NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS

2016 HSC Course Assessment Task 4 (Trial Examination)
Thursday August 4, 2016

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 12)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

BOSTES NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

○ 12MAT.6 – Dr Lawson
○ 12MAT.3 – Mr Lam
○ 12MAT.4 – Mr Wall
○ 12MAT.5 – Mrs Gan

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Mark your answers on the answer grid provided (labelled as page 12).

Questions **Marks**

1. What is the value of $\lim_{x \to 7} \dfrac{x^2 + 5x - 84}{x - 7}$? **1**

(A) -5 (B) 0 (C) 12 (D) 19

2. What are the conditions for the expression $ax^2 + bx + c = 0$ to be *positive definite*? **1**

(A) $c > 0$ and $\Delta < 0$ (C) $a > 0$ and $\Delta < 0$

(B) $c > 0$ and $\Delta > 0$ (D) $a > 0$ and $\Delta > 0$

3. What is the value of $\sum_{k=7}^{10} (3k - 1)$? **1**

(A) 98 (B) 69 (C) 8 (D) 100

4. What is the greatest value of the function: $y = 4 - 2\cos x$? **1**

(A) 2 (B) 4 (C) 6 (D) 8

5. What is the solution to the equation $\log_e(x + 2) - \log_e x = \log_e 4$? **1**

(A) $x = \frac{2}{5}$ (B) $x = \frac{2}{3}$ (C) $x = \frac{3}{2}$ (D) $x = \frac{5}{2}$

6. How many solutions are there to the equation $|2x + 3| = -x + 3$? **1**

(A) None (B) 1 (C) 2 (D) 3

7. Fully simplify: $\dfrac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sin(\pi - \theta)}$. **1**

(A) $\sin\theta$ (B) $\cos\theta$ (C) $\tan\theta$ (D) $\cot\theta$

8. Which expression is the primitive function of $(5x - 4)^7$, excluding the constant of integration? **1**

(A) $30(5x - 4)^6$ (B) $\dfrac{(5x - 4)^8}{5}$ (C) $\dfrac{(5x - 4)^8}{8}$ (D) $\dfrac{(5x - 4)^8}{40}$

9. What is the domain and range of the function $y = \dfrac{1}{\sqrt{x-9}}$? **1**

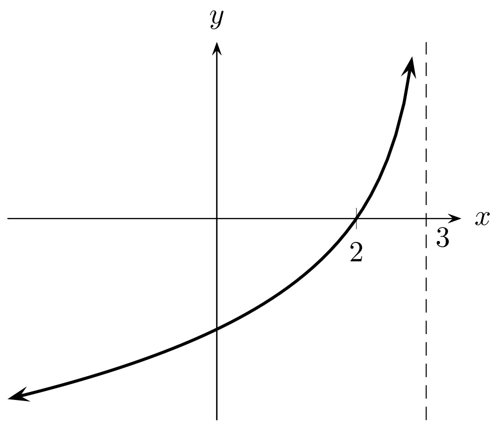
(A) $D = \{x : x \geq 9\}$ and $R = \{y : y > 0\}$

(B) $D = \{x : x > 9\}$ and $R = \{y : y > 0\}$

(C) $D = \{x : -\infty \leq x \leq \infty\}$ and $R = \{y : -\infty \leq y \leq \infty\}$

(D) $D = \{x : x \leq -3 \text{ or } x \geq 3\}$ and $R = \{y : y < 0\}$

10. Which equation below best represents the graph shown? **1**



(A) $y = \log_e(3-x)$

(B) $y = -\log_e(3-x)$

(C) $y = e^{x-2}$

(D) $y = -\log_e(3x-1)$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

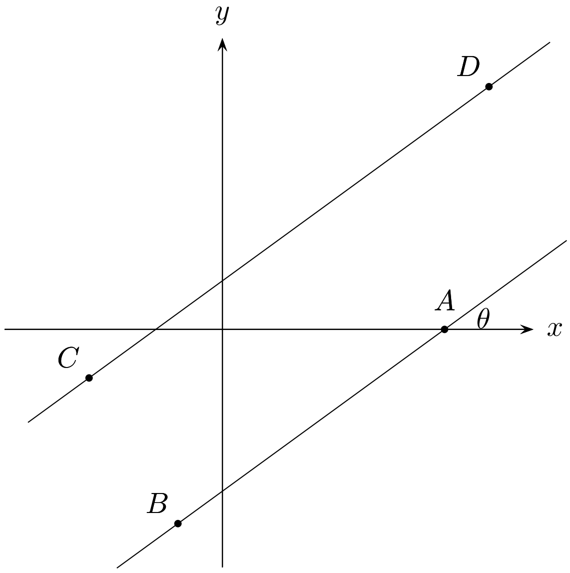
Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) Fully simplify: $\dfrac{5}{x-2} - \dfrac{2}{x-3}$. **2**

(b) Express $\dfrac{7-\sqrt{2}}{3+\sqrt{2}}$ with a rational denominator. **2**

(c) Solve $2\cos x + 1 = 0$ for $0 \leq x \leq 2\pi$. **2**

(d) The diagram shows the points $B(-1,-4)$, $C(-3,-1)$ and $D(6,5)$ in the Cartesian plane. The point $A(a,0)$ is the point where the line AB intersects with the x axis.



