NEWINGTON COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

12 MATHEMATICS

Time allowed - 3 hours

(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- All questions are of equal value.
- All questions may be attempted.
- In every question, show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Approved silent calculators may be used.
- A table of standard integrals is provided for your convenience.
- The answers to the ten questions in this paper are to be returned in separate bundles clearly marked Question 1, Question 2 etc. Start each question in a new booklet.
- Each booklet must show the candidate's computer number.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- Unless otherwise stated candidates should leave their answers in simplest exact form.

Question 1 *(12 Marks)* **Marks**

a) Express 135^0 in radians in terms of π.

b) Factorise $2x^2 - 18$.

c) Solve for x: $|x-3| < 4$.

d) Find the exact value of $\cos\frac{7\pi}{6}$.

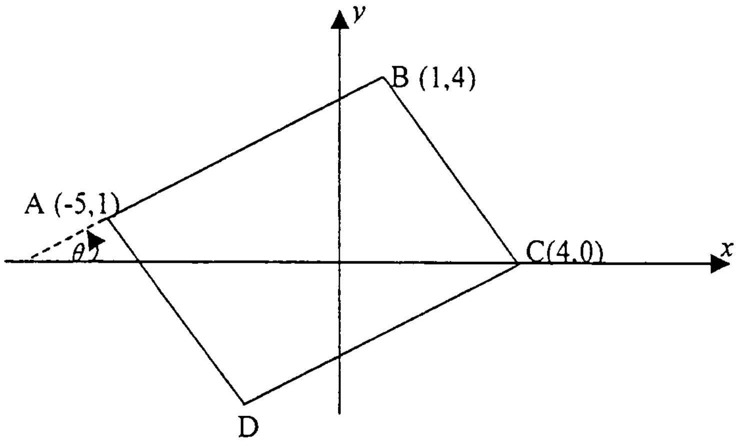
e) Find the value of e^{π}, correct to 3 significant figures.

f) Find integers a and b such that $(2-\sqrt{3})^2 = a + b\sqrt{3}$.

g) Simplify $\log_3\left(\frac{1}{9}\right)$.

Question 2 *(12 marks)* Start this question in a new booklet.

a) **10**



i) Find the gradient of AB.

ii) If BA is extended to the x axis find θ to the nearest degree.

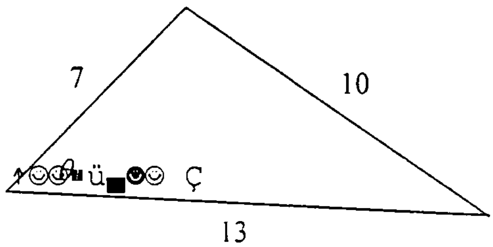
iii) Show that AB has equation $x - 2y + 7 = 0$.

iv) Show that the perpendicular distance from C to AB is $\frac{11}{\sqrt{5}}$ units.

v) If ABCD is a parallelogram find the area of ABCD.

vi) If ABCD is a parallelogram find the coordinates of D.

b) Find θ to the nearest minute. **2**



Question 3 *(12 marks)* Start this question in a new booklet. **Marks**

a) Differentiate **5**

i) $\tan 3x$

ii) $\dfrac{x}{e^{2x}}$

iii) $\log_e(\sin 3x)$

b) Find **4**

i) $\int \cos 3x\,dx$

ii) $\int \dfrac{x^3 + 2x^2 - 3}{x^2}\,dx$

c) Find the equation of the normal to the curve $y = (3x+1)^3$ at the point where $x = 1$. **3**

Question 4 *(12 marks)* Start this question in a new booklet.

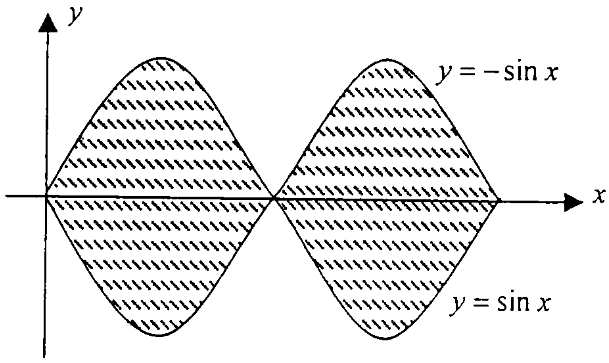
a) Let α and β be the roots of the equation $3x^2 + 5x - 1 = 0$. **3**

Find i) $\alpha + \beta$,

ii) $\dfrac{1}{\alpha\beta}$,

iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$.

b) **4**



Find the area bounded by $y = \sin x$ and $y = -\sin x$, as shown in the diagram above.

