YEAR 12 – TRIAL 2001 – Mathematics

QUESTION 1 — MARKS

a) Find the value of e^2 correct to two decimal places. — 1

b) Factorise $49x^2 - 4$. — 1

c) Find the exact value of $\tan\frac{\pi}{3}$. — 1

d) Express $3\sqrt{18} - 4\sqrt{8}$ in its simplest surd form. — 2

e) Solve $1 - 2x < 5$ and graph the solution on the number line. — 3

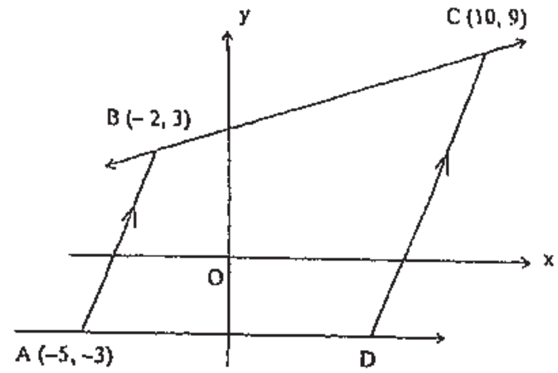
f) Solve the simultaneous equations : — 2

$$x - 2y = 8$$
$$5x + 3y = 1$$

g) Graph the parabola $y = 4 - x^2$ on the number plane, showing its important features. — 2

QUESTION 2 — MARKS

On a number plane the points A $(-5, -3)$, B $(-2, 3)$, C $(10, 9)$ and D form a trapezium, in which AB is parallel to DC. AD is parallel to the x axis.



NOT TO SCALE

a) Show that the gradient of the line DC is 2. — 1

b) Find the equation of the line DC. — 2

c) Show that the coordinates of the point D are $(4, -3)$. — 2

d) Find the distance BC. — 1

e) Find the equation of the circle centred at C with radius BC. — 2

f) Show that the point D lies on the circle. — 1

g) Find the coordinates of the midpoint of BD. — 1

h) The point E is on the line BA produced such that BCDE is a rhombus. Find the coordinates of E. — 2

QUESTION 3 MARKS

a) Differentiate the following functions:

i) $\log_e(3x+1)$ — 1

ii) $e^x \cos x$ — 2

b) Find $\int \frac{4}{2x+1}\,dx$. — 2

c) Evaluate $\int_0^{\frac{\pi}{6}} 2\sec^2 2x\,dx$. — 2

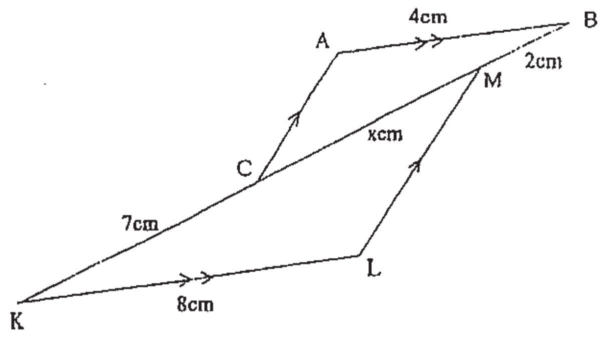
d) Evaluate $\int_0^1 (x^2+1)^2\,dx$. — 2

e) Find the equation of the tangent to the curve $y = e^{2x}$ at the point $P\left(\frac{1}{2}, e\right)$. — 3

109

QUESTION 4 MARKS

a)



NOT TO SCALE

In the diagram AB is parallel to KL and AC is parallel to ML.
AB = 4cm, MB = 2cm and KC = 7cm.

i) Copy or trace the diagram onto your page.

ii) Show that ΔABC is similar to ΔKLM. — 3

iii) Find the length of CM. — 2

b) The discriminant of $x^2 - 2x + p = 0$ is $4 - 4p$.
For what values of p does this equation have real roots? — 2

c) In their first month of operation the 'Computer Experts' sold 200 computers and they increased their sales by 50 computers each month in the first three years of operation.

i) How many computers did they sell in the last month of the third year? — 2

ii) How many computers did they sell over the entire period of three years? — 3

QUESTION 5

MARKS

(a) Consider the curve given by $y = 2x^3 - 3x^2 - 12x$.

