NEWINGTON COLLEGE

IN FIDE SCIENTIAM

Trial Higher School Certificate Examination 1999

12 MATHEMATICS

2/3 UNIT common

Time allowed : *Three Hours*

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES :

All questions are of equal value.

All questions may be attempted.

In every question, show all necessary working.

Marks may not be awarded for careless or badly arranged work.

Approved silent calculators may be used.

A table of standard integrals is provided for your convenience.

The answers to the ten questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc.

Each bundle must show the candidate's computer number.

The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.

Unless otherwise stated candidates should leave their answers in simplest exact form.

Marks

QUESTION ONE *(12 Marks)*

(a) Simplify $|3|-|-4|$. 1

(b) Express $\frac{3}{\sqrt{5}-1}$ with a rational denominator. 1

(c) Factorise a^3-b^3. 2

(d) Find the exact value of $\cos\frac{\pi}{4}$. 1

(e) Solve for x: $|x-2|>5$ 2

(f) Solve for x: $\log_3(x+2)-\log_3(x-1)=2$ 3

(g) Solve for $0\le x\le 2\pi$: $\tan x=\frac{1}{\sqrt{3}}$ 2

QUESTION TWO *(12 Marks)* **Start this question on a new page**

(a) Consider the points $A(1,4)$ and $B(3,-3)$. 10

(i) Show that B lies on the line $l: 6x+5y-3=0$.

(ii) Show that the equation of the line k passing through A with gradient $\frac{1}{3}$ is $x-3y+11=0$.

(iii) The lines k and l intersect at C. Find the coordinates of C.

(iv) Find the perpendicular distance from the point A to the line l.

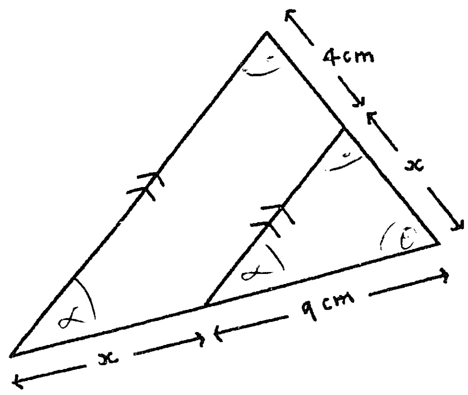
(v) Find the length of the interval BC.

(vi) Hence find the area of ΔABC.

Question two continues ...

Marks

(b) Find the value of x in the diagram below. The diagram is **not** drawn to scale. 2



QUESTION THREE *(12 Marks)* **Start this question on a new page**

(a) Differentiate the following with respect to x: 6

(i) $(x^2+1)^5$,

(ii) $x\tan(x+2)$,

(iii) $\dfrac{\cos x}{x}$.

(b) Evaluate the following integrals: 6

(i) $\int_0^2 e^{2x}\,dx$,

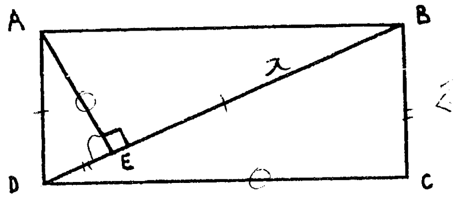
(ii) $\int_1^e \frac{3}{x}\,dx$,

(iii) $\int_0^{\frac{\pi}{12}} \sin 3x\,dx$.

Question four ...

QUESTION FOUR *(12 Marks)* **Start this question on a new page** **Marks**

(a) $ABCD$ is a rectangle and E is a point on the diagonal BD so that AE is perpendicular to BD. 4



(i) Prove that ΔADE is similar to ΔDBC.

(ii) If $AD = 5$ cm and $DE = 2$ cm, find the length of the diagonal BD.

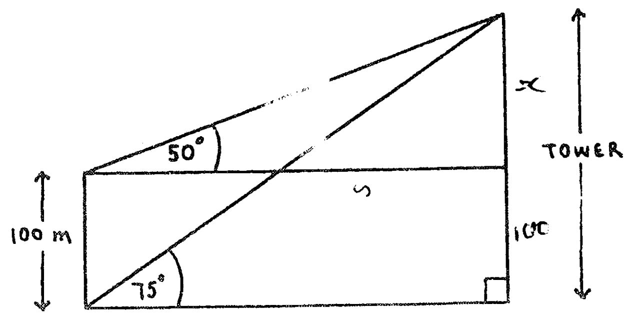
(b) If α and β are the roots of the quadratic equation $x^2 - 5x + 3 = 0$, find the value of: 4

(i) $\alpha + \beta$,

(ii) $\alpha\beta$,

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$.