Question 4 continued. **Marks**

c) A balloon accelerates vertically, so that the change of height (h) in metres is given by $\frac{dh}{dt} = 10 - \frac{1}{3}t$, where t is time in seconds after it takes off. Initially the balloon is 1 metre about the ground. 5

i) Find the height after 6 seconds.
ii) What is the maximum height that the balloon can reach?

Question 5 *(12 marks)* Start this question in a new booklet.

a) In the Snowy Mountains the number of Corroboree frogs (Q) has been in a slow decline because of bush fires and pollution. The rate of decline is given by $\frac{dQ}{dt} = -kQ$. In 1998 the population was estimated at 2100 and in the most recent survey in 2003 it was estimated at 1050. 7

i) Show that the function $Q = Q_0 e^{-kt}$ satisfies the equation $\frac{dQ}{dt} = -kQ$.
ii) Find the value of the constant Q_0.
iii) Show that $k = 0.2 \ln 2$.
iv) In how many years will there be only one frog remaining and hence the frogs become extinct in the Snowy Mountains?

b) The first three terms of an arithmetic series are $25 + 19 + 13 + \ldots\ldots$ 5

i) Find the 20^{th} term.

ii) How many terms will it take for the sum of the terms to become negative?

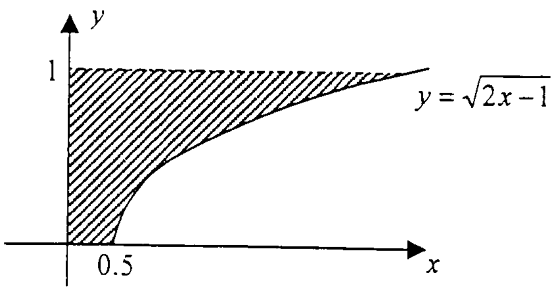
Question 6 *(12 marks)* Start this question in a new booklet.

Consider the curve given by $y = 1 + 3x - x^3$ for $-2 \le x \le 3$.

i) Find the coordinates of the stationary points and determine their nature.
ii) Find the point of inflexion.
iii) Sketch the curve for $-2 \le x \le 3$.
iv) What is the minimum value of y for $-2 \le x \le 3$?

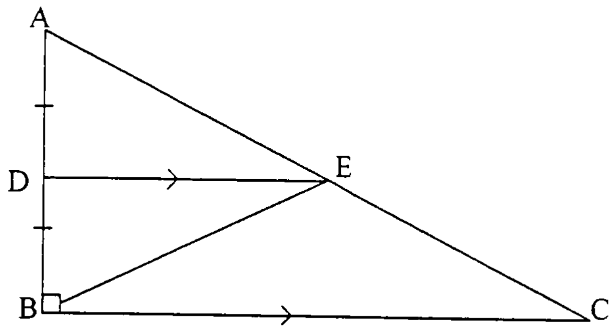
Question 7 *(12 marks)* Start this question in a new booklet. **Marks**

a) 4



The area shown is rotated about the y axis. Find the exact value of the volume of the solid that is generated.

b) 8



In the right angled triangle ABC, D is the mid point of AB

i) Prove that $\angle ADE$ is a right angle.

ii) Prove that $\Delta AED \equiv \Delta BED$.

iii) Prove that BE = EC.

Question 8 *(12 marks)* Start this question in a new booklet.

a) i) Factorise $u^2 - 6u - 16$. 3

ii) Hence or otherwise, solve for x:

$$[\log_2 x]^2 - 6[\log_2 x] - 16 = 0.$$

b) Find an approximate value for $\int_1^3 \log_e x\,dx$ using five function values 3

and the Trapezoidal Rule. (Answers correct to 2 decimal places.)

c) A parabola has equation: $x^2 = 8(4-y)$. 3

Find i) the focal length,

ii) the equation of the directrix,

iii) the coordinates of the focus.