i. Show that the gradient of CD is $\dfrac{2}{3}$. **1**

ii. Find the equation of CD. **2**

iii. If $AB \parallel CD$, find the value of a, the x coordinate of the point A. **2**

iv. Show that the distance BC is $\sqrt{13}$. **1**

v. Find the size of the angle θ, correct to the nearest minute. **1**

vi. Find the perpendicular distance from B to the line CD. **2**

Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) Solve for x: $3^{2x} + 3^x - 12 = 0$. **2**

(b) Find the value of k if the sum of the roots of $x^2 - (k-1)x + 2k = 0$ is equal to the product of the roots. **2**

(c) A parabola has the equation $16y = x^2 - 4x - 12$.

i. Find the coordinates of the vertex. **1**

ii. Find the coordinates of the focus. **1**

iii. Find the equation of the directrix. **1**

(d) Boxes are stacked in layers, where each layer contains one box less than the layer below. There are six boxes in the top layer, seven in the next and so on. There are n layers altogether.

i. Write down the number of boxes in the bottom layer. **1**

ii. Show that there are $\frac{1}{2}n(n+11)$ boxes. **2**

(e) A generous donor to the school donates $\$P$ for the Mathematics Department to set up a fund to print past trial papers for Year 12 students over eight years. At the start of each year, $2 000 is spent printing the trial papers. The interest paid by the bank is 3.6% p.a., compounded annually.

On 1/1/2016, Mr Lam deposits the donation of $\$P$ upon opening the bank account. The first batch of trial papers is printed and paid for on 1/1/2017.

i. Show that the amount in the fund on 1/1/2018 is given by **2**

$$1.036^2 P - 2000(1 + 1.036)$$

ii. Find the value of P. **3**

Examination continues overleaf...

Question 13 (15 Marks) Commence a NEW booklet. **Marks**

(a) Differentiate with respect to x:

i. $x \tan x$. **2**

ii. $\log_e(\cos x)$. **2**

(b) Find:

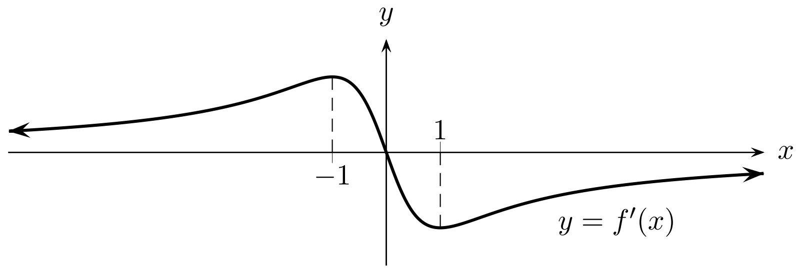
i. $\displaystyle\int \frac{x^2+3}{x}\,dx$ **2**

ii. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$ **2**

(c) Evaluate: $\displaystyle\int_0^{\frac{\pi}{3}} \cos 2x\,dx$. **2**

(d) Find the equation of the normal to the curve $y = x\log_e x$ at the point on the curve where $x = 1$. **2**

(e) The graph of $y = f'(x)$ is shown in the diagram below. $y = f'(x)$ passes through the origin. As $x \to \pm\infty$, $f'(x) \to 0$ and $f(x) \to 0$. **3**

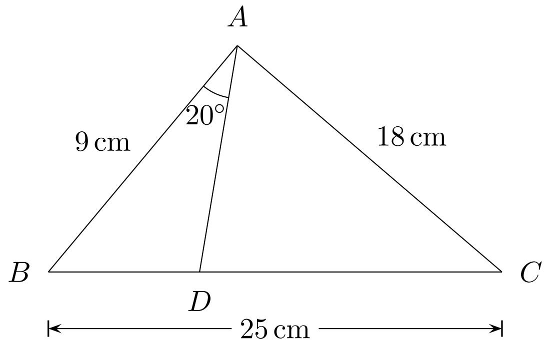


Sketch the graph of $y = f(x)$, given $f(x) > 0$ for all values of x.

Examination continues overleaf...

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) In $\triangle ABC$, $AB = 9$ cm, $AC = 18$ cm and $BC = 25$ cm. Point D lies on BC and $\angle BAD = 20°$, as shown in the diagram.

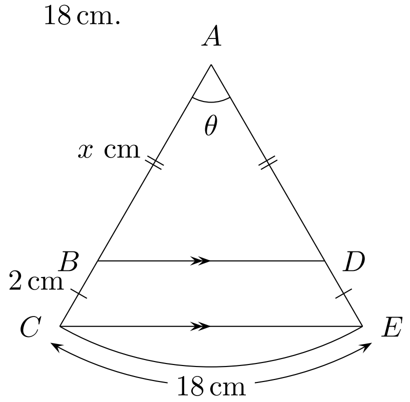


i. Show that $\angle ABC \approx 32°$. **1**

ii. Find the length of BD, correct to 1 decimal place. **2**

iii. Hence or otherwise, find the area of $\triangle ADC$, correct to the nearest cm^2. **2**