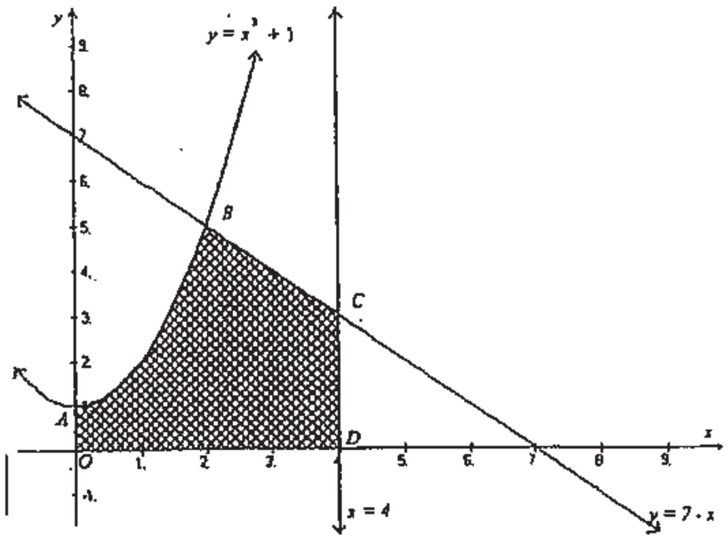
(i) Find $\frac{dy}{dx}$. 1

(ii) Find the coordinates of the two stationary points. 3

(iii) Determine the nature of the stationary points. 2

(iv) Sketch the curve for $-2 \le x \le 3$. 2

(b)



In the diagram, the shaded region $OABCD$ is bounded by $y = x^2 + 1$ the lines $y = 7 - x$, $x = 4$ and the x and y axes.

(i) Show that B has coordinates (2, 5). 1

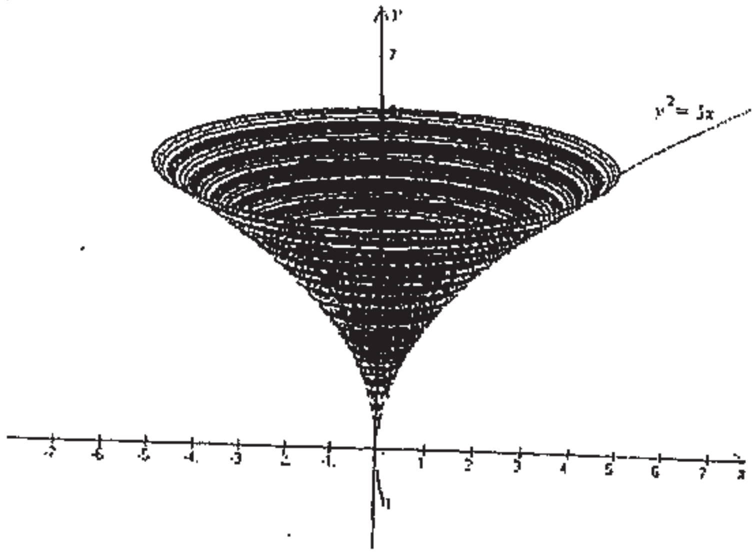
(ii) Use Simpson's rule with 5 function values to estimate the area of the shaded region. 3

110

QUESTION 6

MARKS

a)



The diagram shows the shape of a vessel obtained by rotating about the y axis, the part of the parabola $y^2 = 5x$ between $y = 0$ and $y = 5$. 3

Show that the volume of the vessel is 25π $units^3$.

b) Sketch the parabola $x^2 = 8(y - 2)$ clearly indicating:

i) the vertex 1

ii) the focus 1

iii) give the equation of the directrix 1

c) (i) Show that $x = \frac{2\pi}{3}$ is a solution of $\cos x = \cos 2x$. 1

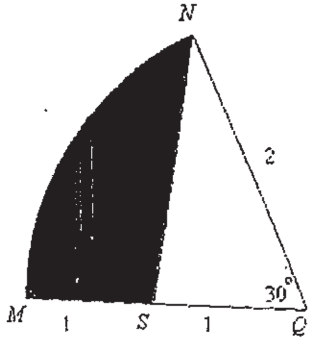
(ii) On the same set of axes, sketch the graphs of $y = \cos x$ and $y = \cos 2x$ for $0 \le x \le \pi$, showing the x coordinate of all points of intersection. 2