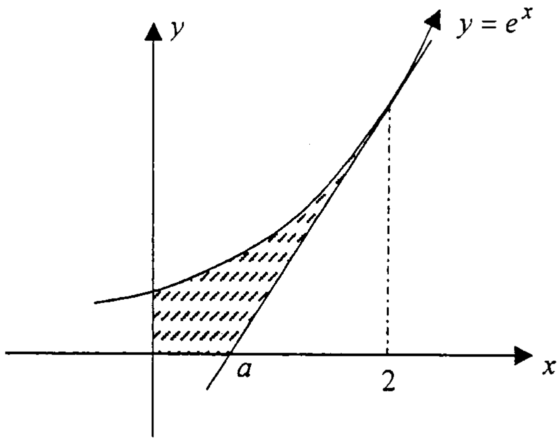
Question 8 Continued **Marks**

d) i) Write down the discriminant of $3x^2 - kx + 3$. **3**

ii) For what values of k does the equation $3x^2 - kx + 3 = 0$ have real and unequal roots?

Question 9 *(12 marks)* Start this question in a new booklet.

a) **6**



i) Find the equation of the tangent to $y = e^x$ at the point where $x = 2$.

ii) Show that the value of a, the point of intersect of the tangent and the x axis is 1.

iii) Find the exact size of the shaded area.

b) Kevin and Sharon borrow \$250 000 to buy an investment property. They agree to a 15 year loan with monthly repayments and interest of 6% p.a., compounding monthly. **6**

i) Show that the amount owed after 3 months is: $A_3 = 250000(1.005)^3 - P(1.005^2 + 1.005 + 1)$, where P is the monthly repayment.

ii) Calculate the monthly repayment to pay the loan off in the 15 years.

iii) After 4 years, they decide to sell the property. How much is still owing on the loan?

Question 10 *(12 marks)* <u>Start this question in a new booklet.</u> **Marks**

a) Solve the equation $2\cos x = \sqrt{3}$, where $0 \le x \le 2\pi$. **2**

b) Consider the function whose derivative is given by **3**

$$\frac{dy}{dx} = 2x^2(x-1)(2x+1).$$

Determine the nature of the stationary point at $x = 0$.

c) You are retrenched from your position as CEO of Pear Computers. You are given a choice of "two payout" options: <u>OPTION 1</u>, a lump sum of $500,000 today or <u>OPTION 2</u>, $50,000 now and two equal payments of $300,000, the first in two year's time and the second in four year's time. **7**

i) Assuming that the interest rate for the period is 6% p.a., compounding annually, show that, by the end of the 4th year, OPTION 2 is worth $68 965.00 (nearest dollar) more than OPTION 1.

ii) What is the minimum rate of interest that makes the OPTION 1 the better choice?

End of Paper

HSC MATHEMATICS TRIAL 2003

QUESTION ONE

(a) $\frac{3\pi}{4}$ ✓

(b) $2(x-3)(x+3)$ ✓✓

(c) The distance from x to 3 is less than 4 units i.e. $-1 < x < 7$ ✓✓

(d) $\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$ ✓✓

(e) $e^{\pi} \doteqdot 23.1$ ✓✓

(f) $(2-\sqrt{3})^2 = 7-4\sqrt{3}$ $a=7$ $b=-4$ ✓✓

(g) $\log_3 \frac{1}{9} = \log_3 3^{-2} = -2$ ✓

MARKING

(b) $2(x^2-9)$ ✓✗

(d) One mark for knowing $\cos\frac{7\pi}{6} < 0$

One mark for knowing $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(e) One mark if calculation correct but incorrectly rounded to 3 significant figures.

QUESTION TWO

(a) (i) $m_{AB} = \frac{4-1}{1+5} = \frac{1}{2}$ ✓

(ii) $27°$ ✓

(iii) $y-4 = \frac{1}{2}(x-1)$ ✓

$2y-8 = x-1$

$x-2y+7=0$ ✓

(iv) $d = \frac{|4-0+7|}{\sqrt{1+4}}$ ✓ $= \frac{11}{\sqrt{5}}$

(v) $d_{AB} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$ ✓

Area of ABCD $= \frac{11}{\sqrt{5}} \times 3\sqrt{5} = 33$ units^2 ✓

(vi) $M_{AC} = \left(-\frac{1}{2}, \frac{1}{2}\right)$ ✓

∴ D has coordinates $(-2, -3)$ ✓✓

(b) $\cos\theta = \frac{7^2+13^2-10^2}{2\times 7\times 13}$ ✓

$\theta \doteqdot 49° 35'$ ✓

Question 3

a) i $\frac{d}{dx}\tan 3x = 3\sec^2 3x$ ✓ — 1 mark answer only.

ii) $\frac{d}{dx}\frac{x}{e^{2x}} = \frac{e^{2x}\cdot 1 - x \times 2e^{2x}}{(e^{2x})^2}$ ✓ — 1 mark substitution into formula.