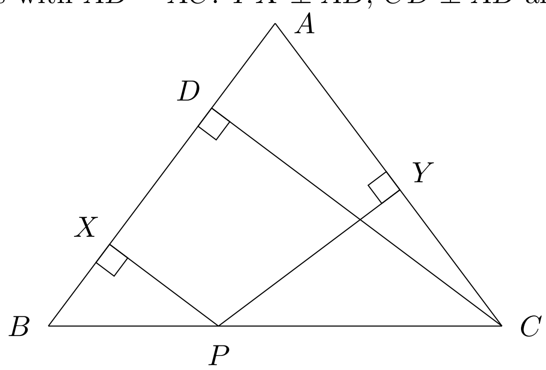
(b) The diagram shows $\triangle ABD$ and $\triangle ACE$, where $BD \parallel CE$, $AB = AD = x$ cm, $BC = DE = 2$ cm and $AD : AE = 3 : 4$. $\triangle ACE$ and arc CE form a sector in a circle of radius $(x + 2)$ cm. The angle of the sector is θ radians, and arc CE is 18 cm.



i. Show that $\theta = \dfrac{9}{4}$ radians. **2**

ii. Calculate the area of the segment cut off by CE, correct to one decimal place. **2**

(c) $\triangle ABC$ is isosceles with $AB = AC$. $PX \perp AB$, $CD \perp AB$ and $PY \perp AC$.



Copy or trace the diagram into your writing booklet.

i. Prove $\triangle PXB \,|||\, \triangle CDB$. **2**

ii. Prove $\triangle CDB \,|||\, \triangle PYC$. **2**

iii. Hence show that $PX + PY = CD$. **2**

Question 15 (15 Marks) Commence a NEW booklet. **Marks**

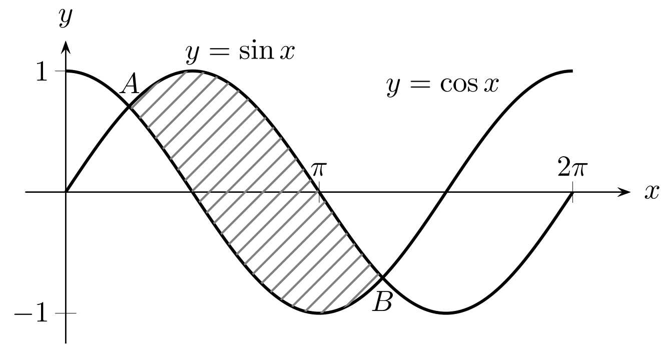
(a) A mattress is sitting in a bedframe as shown. The length of the mattress is 1.8 m, its width is x metres, and height h metres. A blanket is the same length as the mattress and its width needs to be able to cover exactly the width of the mattress and its two sides.

i. If the volume of the mattress is $0.504\,\text{m}^3$, show that the area A of the blanket is given by **2**

$$A = 1.8\left(x + \frac{0.56}{x}\right)$$

ii. Hence or otherwise, find the area of the smallest blanket needed to cover the mattress, correct to 2 decimal places. **3**

(b) The diagram shows the graphs $y = \sin x$ and $y = \cos x$ within the domain $0 \le x \le 2\pi$. The graphs intersect at points A and B.



i. Show that the x coordinates at A and B are $\dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$ respectively. **1**

ii. Find the shaded area enclosed by the two graphs between A and B. **3**

Question 15 continued overleaf. . .

Question 15 continued from previous page...

(c) A curve $y = f(x)$ has derivative $f'(x) = kx^2 + 2$, and a stationary point at $(-1, 3)$.

i. Show that $k = -2$. **1**

ii. Find the equation of $y = f(x)$. **2**

(d) The speed of a train was recorded at intervals of one minute. The times in minutes, and the corresponding speeds v in kilometres per hour, are listed in the table below.

Time (min)	0	1	2	3	4
Speed (km/h)	0	24	35	28	50

i. Explain why the distance x, in *kilometres*, travelled by the train in these four minutes is given by **1**

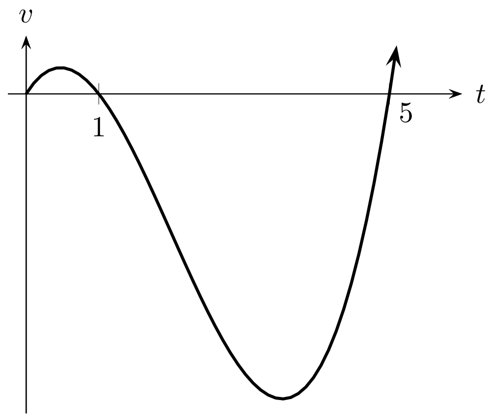
$$x = \int_0^{\frac{1}{15}} v \, dt$$

ii. Estimate the value of x by using Simpson's Rule with five function values. **2**

Examination continues overleaf...

Question 16 (15 Marks) Commence a NEW booklet. **Marks**

(a) A pen moves along the x axis, ruling a line. The diagram shows the graph of the velocity of the tip of the pen as a function of time.



The velocity, in centimetres per second, is given by the equation

$$v = 4t^3 - 24t^2 + 20t$$

where t is the time in seconds. Initially, the pen is 3 cm to the right of the origin.

i. Find an expression for x, the position of the tip of the pen, as a function of time. **2**

ii. What feature will the graph of x as a function of t, have at $t = 1$? Justify your answer. **2**

iii. The pen uses 0.05 mg of ink per centimetre travelled. **3**

How much ink is used between $t = 0$ and $t = 2$?