(c) The angles of elevation of the top of a tower from the top and bottom of a building 100 metres high are 50^o and 75^o respectively. Find the height of the tower, correct to the nearest metre. 4



Question five ...

QUESTION FIVE *(12 Marks)* **Start this question on a new page** **Marks**

(a) (i) Write down the discriminant of $kx^2 - 4x + 3$. 3

(ii) For what values of k does the equation $kx^2 - 4x + 3 = 0$ have real roots?

(b) Sketch the parabola $y^2 = 8(x-2)$, clearly indicating the coordinates of the focus and vertex and the equation of the directrix. 3

(c) $A(4,0)$ and $B(-2,3)$, are fixed points and $P(x, y)$ moves so that its distance from A is always twice its distance from B. 6

(i) Find expressions for PA^2 and PB^2 in terms of x and y.

(ii) Show that the locus of P is a circle with equation $x^2 + y^2 + 8x - 8y + 12 = 0$.

(iii) Find the centre and radius of the circle.

QUESTION SIX *(12 Marks)* **Start this question on a new page**

(a) (i) On the same number plane sketch graphs of the functions $y = |x|$ and $y = x + 3$. 3

(ii) Hence or otherwise solve the equation $|x| = x + 3$.

(b) The sum of the first twelve terms of an arithmetic series is 201 and the sum of the next eight terms is 334. Find the first term and the common difference. 4

Question six continues ...

Marks

(c) A man wishes to have \$30 000 capital in four years time. He invests a fixed amount of money at the beginning of each month during this time. Interest is accumulated at 6% per annum, compounded monthly. 5

(i) Let \$$P$ be the monthly investment. Show that the total amount, \$$A$, after four years is given by $A=P(1.005+1.005^2+1.005^3+\ldots+1.005^{48})$.

(ii) Find, to the nearest dollar, the amount to be invested each month in order to achieve his goal.

QUESTION SEVEN *(12 Marks)* **Start this question on a new page**

(a) The weight W of a baby whale is increasing at a rate proportional to its current weight. The weight in kilograms at time t months after birth is given by $W = Ae^{kt}$ where A and k are constants. 4

(i) Find the value of A if the whale weighed 40 kg at birth.

(ii) Find the exact value of k if the whale weighed 65 kg after one month.

(iii) Calculate the weight of the whale after one year. Give your answer correct to the nearest 100 kg.

(b) Water is flowing out of a sink and the depth of water, d cm, at time t minutes is given by $d = 15 - \frac{t}{8} - \frac{t^3}{4}$. Find the rate at which the depth is decreasing after 2 minutes. 2

Question seven continues ...

Marks

(c) A particle moves in a straight line so that its velocity v metres per second at time t is given by v=2t-8. Initially the particle is at $x = 12$. 6

(i) What is the acceleration of the particle?

(ii) When is the particle at rest?

(iii) Find the displacement as a function of time.

(iv) Find the distance the particle travels in the first 8 seconds.

QUESTION EIGHT *(12 Marks)* **Start this question on a new page**

(a) Consider the function $f(x) = xe^{-x}$. 10

(i) Show that $f'(x) = e^{-x}(1-x)$.

(ii) Show that $f''(x) = e^{-x}(x-2)$.

(iii) Find the coordinates of any stationary points and determine their nature.

(iv) Find the coordinates of any points of inflection.

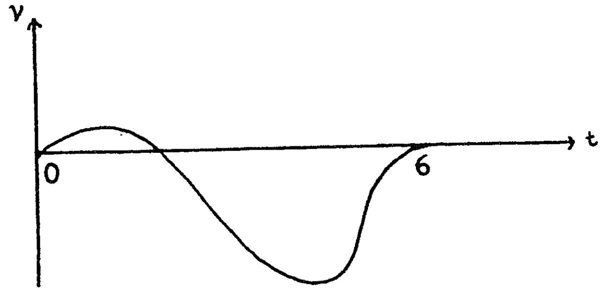
(v) Describe the behaviour of the graph for very large positive and very large negative values of x.

