$ points $(3, 4)$. $(-1, 0)$ [1]

(ii)



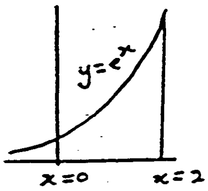
$A = \int_{-1}^{3} (x+1) - (x^2 - x - 2)\, dx$

$= \int_{-1}^{3} -x^2 + 2x + 3\, dx.$

$= -\frac{1}{3}x^3 + x^2 + 3x\Big]_{-1}^{3}$

$\therefore$ area $= 10\frac{2}{3}$ units^2. [4]

b)

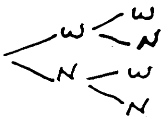


$V = \pi \int_0^2 e^{2x} dx$

$= \pi\, \frac{1}{2}e^{2x}\Big]_0^2$

$\therefore$ vol $= \frac{\pi}{2}(e^4 - 1)$ units^3 [3]

c). Let $P(W) = 0.95$ $P(N) = 0.05$



(i) $P(1\text{ comp}) = WN + NW.$

$= 2(0.95)(0.05)$

$= 0.095$

(ii) $P(\text{at least } 1 . W) = W.W + WN + NW.$

$= (0.95)^2 + 0.095$

$(1 - .05^2)$ $= 0.9975$

(iii) $P(\text{Not work}) = 0.0001 = (0.05)^n$

$\therefore n \log 0.05 = \log 0.0001$

$\therefore n = 3.075$

$\therefore$ least NO. of comp is 4 [4]

Q9. a) $N = 10\,000\, e^{0.05 \times 10}$

(i) Number = 16 487 bacteria

(ii) $2 = e^{0.05t}$ $\therefore 0.05t = \ln 2$

$\therefore$ time is 13.86 mins (4 S.F.)

(iii) α) $\frac{dN}{dt} = 500 e^{0.05 \times 10}$

Rate = 824.4 bacteria p. min (4 S.F.)

β) $\frac{dN}{dt} = kN = 0.05 \times 20\,000$

$\therefore$ Rate is 1000 bac. per min

[4]

b) (i) $v = t^2 - 5t + 6$

when $v = 0$ $(t-3)(t-2) = 0$

$\therefore t = 2$ or $t = 3$

acc $= 2t - 5$

when $t = 2$ acc $= -1$ m sec^{-2}

[2]

(ii) $x = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t + c$.

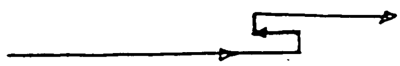
when $t = 0$ $x = -2$ $\therefore c = -2$.

$\therefore x = \frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t - 2$.

when $t = 3$ $x = \frac{5}{2}$ $\therefore$ pos is $\frac{5}{2}$ units (to right)

[2]

(iii)



$t = 0$ $x = -2$

$t = 2$ $x = \frac{8}{3} = 2.67$

$t = 3$ $x = \frac{5}{2}$. 2.5

$t = 4$ $x = 3\frac{1}{3}$.

Dist $= 4\frac{2}{3} + \frac{1}{6} + \frac{5}{6} = 5\frac{2}{3}$ metres

[2]

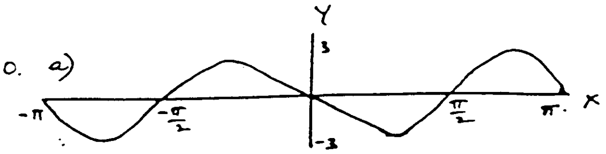
(iv) at $t = 0$ $V = 6$

at $t = 4$ $v = 2$.

$\therefore$ Max speed is when $t = 0$

[2]

Q10. a)



Period $= \frac{2\pi}{2} = \pi$. Amplitude = 3

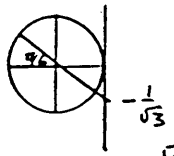
[4]

b) $\tan 2x = -\frac{1}{\sqrt{3}}$ $\quad 0 \le x \le 2\pi$

$\quad 0 \le 2x \le 4\pi$.

$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$

$\therefore x = \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$



[4]

c) (i) $R = (80 + x)(830 - 5x)$

$R = 66400 + 430x - 5x^2$

(ii) $\frac{dR}{dx} = 430 - 10x$ $\quad \frac{d^2R}{dx^2} = -10$.

For max $\frac{dR}{dx} = 0$ and curve concave up.

$430 - 10x = 0$ $\therefore x = 43$.

$\frac{d^2R}{dx^2} = -10 < 0$ $\therefore$ concave down $\therefore$ max.

$\therefore$ max No of students $80 + 43 = 123$

(iii) Cost per student $= 830 - 5(43)$

$= \$615$.

[4]

[120]