$= \frac{e^{2x} - 2xe^{2x}}{e^{4x}}$ ✓ — 1 mark some type of simplification.

$= \frac{e^{2x}(1-2x)}{e^{4x}}$ — = 2 marks.

$= \frac{1-2x}{e^{2x}}$

iii) $\frac{d}{dx}\log_e(\sin 3x) = \frac{1}{\sin 3x} \times 3\cos 3x$ ✓

$= \frac{3\cos 3x}{\sin 3x}$ ✓

1 mark the fraction

1 mark the correct differentiatio (2 marks).

* 1 mark was deducted from a student who added "c" to all answers.

b) i) $\int \cos 3x\, dx = \frac{1}{3}\sin 3x + c$ ✓ — 1 mark integra[tion]; c was not needed for the mark

ii) $\int \frac{x^3 + 2x^2 - 3}{x^2}\, dx$

$= \int \frac{x^3}{x^2} + \frac{2x^2}{x^2} - \frac{3}{x^2}\, dx$ ✓ — 1 mark for seperating into correct fractions or using factors.

$= \int x + 2 - \frac{3}{x^2}\, dx$

Question 3

$= \frac{x^2}{2} + 2x + \frac{3}{x} + C$ ✓ ✓

1 mark integrations. all correctly done.

1 mark for $+C$ with 2 correct integrations.

= 3 marks

(c) $y = (3x+1)^3$

$\frac{dy}{dx} = 3(3x+1)^2 \times 3$

$\therefore \frac{dy}{dx} = 9(3x+1)^2$ ✓

1 mark for the differentiat

gradient of tangent at $x=1$

$m = 9(3\times1+1)^2$

$= 9 \times 16$

$= 144$

gradient of normal at $x=1$

$m = \frac{-1}{144}$

1 mark for the gradient of the normal

equation of the normal

$y - y_1 = m(x - x_1)$

$y = (3\times1+1)^3$

$y = 64$

$y - 64 = \frac{-1}{144}(x-1)$ ✓

1 mark correct substitution into the formula

= (3 marks)

$144y - 9216 = -x + 1$

$x + 144y - 9217 = 0$ is the required equation.

Question 4

a) $3x^2 + 5x - 1 = 0$

(i) $\alpha + \beta = \frac{-b}{a}$

$= -\frac{5}{3}$ (1)

1 mark each

(ii) $\alpha\beta = \frac{c}{a}$

$= -\frac{1}{3}$

$\frac{1}{\alpha\beta} = -3$ (1)

(iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$

$= \frac{-\frac{5}{3}}{-\frac{1}{3}}$ (1)

$= 5$ (1)

(b) $A = 4\int_0^{\pi} \sin x \, dx$ (1) (1)

1 mark correct integrand + limits
1 mark multiplying effect

$= 4\left[-\cos x\right]_0^{\pi}$ (1)

1 mark integration

$= 4\left[-\cos\pi + \cos 0\right]$

$= 4\left[+1 + 1\right]$ (1)

1 mark evaluation.

$= 8$ sq units

c) $\frac{dh}{dt} = 10 - \frac{1}{3}t$

$h = 10t - \frac{1}{3}\frac{t^2}{2} + c$ (1)

$h = 10t - \frac{t^2}{6} + c$

when $t = 0$ $h = 1$

$1 = c$

Question 8

$\therefore h = 10t - \frac{t^2}{6} + 1$

$h = ?\ t = 6$

$h = 10 \times 6 - \frac{6^2}{6} + 1$

$= 60 - 6 + 1$

$\therefore h = 55m$ (1)

ii) $h = 10t - \frac{t^2}{6} + c$

$\frac{dh}{dt} = 10 - \frac{1}{3}t$

$10 - \frac{1}{3}t = 0$ (1) 1 mark showing $\frac{dh}{dt} = 0$

$30 - t = 0$

$t = 30$

$\frac{d^2h}{dt^2} = -\frac{1}{3}$

< 0 (1) 1 mark showing it is a max

$\therefore$ max at $t = 30$

$\therefore$ max height

$h = 10 \times 30 - \frac{30^2}{6} + 1$

$= 300 - 150 + 1$

$= 151\ m$ (1) 1 mark for answer.