(iii) Find the exact area of the region bounded by the curves $y = \cos x$ and $y = \cos 2x$ over the interval $0 \le x \le \frac{2\pi}{3}$ 3

QUESTION 7

MARKS

a) Solve $\log_e x = \log_e (2x-1)$. 2



Not to scale

In the figure, MNQ is the sector of a circle, $\angle MQN = 30^\circ$, $NQ = 2$ cm and $MS = SQ = 1$ cm.

b) (i) Calculate the **exact** length of NS. 2

(ii) Find the **perimeter** of the shaded region MSN. 2

c) If $\log_x a = 3.6$ and $\log_x b = 2$ find :

(i) $\log_x \sqrt[3]{a}$ 1

(ii) $\log_x ab$ 1

(iii) $\log_x \dfrac{b}{a}$ 1

d) A point $P(x,y)$ moves so that its distance from $M(3,0)$ is always twice its distance from the point $N(0,3)$.

Show that the equation of the locus of all points $P(x,y)$ is:

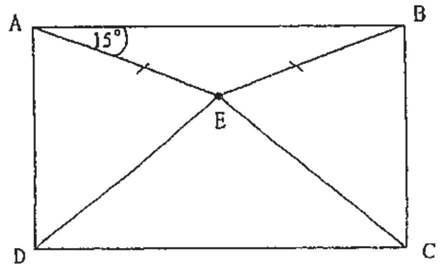
$x^2 + 2x + y^2 - 8y + 9 = 0$ 3

111

QUESTION 8

MARKS

a) E is a point inside the rectangle ABCD such that AE = BE and $\angle$ EAB = 15°.



NOT TO SCALE

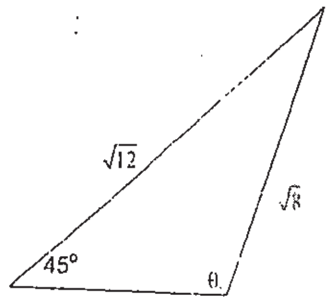
i) Copy or trace the diagram onto your page.

ii) Explain why $\angle$ DAE = $\angle$ CBE. 2

iii) Prove that Δ ADE is congruent to Δ BCE. 3

iv) Hence, prove that Δ DEC is isosceles. 1

b)



NOT TO SCALE

Use the sine rule to find the value of θ where θ is obtuse. 3

c) The geometric series $a + ar + ar^2 + \ldots$ has a second term of $\frac{1}{4}$ and has a limiting sum of 1.

(i) Show that $a = 1 - r$. 1

(ii) Solve a pair of simultaneous equations to find r. 2

QUESTION 9

MARKS

a) (i) Without using calculus, sketch the graph of $y = e^x - 3$. **2**

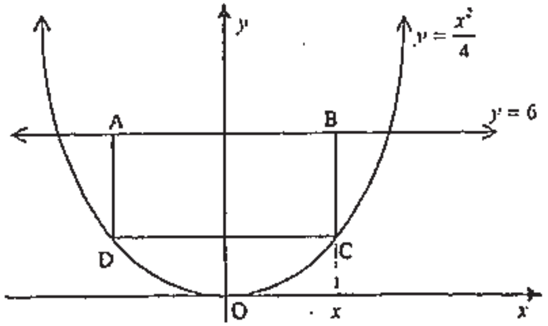
(ii) On the same sketch, find graphically the number of solutions of the equation $e^x - 3 = -x^2$. **1**

b) For the quadratic function $f(x) = Ax^2 - 7x + 3$, $f(2) = -3$.