(vi) Sketch the graph of the function $y = f(x)$ indicating all important features.

Question eight continues ...

Marks

(b) A particle moves in a straight line in the period between $t = 0$ and $t = 6$. Its velocity v at time t is shown on the graph below. Copy or trace this graph. 2

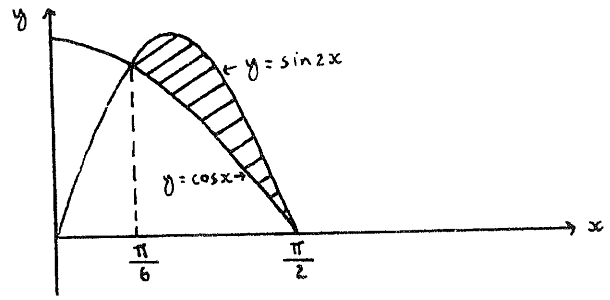


(i) On the time axis, label with the letter A the times at which the acceleration of the particle is zero.

(ii) On the time axis, label with the letter M the time at which the acceleration is a maximum.

QUESTION NINE *(12 Marks)* **Start this question on a new page**

(a) The diagram below shows the graphs of the functions $y = \sin 2x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$. The two graphs intersect at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$. Calculate the area of the shaded region. 3



(b) (i) Sketch the region bounded by the curve $y = e^x + 1$ and the x-axis, $0 \le x \le 2$. 5

(ii) Find the volume of the solid formed when this area is rotated about the x-axis.

Question nine continues ...

Marks

(c) (i) Sketch the curve $y = 3\sin 4x$ for $0 \le x \le \pi$. 4

(ii) Consider the line $y = mx + 1$. Through what point will the line always pass irrespective of the value of m?

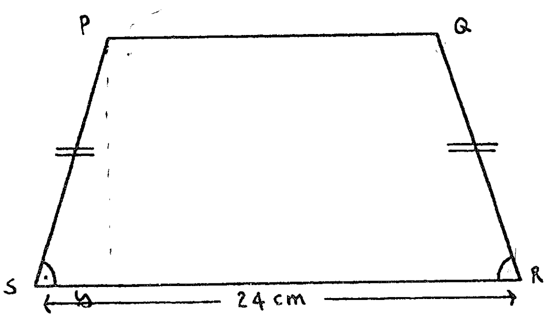
(iii) What is the maximum number of solutions the equation $3\sin 4x = mx + 1$ can have in the domain $0 \le x \le \pi$? Give a reason for your answer.

QUESTION TEN *(12 Marks)* **Start this question on a new page**

(a) (i) Find the derivative of $2x\sin 2x + \cos 2x$. 5

(ii) Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$.

(b) $PQRS$ is a trapezium in which RS has length 24 cm, $QR = PS$, $\angle PSR = \angle QRS$ and $SP + PQ + QR = 42$ cm. 7



(i) Let $QR = PS = x$ cm. Show that the trapezium has perpendicular height $\sqrt{18x - 81}$ cm.

(ii) Show that the area of the trapezium is given by $A = (33 - x)\sqrt{18x - 81}$.

(iii) Hence find the maximum possible area of the trapezium.

END OF PAPER

2U TRIAL SOLUTIONS

QUESTION ONE

(a) $|3| - |-4| = -1$

(b) $\dfrac{3}{\sqrt{5}-1} = \dfrac{3\sqrt{5}+3}{4}$

(c) $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

(d) $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

(e) $|x-2| > 5 \quad \therefore \; x > 7 \text{ or } x < -3$

(f) $\log_3(x+2) - \log_3(x-1) = 2$

$\therefore \; \log_3\left(\frac{x+2}{x-1}\right) = 2$

$\therefore \; \frac{x+2}{x-1} = 9$

$\therefore \; x + 2 = 9x - 9$

$\therefore \; x = \frac{11}{8}$

(g) $\tan x = \frac{1}{\sqrt{3}} \quad \therefore \; x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$

QUESTION TWO

(a) $A(1,4) \quad B(3,-3)$

(i) $\text{LHS} = 6x + 5y - 3$

$= 18 - 15 - 3$

$= 0 = \text{RHS} \quad \therefore$ B lies on l.