$\therefore$ max height is 151m.

Question 5

a) i) $Q = Q_0 e^{-kt}$

$\frac{dQ}{dt} = \frac{d}{dt}\left(Q_0 e^{-kt}\right)$

$= Q_0 \times -k e^{-kt}$ ✓

$= -k Q_0 e^{-kt}$

$= -k Q$ ✓ (2)

ii) $t = 0 \quad Q = 2100$

$2100 = Q_0 e^{-k \times 0}$ ✓

$\therefore Q_0 = 2100$ (1)

iii) $t = 6 \quad Q = 1050$

$1050 = 2100 e^{-6k}$

$\frac{1}{2} = e^{-6k}$ ✓

$-6k = \ln \frac{1}{2}$

$5k = \ln 2$

$k = \frac{1}{5} \ln 2$ ✓ (2)

$= 0.2 \ln 2$

must show $\ln \frac{1}{2} = -\ln 2$

iv) $1 = 2100 e^{-0.2 \ln 2\, t}$

$e^{-0.2 \ln 2\, t} = \frac{1}{2100}$ ✓

$-0.2 \ln 2\, t = -\ln 2100$

$t = \frac{\ln 2100}{0.2 \ln 2}$ ✓ (2)

≈ 55 years

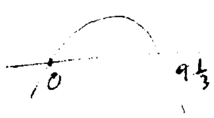
b) i) $a = 25,\ d = -6$ ✓

$T_{20} = 25 + 19 \times -6$ ✓ (2)

$= -89$

ii) $S_n < 0 \quad \frac{n}{2}(50 + (n-1) \times -6) < 0$ ✓

$n(56 - 6n) < 0$ ✓



$n < 0$ or $n > 9\frac{1}{3}$ ✓

$\therefore n = 10 \quad (n \neq 0)$ (3)

Q6

i) $y' = 3 - 3x^2$ ✓

stationary points $y' = 0$

$0 = 3(1 - x^2)$ ✓

$x = \pm 1$

$x = 1 \quad y = 1 + 3 - 1 = 3$

$x = -1 \quad y = 1 - 3 + 1 = -1$

$(1, 3) \quad (-1, -1)$ ✓✓

$y'' = -6x$

$x = 1 \quad y'' = -6 < 0$

$\therefore$ max $(1, 3)$ ✓

$x = -1 \quad y'' = 6 > 0$

$\therefore$ min $(-1, -1)$ ✓

ii) $0 = -6x$

$x = 0 \quad y = 1$ ✓

$(0, 1)$

check

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y''	3	0	-3

✓

change in concavity.