Question 16 continued overleaf...

Question 16 continued from previous page...

(b) The mean Blood Alcohol Concentration (BAC) of a large sample of male adults of similar metabolism and mass were measured after fasting and subsequent rapid consumption of two standard drinks (One standard drink is approximately one 375 mL can of "mid strength beer", 3.5% alcohol content).

The BAC measured is given by

$$BAC = 0.7e^{-kt}$$

where k is a constant, and t is the time after the consumption of the two standard drinks, measured in hours.

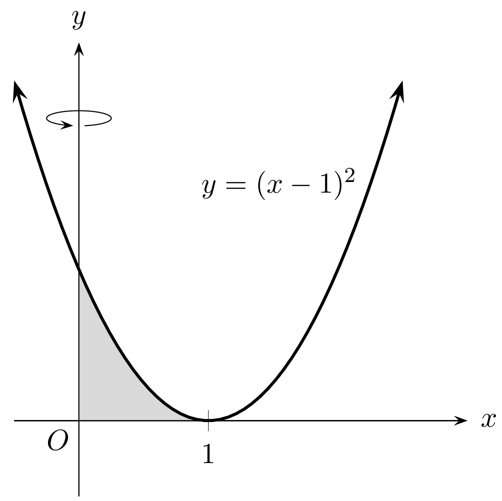
i. Find the value of k (correct to 4 decimal places) if the first breath test returns $BAC = 0.33$, one hour after all of the alcohol is consumed. **1**

ii. Breathalyser instrumentation has accuracy up to 3 decimal places, i.e. an actual BAC reading of 0.0007 is displayed as $BAC = 0.001$. **3**

How long (correct to the nearest minute) would it take for the BAC to drop below $BAC = 0.0005$, so that the reading on the instrumentation shows an effective $BAC = 0$, the Australian legal limit for restricted licenses (Learner, Provisional P1/P2)?

(**Safety message:** do not drink and drive)

(c) The region bounded by the x axis, the y axis and the parabola $y = (x-1)^2$ is rotated about the y axis to form a solid. **4**



Find the volume of the solid.

End of paper.

2016 Mathematics HSC Course Assessment Task 4 (Trial Examination) STUDENT SELF REFLECTION

1. In hindsight, did I do the best I can? Why or why not?

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2. Which topics did I need more help with, and what parts specifically?

- Q1, 8 - Miscellaneous calculus

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- Q6, 9, 11 (Prelim) - Functions, Basic Arith/Algebra, Linear Function

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- Q2, 3, 12 - Quadratic Polynomial, Locus & parabola, Series and Applications

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- Q10, 13 - Calculus of Exponential, Logarithmic and Trigonometric functions

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- Q4, 7, 14 - Preliminary Trigonometry, Euclidean Geometry

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- Q15 - Geometric applications of differential/integral calculus

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- Q16(a)-(c) - Applications of Calculus to the Physical World

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- Q16(d) - Harder integral calculus

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3. What other parts from the feedback session can I take away to refine my solutions for future reference?

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Suggested Band 6 Responses

Section I

1. (D) **2.** (C) **3.** (A) **4.** (C) **5.** (B)
6. (C) **7.** (D) **8.** (D) **9.** (B) **10.** (B)

Section II

Question 11 (Gan)

(a) (2 marks)

- ✓ [1] for common denominator and single fraction.
- ✓ [1] for final answer.

$$\frac{5}{x-2} - \frac{2}{x-3} = \frac{5(x-3)-2(x-2)}{(x-2)(x-3)} = \frac{3x-11}{(x-2)(x-3)}$$

(b) (2 marks)

- ✓ [1] for multiplication by conjugate in the numerator and denominator.
- ✓ [1] for final answer.

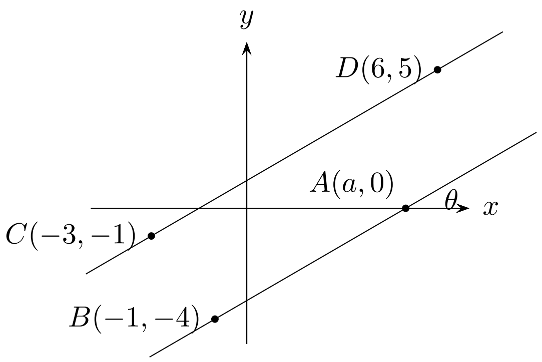
$$\frac{7-\sqrt{2}}{3+\sqrt{2}} \frac{\times(3-\sqrt{2})}{\times(3-\sqrt{2})} = \frac{21-10\sqrt{2}+2}{9-2} = \frac{23-10\sqrt{2}}{7}$$

(c) (2 marks)

- ✓ [1] for each correct angle.

$$2\cos x + 1 = 0$$
$$\cos x = -\frac{1}{2}$$
$$x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

(d) i. (1 mark)



$$m_{CD} = \frac{5-(-1)}{6-(-3)} = \frac{2}{3}$$

ii. (2 marks)

- ✓ [1] for correct point-gradient formula application
- ✓ [1] for general form

$$\frac{y-(-1)}{x-(-3)} = \frac{2}{3}$$
$$3y + 3 = 2x + 6$$
$$\therefore 2x - 3y + 3 = 0$$

iii. (2 marks)