(i) Find the value of A. **1**

(ii) If the two roots of the equation $f(x) = 0$ are α and β, find the value of $\alpha^2 + \beta^2$. **2**

c) The diagram shows a rectangle ABCD inscribed in the region bounded by the parabola $y = \frac{x^2}{4}$ and the line $y = 6$.



Not to scale

(i) Show that the area of ABCD is given by $A = 12x - \frac{x^3}{2}$. **2**

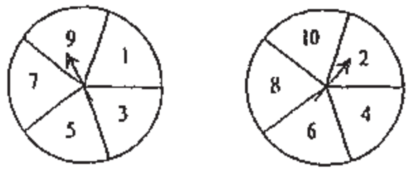
(ii) Find the dimensions of the rectangle so that its area is a maximum. **4**

QUESTION 10

MARKS

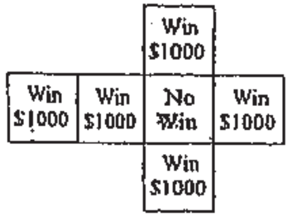
a) A game of chance has two stages.

In stage 1, a player spins the two spinners shown below.



If the sum of the two numbers indicated by the spinners is greater than 11, the player has the chance to move to stage 2.

In stage 2, the player throws the die whose net is shown.



John is to play the game once.

i) What is the probability that he will move to stage 2? **2**

ii) What is the probability that he will win $1000 in the game? **1**

QUESTION 10 (Continued) MARKS

b) A box contains twelve chocolates of exactly the same appearance. Four of the chocolates are hard and eight are soft. Kerrie eats three of the chocolates chosen randomly from the box. Using a tree diagram, or otherwise, find the probability that:

(i) The first chocolate Kerrie eats is hard. 1

(ii) Kerrie eats three hard chocolates. 1

(iii) Kerrie eats exactly one hard chocolate. 2

(c) A man wishes to have $30 000 capital in four years time. He invests a fixed amount of money at the beginning of each month during this time. Interest is accumulated at 6% per annum, compounded monthly.

(i) Let $P be the monthly investment. Show that the total amount, $A, after four years is given by 2

$A = P(1.005 + 1.005^2 + 1.005^3 + \dots + 1.005^{48})$.

(ii) Find, to the nearest dollar, the amount to be invested each month in order to achieve his goal. 3

(a) $e^2 = 7.39$

(b) $49x^2 - 4$

$= (7x-2)(7x+2)$

(c) $\tan\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

(d) $3\sqrt{18} - 4\sqrt{8}$

$= 3.\sqrt{9\times 2} - 4\sqrt{4\times 2}$

$= 3.3\sqrt{2} - 4\times 2\sqrt{2}$

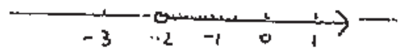
$= 9\sqrt{2} - 8\sqrt{2}$

$= \sqrt{2}$

(e) $1 - 2x < 5$

$-2x < 4$

$x > -2$



(f) $x - 2y = 8 \quad \dots (1)$

$5x + 3y = 1 \quad \dots (2)$

$(1)\times 5$

$5x - 10y = 40 \quad \dots (3)$

$(3) - (2)$

$-13y = 39$

$y = -3$

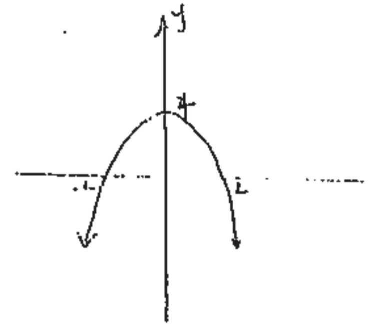
$x - 2\times -3 = 8$

$x + 6 = 8$

$x = 2$

The solution is $x = 2, y = -3$

(g)



Question 2

(a) m of DC = m of AB

$= \frac{3 - -3}{-2 - -5}$

$= \frac{6}{3}$

$= 2$

(b) $y - 9 = 2(x - 10)$

$y - 9 = 2x - 20$

$2x - y - 11 = 0$

(c) As AB is horizontal then y value of D is -3.