$\therefore$ point of inflexion

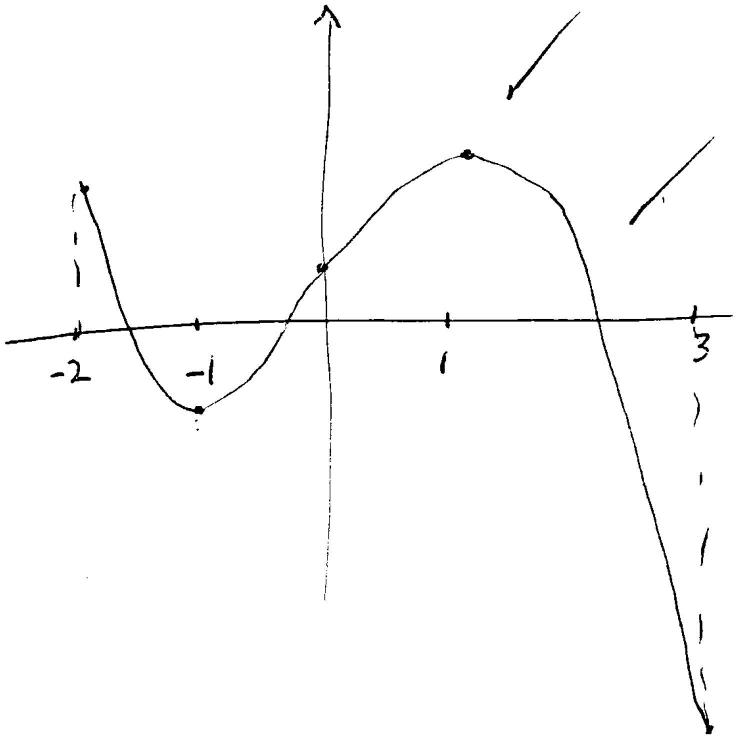
iii) $x = -2 \quad y = 1 - 6 + 8 = 3$

$(-2, 3)$ ✓

$x = 3 \quad y = 1 + 9 - 27$

$= -17$

$(3, -17)$



✓ ✓

iv) 17 ✓

7/(a) $y = \sqrt{2x-1}$

$y^2 + 1 = 2x$

$x^2 = \frac{(y^2+1)^2}{4}$

$V = \pi \int_0^1 x^2 \, dy$

$= \frac{\pi}{4} \int_0^1 (y^4 + 2y^2 + 1) \, dy$

$= \frac{\pi}{4} \left[\frac{1}{5}y^5 + \frac{2}{3}y^3 + y \right]_0^1$

$= \frac{\pi}{4} \left(\frac{1}{5} + \frac{2}{3} + 1 \right)$

$= \frac{7\pi}{15}$ units^3

(b)

(i) $A\hat{D}E = A\hat{B}C$ (corresponding ∠s, DE∥BC)

$= 90°$

(ii) $B\hat{D}E = 180° - A\hat{D}E$ ($A\hat{D}B$ straight ∠)

$= 90°$

DE is common

AD = BD (given)

∴ △AED ≡ △BED (SAS)

(iii) $A\hat{E}D = E\hat{C}B$ (corresponding ∠s, DE∥BC)

$B\hat{E}D = E\hat{B}C$ (alternate ∠s, DE∥BC)

But $A\hat{E}D = B\hat{E}D$ (equal corresponding ∠s in congruent △s AED,

∴ $E\hat{B}C = E\hat{C}B$

∴ △EBC is isosceles (equal base ∠s)

∴ BE = EC

8/ (a) (i) $(u-8)(u+2)$

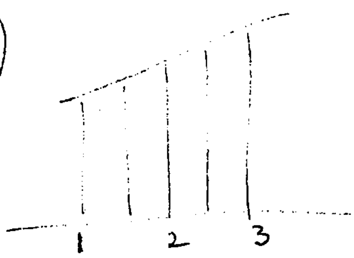
(ii) ~~let u = log₂x~~

$\log_2 x = 8$ or $\log_2 x = -2$

$x = 2^8$ $\quad$ $x = 2^{-2}$

$= 256$ $\quad$ $= \frac{1}{4}$

(b)



$$\int_1^3 \log x\,dx = \frac{1}{4}\left[f(1)+f(3)+2\left[f(1.5)+f(2)+f(2.5)\right]\right]$$

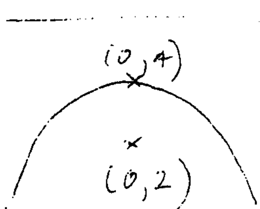
$$= \frac{1}{4}\left(0+\log 3+2\left(\log 1.5+\log 2+\log 2.5\right)\right)$$

$$= \frac{1}{4}\log 168.75$$

$= 1.28$ (2 decimal places)

(c) (i) 2

(ii)



(iii) S (0, 2)

(d) (i) $k^2 - 4.3.3$

(ii) $k^2 - 36 > 0$

$k < -6$ or $k > 6$

QUESTION 9

a) i) $\frac{dy}{dx} = e^x$

$\therefore m = e^2$ at $(2, e^2)$

$y - e^2 = e^2(x-2)$

$y - e^2 = e^2x - 2e^2$

$e^2x - y - e^2 = 0$

(or $y = e^2x - e^2$)

1 - find gradient

1 - equation of line

ii) when $y = 0$

$e^2x - e^2 = 0$

$\therefore x = 1$

1 - attempt to solve for x when $y=0$.