- ✓ [1] for significant progress towards answer.
- ✓ [1] for final answer.

$$m_{AB} = \frac{0-(-4)}{a-(-1)} = \frac{2}{3}$$
$$\frac{4}{a+1} = \frac{2}{3}$$
$$a + 1 = 6$$
$$a = 5$$

iv. (1 mark)

$$BC = \sqrt{(-1-(-4))^2 + (-3-(-1))^2} = \sqrt{9+4} = \sqrt{13}$$

v. (1 mark)

$$m = \tan\theta = \frac{2}{3}$$
$$\therefore \theta = 33^\circ 41'$$

vi. (2 marks)

$$d_\perp = \frac{|2(-1)-3(-4)+3|}{\sqrt{2^2+(-3)^2}} = \frac{13}{\sqrt{13}} = \sqrt{13} \text{ units}$$

Question 12 (Gan)

(a) (2 marks)

✓ [1] for correct factorisation.

✓ [1] for final answer.

$$3^{2x} + 3^x - 12 = 0$$

Let $m = 3^x$,

$$\begin{aligned} m^2 + m - 12 &= 0 \\ (m+4)(m-3) &= 0 \\ m &= -4, 3 \end{aligned}$$

However, $3^x > 0\,\forall x$,

$$\begin{aligned} \therefore 3^x &= 3 \text{ only} \\ x &= 1 \end{aligned}$$

(b) (2 marks)

✓ [1] for correctly setting up the sum/prod of roots.

✓ [1] for final answer.

$$\begin{aligned} x^2 - (k-1)x + 2k &= 0 \\ \text{Sum of roots:} \alpha + \beta &= \frac{(k-1)}{1} = k - 1 \\ \text{Prod of roots:} \alpha\beta &= \frac{2k}{1} = 2k \\ \therefore k - 1 &= 2k \\ k &= -1 \end{aligned}$$

(c) i. (1 mark)

$$\begin{aligned} \frac{1}{16}y &= \left(x^2 - 4x + 4 - 4\right) - \frac{12}{16} \\ &= \left((x-2)^2\right) - \frac{1}{4} - \frac{3}{4} \\ &= (x-2)^2 - 1 \\ \therefore (x-2)^2 &= 16(y+1) \\ (x-2)^2 &= 4(4)(y+1) \\ &\therefore V(2,-1) \end{aligned}$$

ii. (1 mark)

$$a = 4 \longrightarrow S(2,3)$$

iii. (1 mark)

$$y = -5$$

(d) i. (1 mark)

$$5 + n$$

ii. (2 marks)

$$\begin{aligned} S_n &= \frac{n}{2}(a + \ell) \\ &= \frac{n}{2}(6 + (5+n)) \\ &= \frac{1}{2}n(11+n) \end{aligned}$$

(e) i. (2 marks)

✓ [1] for A_1.

✓ [1] for obtaining given expression with full working.

At the end of 2016/start of 2017:

$$A_1 = P \times 1.036 - 2\,000$$

End of 2017/start of 2018:

$$\begin{aligned} A_2 &= A_1 \times 1.036 - 2\,000 \\ &= (P \times 1.036 - 2\,000) \times 1.036 \\ &\quad - 2\,000 \\ &= 1.036^2 P - 2\,000 \times 1.036 - 2\,000 \\ &= 1.036^2 P - 2\,000\,(1 + 1.036) \end{aligned}$$

ii. (3 marks)

✓ [1] for correct eqns in terms of P.

✓ [1] for converting GP into single expression.

✓ [1] for final answer.

$$\begin{aligned} A_3 &= 1.036^3 P - 2\,000\left(1 + 1.036 + 1.03^2\right) \\ A_8 &= 1.036^8 P \\ &\quad - 2\,000\underbrace{\left(1 + 1.036 + \cdots + 1.036^7\right)}_{=S_8} \end{aligned}$$

$$S_8 = \frac{1.036^8 - 1}{1.036 - 1} = \frac{1.036^8 - 1}{0.036}$$

Since $A_8 = 0$ as amount of cash runs out,

$$\begin{aligned} 0 &= 1.036^8 P - 2\,000\frac{1.036^8 - 1}{0.036} \\ P &= \frac{2\,000\left(1.036^8 - 1\right)}{0.036 \times 1.036^8} \\ &= \$13\,690.71 \end{aligned}$$

Question 13 (Lawson)

(a) i. (2 marks)

$$u = x \qquad v = \tan x$$
$$u' = 1 \qquad v' = \sec^2 x$$
$$\therefore \frac{d}{dx}(x \tan x) = x \sec^2 x + \tan x$$

ii. (2 marks)

$$\frac{d}{dx}(\log_e \cos x) = \frac{-\sin x}{\cos x}$$
$$(= -\tan x)$$

(b) i. (2 marks)

$$\int \frac{x^2+3}{x}\,dx = \int \left(x + \frac{3}{x}\right)\,dx$$
$$= \frac{1}{2}x^2 + 3\ln x + C$$

ii. (2 marks)

$$\int \frac{e^x}{1+e^x}\,dx = \log_e(1+e^x) + C$$

(c) (2 marks)

$$\int_0^{\frac{\pi}{3}} \cos 2x\,dx = \frac{1}{2}\Big[\sin 2x\Big]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\left(\sin\frac{2\pi}{3} - 0\right)$$
$$= \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