Substitute into $2x - y - 11 = 0$

$2x - -3 - 11 = 0$

$2x + 3 - 11 = 0$

$2x = 8$

$x = 4$

D is $(4, -3)$

(d) $BC = \sqrt{(10 - -2)^2 + (9-3)^2}$

$= \sqrt{12^2 + 6^2}$

(e) $(x-10)^2 + (y-9)^2 = 180$

(f) Substitute $(4, -3)$

$(4-10)^2 + (-3-9)^2 = 180.$

$36 + 12^2 = 180$

$180 = 180$

True

$\therefore$ $(4, -3)$ lies on the circle.

(g) Midpoint $= \frac{-2+4}{2}, \frac{3+-3}{2}$

$= (1, 0)$

(h) E is $(-8, -9)$

Question 3

a) (i) $\frac{d}{dx}\left(\log_e(3x+1)\right)$

$= \frac{3}{3x+1}$

) $y = e^x \cos x$

$y' = e^x \cdot -\sin x + \cos x \cdot e^x$

$= (\cos x - \sin x)e^x$

) $\int \frac{4}{2x+1}\,dx$

$= 2\int \frac{2}{2x+1}\,dx$

(c) $\int_0^{\pi/6} 2\sec^2 2x\,dx$

$= \left[\tan 2x\right]_0^{\pi/6}$

$= \tan\frac{\pi}{3} - \tan 0$

$= \sqrt{3}$

(d) $\int_0^1 (x^2+1)^2\,dx$

$= \int_0^1 (x^4 + 2x^2 + 1)\,dx$

$= \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1$

$= \frac{1}{5} + \frac{2}{3} + 1$

$= 1\frac{13}{15}.$

(e) $y = e^{2x}$

$y' = 2e^{2x} = 2e^{2\times\frac{1}{2}} = 2e$

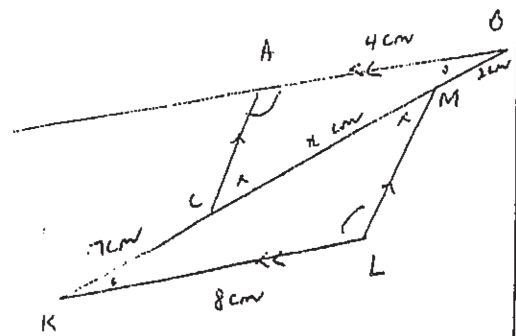
$y - e = 2e\left(x - \frac{1}{2}\right)$

$y - e = 2ex - e$

$y = 2ex$

Question 4 (a)

(i)



(ii) In $\triangle ABC$, $\triangle KLM$

$\angle ACM = \angle KML$ (alternate angles on parallel lines AC, ML)

$\angle BAM = \angle MKL$ (alternate angles on parallel lines AB, KL)

$\therefore \triangle ABC \parallel\!| \triangle KLM$ (AA)

$\frac{BC}{KM} = \frac{AB}{KL}$ (corresponding sides of similar triangles are same ratio)

$\frac{x+2}{x+7} = \frac{4}{8}$

$4x + 28 = 8x + 16$

$12 = 4x$

$x = 3$

$\therefore$ CM = 3 cm

(b) $4 - 4p \geq 0$

$-4p \geq -4$

$p \leq 1$

(c) (i)