(ii) $y - 4 = \frac{1}{3}(x-1)$

$3y - 12 = x - 1$

$x - 3y + 11 = 0$

(iii) $6x + 5y = 3$

$6x - 18y = -66$

$23y = 69 \quad \therefore \; y = 3,\; x = -2 \quad \therefore \; C(-2,3)$

(iv) $d = \dfrac{|6 + 20 - 3|}{\sqrt{61}} = \dfrac{23}{\sqrt{61}}$ units

(v) $d_{BC} = \sqrt{36 + 25} = \sqrt{61}$ units

(vi) $\text{Area} = \frac{1}{2} \times \sqrt{61} \times \frac{23}{\sqrt{61}} = 11.5 \text{ units}^2$.

(b) $\frac{x}{4} = \frac{9}{x}$ (ratio of intercepts on transversals across parallel lines equal.)

$\therefore \; x^2 = 36$

$\therefore \; x = 6$

QUESTION THREE

(a) (i) $\frac{d}{dx}\left[(x^2+1)^5\right] = 10x(x^2+1)^4$

(ii) $\frac{d}{dx}\left\{x\tan(x+2)\right\} = \tan(x+2) + x\sec^2(x+2)$

(iii) $\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \dfrac{-x\sin x - \cos x}{x^2}$

(b) (i) $\int_0^2 e^{2x}\,dx = \frac{1}{2}\left[e^{2x}\right]_0^2 = \frac{1}{2}(e^4 - 1)$

(ii) $\int_1^e \frac{3}{x}\,dx = 3\left[\ln x\right]_1^e = 3$

(iii) $\int_0^{\frac{\pi}{12}} \sin 3x\,dx = -\frac{1}{3}\left[\cos 3x\right]_0^{\frac{\pi}{12}} = \frac{2-\sqrt{2}}{6}$

QUESTION FOUR

(a)(i) In Δ's ADE and DBC:

$\angle AED = \angle DCB = 90°$ (given)

$\angle ADE = \angle DBC$ (alternate angles, $AB \parallel DC$)

$\therefore \; \Delta ADE \;|||\; \Delta DBC$ (equiangular)

(ii) $\frac{AD}{BD} = \frac{DE}{BC}$ (corresponding sides of similar Δ's in the same ratio)

$\therefore \; \frac{5}{BD} = \frac{2}{5}$

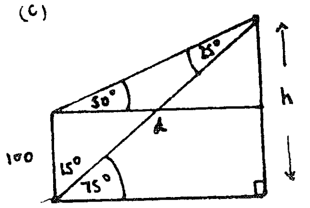
$\therefore \; BD = 12\frac{1}{2}$ cm

(b) (i) $\alpha + \beta = 5$

(ii) $\alpha\beta = 3$

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{5}{3}$

(c)



$\frac{d}{\sin 140°} = \frac{100}{\sin 25°}$

$d = \frac{100 \sin 140°}{\sin 25°}$

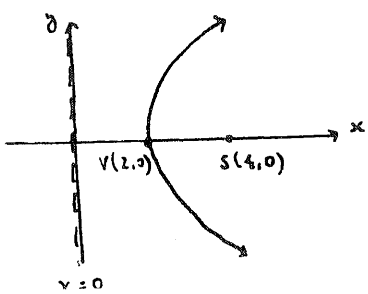
$h = d\sin 75° \doteqdot 147$ m.