(d) (2 marks)

$$y = x \log_e x$$
$$u = x \qquad v = \ln x$$
$$u' = 1 \qquad v' = \frac{1}{x}$$
$$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x$$
$$= 1 + \ln x\Big|_{x=1}$$
$$= 1 + \ln 1 = 1$$
$$\therefore m_\perp = -1$$

Applying point-gradient formula at $(1, 0)$ with gradient -1:

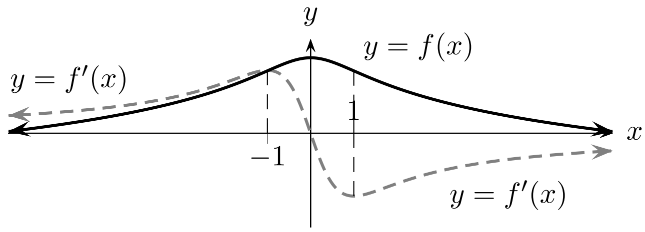
$$\frac{y-0}{x-1} = -1$$
$$y = -x + 1$$

(e) (3 marks)

✓ [1] for points of inflexion at $x = \pm 1$

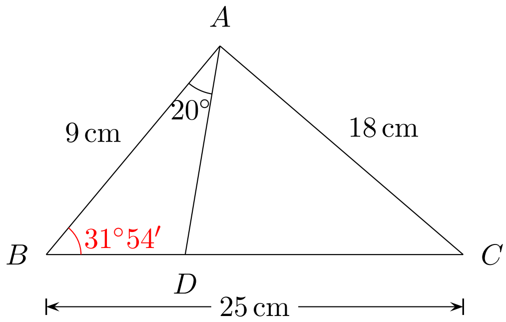
✓ [1] for local max at $x = 0$

✓ [1] for shape



Question 14 (Lawson)

(a) i. (1 mark)



Applying the cosine rule in $\triangle ABC$,

$$\cos B = \frac{9^2 + 25^2 - 18^2}{2 \times 9 \times 25}$$
$$\approx 0.848\cdots$$
$$\therefore B = 31°54' \approx 32°$$

ii. (2 marks)

✓ [1] for use of the sine rule.

✓ [1] for final answer.

$$\angle ADB = 180° - 20° - 31°54'$$
$$= 128°6'$$

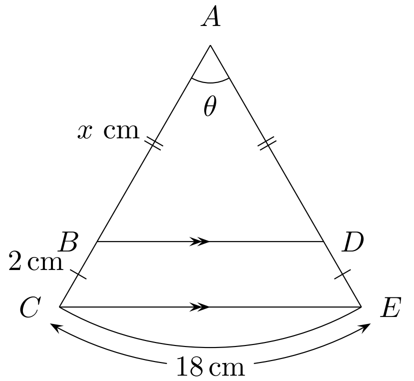
In $\triangle ABD$,

$$\frac{BD}{\sin 20°} = \frac{9}{\sin 128°6'}$$
$$\therefore BD = \frac{9 \sin 20°}{\sin 128°6'}$$
$$= 3.9\,\text{cm (1 d.p.)}$$

iii. (2 marks)

$$\begin{aligned}
A_{\triangle ADC} &= A_{\triangle ABC} - A_{\triangle ABD} \\
A_{\triangle ABC} &= \frac{1}{2} \times 9 \times 25 \sin 31°54' \\
A_{\triangle ABD} &= \frac{1}{2} \times 9 \times 3.9 \sin 31°54' \\
\therefore A_{\triangle ADC} &= \frac{1}{2} \times 9 \times (25 - 3.9) \sin 31°54' \\
&= 50.17\,\text{cm}^2 \\
&= 50\,\text{cm (nearest cm)}
\end{aligned}$$

(b) i. (2 marks)



By ratios of similar triangles where $AD : AE = 3 : 4$,

$$\begin{aligned}
\frac{x}{x+2} &= \frac{3}{4} \\
4x &= 3x + 6 \\
\therefore x &= 6
\end{aligned}$$

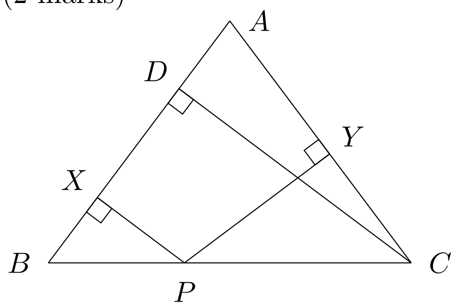
Evaluating the arc length,

$$\begin{aligned}
\ell &= r\theta \\
18 &= 8\theta \\
\therefore \theta &= \frac{9}{4}\text{ rad}
\end{aligned}$$

ii. (2 marks)

$$\begin{aligned}
A &= \frac{1}{2} r^2 (\theta - \sin\theta) \\
&= \frac{1}{2} \times 8^2 \times \left(\frac{9}{4} - \sin\frac{9}{4}\right) \\
&\approx 47.1\,\text{cm}^2
\end{aligned}$$

(c) i. (2 marks)



In $\triangle PXB$ and $\triangle CDB$,

- $\angle B$ is common
- $\angle BXP = \angle BDC$ (given)

$\therefore \triangle PXB \parallel\!| \triangle CDB$ (equiangular)

ii. (2 marks)
In $\triangle CDB$ and $\triangle PYC$,

- $\angle DBC = \angle YCP$
  (base $\angle$ of isosceles $\triangle ABC$)
- $\angle BDC = \angle PYC$ (given)

$\therefore \triangle CDB \parallel\!| \triangle PYC$ (equiangular)

iii. (2 marks)

✓ [1] for PY

✓ [1] for final answer.