$200 + 250 + 300 + \cdots$

A.P

$a = 200 \quad d = 50 \quad n = 36$

$T_{36} = a + (n-1)d$

$= 200 + 35 \times 50$

$= 1950$

(ii) $S_{36} = \frac{n}{2}\left(2a + (n-1)d\right)$

$= 18\left(400 + 35 \times 50\right)$

$= 38700$

Question 5

(a) $y = 2x^3 - 3x^2 - 12x$

$= x(2x^2 - 3x - 12)$

(i) $y' = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x-2)(x+1)$

(ii) $y' = 0$

when $x = 2$ or -1

when $x = 2$, $y = 2\times8 - 3\times4 - 24$

$= -20$

when $x = -1$, $y = (2\times-1) - (3\times1) + 12$

$y = 7$

Stationary pts are $(2, -20)$

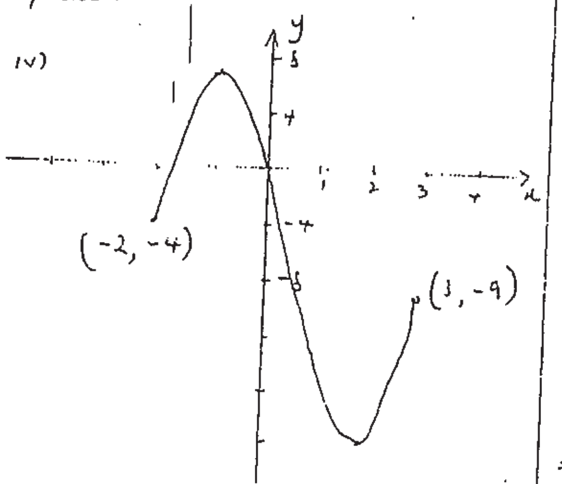
(iii)

x	-2	-1	0	1	2	3
y'	24	0	-12	-12	0	24

/ — \ — /

∴ At $x = -1$ there exists a relative maximum stationary point and at $x = 2$ there is a relative minimum turning point.

(iv)



when $x = -2$

$y = (2\times -8) - (3\times 4) + 24$

$= -4$

when $x = 3$

$y = 2\times 27 - 3\times 9 - 36$

$= -9$

(c)

(i) $y = x^2 + 1$

$y = 7 - x$

sub $(2, 5)$ into $y = x^2 + 1$

when $x = 2$, $y = 2^2 + 1 = 5$

Sub $(2, 5)$ into $y = 7 - x$

$5 = 7 - 2$

∴ $(2, 5)$ satisfies both equations

(ii)

$\frac{2-0}{6}\left(1 + 4\times 2 + 5\right)$

$+ \frac{4-2}{6}\left(5 + 4\times 8 + 3\right)$

$= \frac{2}{6}(1 + 8 + 5) + \frac{2}{6}(24)$

$= \frac{14}{3} + 8$

$= 12\frac{2}{3}$ sq units

(a) $V = \pi\int x^2\,dy$

$= \pi\int_0^5 \left(\frac{y^2}{5}\right)^2 dy = \pi\int_0^5 \frac{y^4}{25}\,dy$

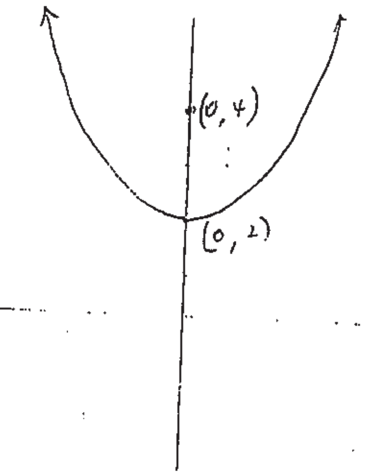
$= \pi\left[\frac{y^5}{125}\right]_0^5$

$= \pi\left[\frac{5^5}{5^3}\right]$

$= 25\pi$ cubic units

(b) $x^2 = 8(y-2)$

$(x-0)^2 = 4a(y-2)$



(i) Focus is $(0, 4)$

(ii) vertex is $(0, 2)$

(iii) Directrix is $y = 0$

(i) $\cos x = \cos 2x$

LHS $= \cos\frac{2\pi}{3}$

$= -\frac{1}{2}$

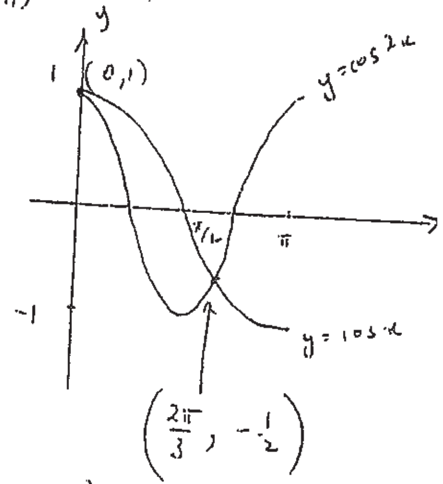
RHS $= \cos 2\times\frac{2\pi}{3}$

$= \cos\frac{4\pi}{3}$

$= -\frac{1}{2}$

∴ $x = \frac{2\pi}{3}$ is a solution

(ii)