iii) Shaded Area $= \int_0^2 e^x dx - \frac{1}{2} \times 1 \times e^2$

(or $\int_0^1 e^x dx - \int_1^2 e^x - (e^2x - e^2)dx$)

$= [e^x]_0^2 - \frac{1}{2}e^2$

$= e^2 - 1 - \frac{1}{2}e^2$

$= \frac{1}{2}e^2 - 1$ units^2

2 - correct area expression (2 parts)

NOTE: 1 mk given for $\int_0^2 e^x - (e^2x - e^x)dx$

1 - answer in exact form

b) i) A_n = amount owed after n months $\quad r = \frac{0.06}{12} = 0.005$

$A_1 = 250000 \times 1.005 - P$

$A_2 = (250000 \times 1.005 - P) \times 1.005 - P$

$= 250000 \times 1.005^2 - P(1 + 1.005)$

$\therefore A_3 = 250000 \times 1.005^3 - P(1 + 1.005 + 1.005^2)$

1 - adjust rate

1 - derive expression showing expansion.

ii) $n = 15 \times 12 = 180$ months $\quad A_{180} = 0$.

$A_{180} = 250000(1.005)^{180} - P(1 + 1.005 + \ldots + 1.005^{179})$

$250000(1.005)^{180} = P\left(\frac{1(1.005)^{180} - 1}{1.005 - 1}\right)$

$P = 250000(1.005)^{180} \times \frac{0.005}{1.005^{180} - 1}$

$= \underline{\$2109.64}$

1 - $A_{180} = 0$

1 - GP sum

1 - correct solution.

iii) 4yrs = 48 mth

$A_{48} = 250000(1.005)^{48} - 2109.64\left(\frac{1.005^{48} - 1}{0.005}\right)$

$= \$203495.23$

1 - correct subst.

QUESTION 10

a) $2\cos x = \sqrt{3}$

$\cos x = \frac{\sqrt{3}}{2}$

related $x = \pi/6$ — 1 - related angle

$\therefore x = \pi/6,\ 11\pi/6$ — 1 - correct quadrants & expressed in radians.

b) $\frac{dy}{dx} = 2x^2(x-1)(2x+1) \quad (= 2x^2(2x^2-x-1))$

$\frac{d^2y}{dx^2} = 4x(2x^2-x-1) + 2x^2(4x-1)$ — 1 - $\frac{d^2y}{dx^2}=0$ or check any points

when $x=0$, $\frac{d^2y}{dx^2}=0$

$\therefore$ possible horizontal inflexion. — 1 - check pts between stationary points

Stat pts $\Rightarrow 0, 1, -\frac{1}{2}$

check

x	$-1/4$	0	$1/2$
$\frac{dy}{dx}$	$-$	0	$-$
	\	.	\

1 - correct conclusion from info gained.

$\therefore$ horizontal inflexion at $x=0$.

c) i/ Option 1 $\Rightarrow A_4 = 500\,000 \times 1.06^4$ — 1 - calculate Opt 1

$= \$631\,238.48$

Option 2 — 1 - expression for Opt 2

$A_4 = 50\,000 \times 1.06^4 + 300\,000 \times 1.06^2 + 300\,000$ — 1 - calculate Opt 2

$= \$700\,203.85$

Opt 2 - Opt 1 $= \$68\,965$ — 1 - subtraction.

ii/ $500\,000 r^4 > 50\,000 r^4 + 300\,000 r^2 + 300\,000$ — 1 - inequality statement

$450\,000 r^4 - 300\,000 r^2 - 300\,000 > 0$.

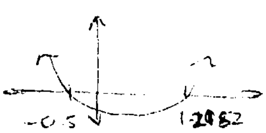
$45r^4 - 30r^2 - 30 > 0$.

Let $u = r^2$

$u = \frac{30 \pm \sqrt{900 + 5400}}{2 \times 45}$

$= \frac{30 \pm 30\sqrt{7}}{90}$ — 1 - solve quadratic

$= \frac{1 \pm \sqrt{7}}{3}$



≈ 1.2152 or $-0.5485\ldots$ (reject negative)

$\therefore r = \sqrt{1.2152}$ — 1 - correct answer

$= 1.1023$

$\therefore$ minimum rate is 10.2%