- By corresponding sides of similar $\triangle PXB$ and $\triangle CDB$,

$$\begin{aligned}
\frac{PX}{CD} &= \frac{BP}{BC} \\
\therefore PX &= \frac{BP}{BC} \times CD \qquad (14.1)
\end{aligned}$$

- Similarly, in $\triangle CDB$ and $\triangle PYC$,

$$\begin{aligned}
\frac{PY}{CD} &= \frac{CP}{BC} \\
PY &= \frac{CP}{BC} \times CD \qquad (14.2)
\end{aligned}$$

- (14.1) + (14.2):

$$\begin{aligned}
PX + PY &= \frac{BP}{BC} \times CD + \frac{CP}{BC} \times CD \\
&= CD\left(\frac{BP + CP}{CD}\right) \\
&= CD \times \frac{BC}{BC} \\
&= CD
\end{aligned}$$

Question 15 (Lawson)

(a) i. (2 marks)

- ✓ [1] for finding h in terms of other variables.
- ✓ [1] for substituting (15.1) into area expression.

$$V = 0.504 = 1.8xh$$

$$\therefore h = \frac{0.504}{1.8x} \qquad (15.1)$$

Finding the area which the sheet is to cover:

$$\begin{aligned} A &= 1.8x + 2(1.8h) \\ &= 1.8\,(x + 2h) \end{aligned}$$

Substituting (15.1),

$$\begin{aligned} A &= 1.8\left(x + 2 \times \frac{0.508}{1.8x}\right) \\ &= 1.8\left(x + \frac{0.56}{x}\right) \end{aligned}$$

ii. (3 marks)

- ✓ [1] for differentiating and finding $x = \sqrt{0.56}$ as stationary point.
- ✓ [1] for classifying the type of stationary point.
- ✓ [1] for final answer.

$$A = 1.8\left(x + 0.56x^{-1}\right)$$

Stationary points occur when $\frac{dA}{dx} = 0$:

$$\begin{aligned} \frac{dA}{dx} &= 1.8\left(1 - 0.56x^{-2}\right) = 0 \\ 1 &= \frac{0.56}{x^2} \\ \therefore x^2 &= 0.56 \\ x &= \sqrt{0.56} \text{ (Note } x > 0) \end{aligned}$$

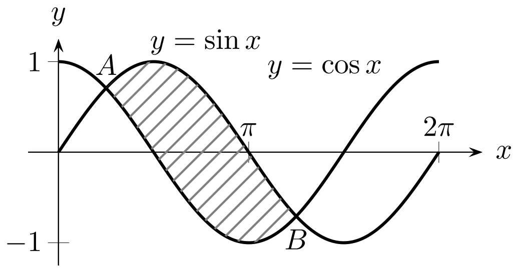
Testing for local min/max with second derivative:

$$\begin{aligned} \frac{d^2A}{dx^2} &= 1.8\left(1.12x^{-3}\right)\Big|_{x=\sqrt{0.56}} \\ &= \frac{2.016}{0.56^{\frac{3}{2}}} > 0 \end{aligned}$$

Hence $x = \sqrt{0.56}$ produces the least area.

$$A_{\text{min}} = 1.8\left(\sqrt{0.56} + \frac{0.56}{\sqrt{0.56}}\right) = 2.69\,\text{cm}^2$$

(b) i. (1 mark)



$$\begin{aligned} \underset{\div \cos x}{\sin x} &= \underset{\div \cos x}{\cos x} \\ \tan x &= 1 \\ \therefore x &= \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

ii. (3 marks)

- ✓ [1] for correct primitive.
- ✓ [1] for further working to simplify trig ratios
- ✓ [1] for final answer.

$$\begin{aligned} A &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\; dx \\ &= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -\left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4} - \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \\ &= -\left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4} - \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \\ &= 2\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) \\ &= 2 \times 2 \times \frac{1}{\sqrt{2}} \\ &= \frac{4}{\sqrt{2}} \text{ units}^2 \end{aligned}$$

(c) i. (1 mark)

$$f'(x) = kx^2 + 2$$

Stationary point occurs at $x = -1$, i.e. $f'(-1) = 0$:

$$\begin{aligned} k(-1)^2 + 2 &= 0 \\ k + 2 &= 0 \\ k &= -2 \end{aligned}$$

ii. (2 marks)

✓ [1] for the primitive

✓ [1] for final answer.

$$\begin{aligned} f'(x) &= -2x^2 + 2 \\ \therefore f(x) &= \int \left(-2x^2 + 2\right)\, dx \\ &= -\frac{2}{3}x^3 + 2x + C \end{aligned}$$

When $x = -1$, $y = 3$:

$$\begin{aligned} 3 &= -\frac{2}{3}(-1) + 2(-1) + C \\ C &= \frac{13}{3} \\ \therefore f(x) &= -\frac{2}{3}x^2 + 2x + \frac{13}{3} \end{aligned}$$

(d) i. (1 mark)

- The distance travelled is the area enclosed by the velocity-time graph and the t axis. Also, 4 min is $\frac{1}{15}$ of one hour. Hence

$$x = \int_0^{\frac{1}{15}} v\, dt$$

ii. (2 marks)

✓ [1] for converting minutes into hours.