(iii) $\int_0^{2\pi/3} (\cos x - \cos 2x)\,dx$

$= \left[\sin x - \frac{1}{2}\sin 2x\right]_0^{2\pi/3}$

$= \sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}\times -\frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}$

$= \frac{2\sqrt{3} + \sqrt{3}}{4} = \frac{3\sqrt{3}}{4}$ sq units

) $\log_e x = \log_e 2x-1$

$x = 2x-1$

$1 = x$

(ii)

$NS)^2 = 2^2+1^2 - 2\times2\times1\cos 30^\circ$

$= 4+1 - 2\times 2\times \frac{\sqrt3}{2}$

$S)^2 = 5-2\sqrt3$

$\sqrt S = \sqrt{5-2\sqrt3}$

$NM = 2\cdot\frac{\pi}{6} = \frac{\pi}{3}$

perimeter $= 1 + \frac{\pi}{3} + \sqrt{5-2\sqrt3}$

ii $\log_x a^{1/3}$

$= \frac13 \log_x a$

$= \frac13 \times 3.6$

$= 1.2$

$\log_x ab$

$= \log_x a + \log_x b$

$= 3.6 + 2$

$= 5.6$

$\log_x b - \log_x a$

$= 2 - 3.6$

(d)

$PM = 2PN$

$PM^2 = 4PN^2$

$(x-3)^2 + (y-0)^2 = 4\left(x^2+(y-3)^2\right)$

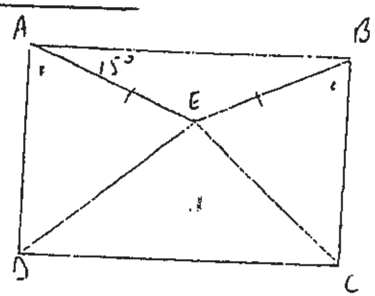
$(x-3)^2+y^2 = 4x^2+4(y-3)^2$

$\cancel{x^2}-6x+9+\cancel{y^2} = 4x^2+4y^2-24y+$

$3x^2+3y^2-24y+6x+27=$

$x^2+y^2-8y+2x+9=0$

Question 8



(ii)

$\angle BAD = \angle ABC = 90^\circ$ (angles of a rectangle)

$\therefore \angle DAE = 90-15^\circ$ (given $B\hat{A}E = 15^\circ$)

$= 75^\circ$

$\angle ABE = 15^\circ$ (base angles of isosceles triangle)

$\therefore \angle CBE = 90^\circ - 15^\circ$

$= 75^\circ$

Hence $\angle CBE = \angle DAE$.

(iii) In ΔADE, ΔBCE

$AE = EB$ (given)

$\angle DAE = \angle EBC$ (above)

$AD = BC$ (sides of a rectangle)

$\therefore \Delta ADE \equiv \Delta BCE$ (SAS)

(iv) $DE = EC$ (corresponding sides of congruent triangles)

$\therefore \Delta DEC$ is isosceles (two sides equal)

(b)