QUESTION FIVE

(a) (i) $\Delta = 16 - 12k$

(ii) $16 - 12k \geq 0 \quad \therefore \; k \leq \frac{4}{3}$

(b)



(c)(i) $A(4,0)$ $B(-2,3)$ $P(x,y)$

$PA^2 = (x-4)^2 + y^2$

$PB^2 = (x+2)^2 + (y-3)^2$

(ii) $PA = 2PB$

$\therefore PA^2 = 4PB^2$

$\therefore (x-4)^2 + y^2 = 4(x+2)^2 + 4(y-3)^2$

$x^2 - 8x + 16 + y^2 = 4x^2 + 16x + 16 + 4y^2 - 24y + 36$

$3x^2 + 3y^2 + 24x - 24y + 36 = 0$

$x^2 + y^2 + 8x - 8y + 12 = 0$

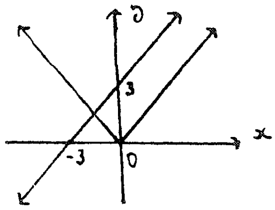
(iii) $x^2 + 8x + 16 + y^2 - 8y + 16 = -12 + 16 + 16$

$(x+4)^2 + (y-4)^2 = 20$

$\therefore C(-4,4)$ $r = 2\sqrt{5}$

QUESTION SIX

(a) (i)



(ii) $-x = x + 3$

$\therefore x = -\frac{3}{2}$

(b) $S_{12} = 201$ $S_{20} = 535$

$201 = 6(2a + 11d)$ $\therefore 67 = 4a + 22d$

$535 = 10(2a + 19d)$ $\therefore 107 = 4a + 38d$

$40 = 16d$

$\therefore d = \frac{5}{2}$, $a = 3$

(c) (i) After first month he has $P(1.005)$

After second month he has $[P(1.005) + P](1.005)$

i.e. after the 2nd month he has $P(1.005)^2 + P(1.005)$

$\vdots$

after 48 months he has $P(1.005)^{48} + P(1.005)^{47} + \ldots$

$\ldots + P(1.005)$

(ii) $30\,000 = P(1.005)(1 + 0.005 + \ldots + 0.005^{47})$

$30\,000 = P(1.005)\left[\frac{1 - 1.005^{48}}{1 - 1.005}\right]$

$\therefore P \doteqdot \$552$

QUESTION SEVEN

(a) (i) $A = 40$

(ii) $W = 40e^{kt}$

$\therefore 65 = 40e^{k}$

$\therefore e^{k} = \frac{13}{8}$

$\therefore k = \log_e\left(\frac{13}{8}\right)$

(iii) $W = 40e^{12k}$

$\doteqdot 13\,600$ kg

(b) $d = 15 - \frac{t}{8} - \frac{t^3}{4}$

$d' = -\frac{1}{8} - \frac{3t^2}{4}$

when $t = 2$ $d' = -\frac{1}{8} - 3 = -3\frac{1}{8}$

$\therefore$ depth decreasing at $3\frac{1}{8}$ cm/min.

(c) $v = 2t - 8$ $t = 0$ $x = 12$

(i) $a = 2$ m/s^2

(ii) $t = 4$ seconds

(iii) $x = t^2 - 8t + c$

when $t = 0$ $x = 12$ $\therefore c = 12$

$\therefore x = t^2 - 8t + 12$

(iv) when $t = 0$ $x = 12$

when $t = 4$ $x = -4$

when $t = 8$ $x = 12$

$\therefore$ particle travels 32m in the first 8 seconds.

QUESTION EIGHT

(a) $f(x) = xe^{-x}$

(i) $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$

(ii) $f''(x) = -e^{-x}(1-x) - e^{-x}$

$= -e^{-x}(1 - x + 1)$

$= e^{-x}(x - 2)$

(iii) $f'(x) = 0$ when $x = 1$

$\therefore$ stat. point at $(1, \frac{1}{e})$

$f''(1) = \frac{1}{e}(1-2) = -\frac{1}{e} < 0$

$\therefore$ maximum turning point at $(1, \frac{1}{e})$

(iv) $f''(x) = 0$ when $x = 2$

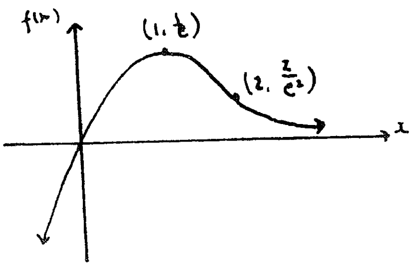
x	1	2	3
$f''(x)$	$-\frac{1}{e}$	0	$\frac{1}{e^3}$
	$\cap$		$\cup$