$\frac{\sqrt{12}}{\sin\theta} = \frac{\sqrt3}{\sin 45^\circ}$

$\sqrt3 \sin\theta = \sqrt{12}\times\frac{1}{\sqrt2}$

$\sqrt3\sin\theta = \sqrt6$

$\sin\theta = \frac{\sqrt6}{\sqrt8}$

$\sin\theta = \sqrt{\frac34}$

$= \frac{\sqrt3}{2}$

$\therefore \theta = 120^\circ$

(c) (i) $T_2 = ar = \frac14$

$\frac{a}{1-r} = 1$

$a = 1-r.$

(ii) $a = 1-r$

$ar = \frac14$

$r = \frac{1}{4a}$

$a = 1-\frac{1}{4a}$

$4a^2 = 4a-1$

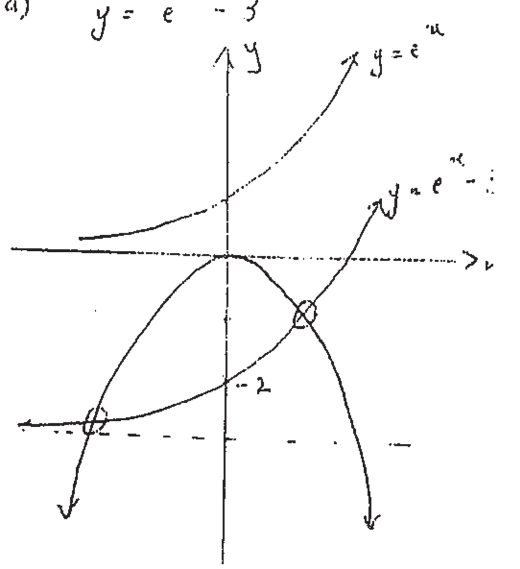
$4a^2-4a+1=0$

$(2a-1)(2a-1)=0$

$a = \frac12$

Question 9

(a) $y = e^x - 3$



(ii) There are 2 solutions

(b) (i) $f(x) = Ax^2-7x+3$

$f(2) = -3.$

$4A-14+3 = -3$

$4A = 8$

$A = 2$

(ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{+7}{2}\right)^2 - 2\times\frac32$

$= \frac{49}{4} - 3$

$= 9\frac14.$

11/2

$$= \left| 12x - \frac{2x^3}{4} \right.$$

$$= 12x - \frac{x^3}{2}$$

(ii) $A' = 12 - \frac{3x^2}{2}$

$0 = 12 - \frac{3x^2}{2}$

$\frac{3x^2}{2} = 12$

$3x^2 = 24$

$x^2 = 8$

$x = \pm\sqrt{8}$ but disregarding −ve as $x > 0$

$x = \sqrt{8}$

x	2	$\sqrt{8}$	3
A'	6	0	$-1\frac{1}{2}$

∴ At $x = \sqrt{8}$ a relative maximum turning point exists.

As the curve is cts and there are no more turning pts in the domain $x > 0$ the rel. max is absolute

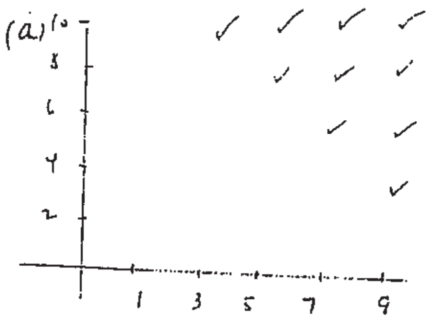
... dimensions are

$2\sqrt{8}$ units $\times \; 6 - \frac{8}{4}$

$= 2\sqrt{8}$ units $\times$ 4 units.

Question 10

(a)



(i) $P(\text{Stage } 2) = \frac{10}{25}$

$= \frac{2}{5}$

(ii) $P(\$1000) = \frac{2}{5} \times \frac{5}{6}$

$= \frac{1}{3}$.

(b)

(i) $P(H) = \frac{4}{12} = \frac{1}{3}$

(ii) $P(HHH) = \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$

$= \frac{1}{55}$

(iii) $P(\text{one hard}) = 3 \times \frac{4}{12} \times \frac{8}{11} \times \frac{7}{10}$

$= \frac{56}{110} = \frac{28}{55}$

117

(c)

	Beg of Month	end of Month
Month 1	P	$P \times 1.005$
Month 2	$P \times 1.005 + P$ $= P(1 + 1.005)$	$P(1 + 1.005) \times 1.005$ $= P(1.005 + 1.005^2)$

Generalising at end of Month 48

$$A = P(1.005 + 1.005^2 + \dots + 1.005^{48})$$

$$A = P \times \frac{1.005(1 - 1.005^{48})}{1 - 1.005}$$

$$30{,}000 = P \times \frac{1.005(1 - 1.005^{48})}{-0.005}$$

$$P = \frac{30{,}000 \times -0.005}{1.005(1 - 1.005^{48})}$$

$= \$551.79$

∴ \$552 to nearest dollar