$\therefore$ point of inflection at $(2, \frac{2}{e^2})$

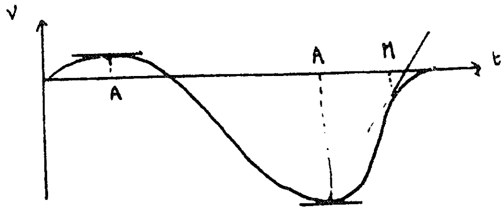
(v) as $x \to \infty$ $f(x) \to 0$

as $x \to -\infty$ $f(x) \to -\infty$

(vi)



(b)



QUESTION NINE

(a) Area $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)\, dx$

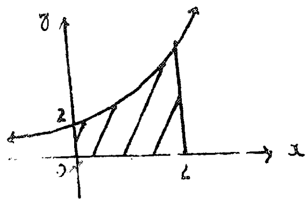
$= \left[-\frac{\cos 2x}{2} - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= -\frac{\cos \pi}{2} - \sin\frac{\pi}{2} + \frac{\cos\frac{\pi}{3}}{2} + \sin\frac{\pi}{6}$

$= \frac{1}{2} - 1 + \frac{1}{4} + \frac{1}{2}$

$= \frac{1}{4}$ units^2

(b)



(ii) $V = \pi \int_0^2 (e^x + 1)^2\, dx$

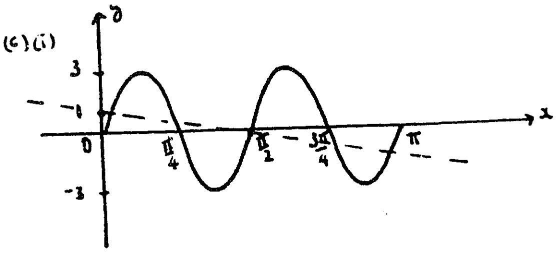
$= \pi \int_0^2 (e^{2x} + 2e^x + 1)\, dx$

$= \pi \left[\frac{e^{2x}}{2} + 2e^x + x \right]_0^2$

$= \pi \left(\frac{e^4}{2} + 2e^2 + 2 - \frac{1}{2} - 2 \right)$

$= \frac{\pi}{2}(e^4 + 4e^2 - 1)$ units^3.

(c)(i)



(ii) $(0, 1)$

(iii) maximum of 5 solutions.

QUESTION TEN

(a) (i) $\frac{d}{dx}(2x\sin 2x + \cos 2x)$

$= 2x(2\cos 2x) + 2\sin 2x - 2\sin 2x$

$= 4x\cos 2x$

(ii) $\int_0^{\frac{\pi}{2}} x\cos 2x\, dx$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 4x\cos 2x\, dx$

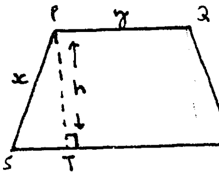
$= \frac{1}{4}\left[2x\sin 2x + \cos 2x \right]_0^{\frac{\pi}{2}}$

$= \frac{1}{4}(\pi \sin\pi + \cos\pi - \cos 0)$

$= -\frac{1}{2}$

(b)(i) let $PQ = y$ $\therefore 2x + y = 42$

$\therefore y = 42 - 2x$



$ST = \frac{24 - y}{2} = 12 - \frac{y}{2} = 12 - 21 + x = x - 9$

$x^2 = (x-9)^2 + h^2$

$h^2 = x^2 - (x-9)^2 = 18x - 81$ $\therefore h = \sqrt{18x - 81}$

(ii) Area $= \frac{\sqrt{18x-81}}{2}(24 + y)$

$= \frac{\sqrt{18x-81}}{2}(24 + 42 - 2x)$

$= \sqrt{18x-81}\,(33 - x)$

(iii) $\frac{dA}{dx} = -\sqrt{18x-81} + \frac{9(33-x)}{\sqrt{18x-81}}$

$= \frac{9(33-x) - (18x-81)}{\sqrt{18x-81}}$

$$= \frac{378 - 27x}{\sqrt{18x - 81}}$$

$= 0$ when $x = 14$

x	13	14	15
$\frac{dA}{dx}$	$\frac{27}{\sqrt{153}}$	0	$\frac{-27}{\sqrt{189}}$

+ve 0 -ve

$\therefore$ max. when $x = 14$

Maximum possible area is $57\sqrt{19}$ units^2.