✓ [1] for final answer.

Time (min)	0	1	2	3	4
Time (hrs)	0	$\frac{1}{60}$	$\frac{2}{60}$	$\frac{3}{60}$	$\frac{4}{60}$
Speed (km/h)	0	24	35	28	50

$$\begin{aligned} x &\approx \frac{\frac{1}{60}}{3}\left(0 + 4(24+28) + 2(35) + 50\right) \\ &= \frac{82}{45} \approx 1.822 \text{ km} \end{aligned}$$

Question 16 (Lawson)

(a) i. (2 marks)

✓ [1] for correct primitive

✓ [1] for finding C.

$$\begin{aligned} x &= \int v\, dt \\ &= \int \left(4t^3 - 24t^2 + 20t\right)\, dt \\ &= t^4 - 8t^3 + 10t^2 + C \end{aligned}$$

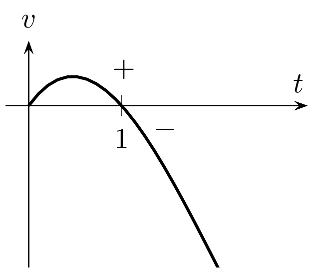
When $t = 0$, $x = 3$:

$$\therefore C = 3$$
$$x = t^4 - 8t^3 + 10t^2 + 3$$

ii. (2 marks)

✓ [1] for reasoning.

✓ [1] for identifying a local max.



- Pen at rest when $t = 1$ as $v = 0$.
- Also, for $0 < t < 1$, $v > 0$ and $1 < t < 5$, $v < 0$.
- Local maximum for v-t graph at $t = 1$.

iii. (3 marks)

✓ [1] for identifying distance being the area enclosed, as opposed to straight integral.

✓ [1] for correct dist travelled.

✓ [1] for amount of ink.

- Distance travelled: area enclosed between the graph and t axis.

$$\begin{aligned} x &= \int_0^1 v\,dt + \left|\int_1^2 v\,dt\right| \\ &= \left[t^4 - 8t^3 + 10t^2\right]_0^1 \\ &\quad + \left|\left[t^4 - 8t^3 + 10t^2\right]_1^2\right| \\ &= (1 - 8 + 10) \\ &\quad + \left|16 - 64 + 40 - (1 - 8 + 10)\right| \\ &= 3 + |-11| \\ &= 14\,\text{cm} \end{aligned}$$

Amount of ink used:

$$14 \times 0.05\,\text{mg} = 0.7\,\text{mg}$$

(b) i. (1 mark)

$$BAC = 0.7e^{-kt}$$

When $t = 1$, $BAC = 0.33$:

$$0.33 = 0.7e^{-k}$$

$$\therefore k = -\ln\frac{0.33}{0.7} \approx 0.7520$$

ii. (3 marks)

✓ [1] for manipulation to obtain t as subject, after substitution.

✓ [1] for $t = 9.633\cdots$

✓ [1] for nearest min

$$\begin{aligned} 0.0005 &= 0.7e^{-kt} \\ e^{-kt} &= \frac{0.0005}{0.7} \\ -kt &= \ln\frac{0.0005}{0.7} \\ t &= \frac{\ln\frac{0.0005}{0.7}}{\ln\frac{0.33}{0.7}} \\ &= 9.63328 \end{aligned}$$

Hence it takes approx 9.633 hrs (9 hr 38 min) for BAC to reach effective zero.

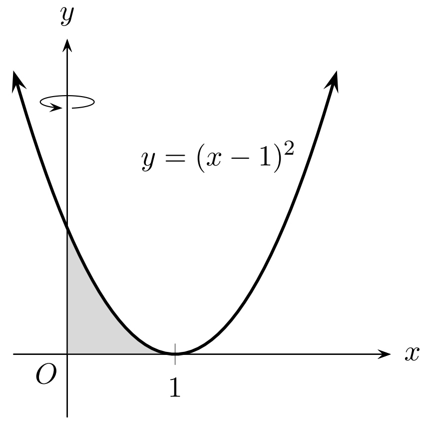
(c) (4 marks)

✓ [1] for changing subject of the parabola to x with correct square root.

✓ [1] for correctly squaring integrand and volume integral

✓ [1] for correct primitive

✓ [1] for final answer.



Volume integral requires x as subject:

$$V = \int_0^1 x^2\,dy$$

Change subject of $y = (x-1)^2$ to x:

$$y = (x-1)^2$$

Note that region being rotated is the 'negative' branch of the parabola, hence negative root required.

$$\begin{aligned} x - 1 &= -\sqrt{y} \\ x &= 1 - \sqrt{y} \end{aligned}$$

Substitute into the volume integral,

$$\begin{aligned} V &= \pi\int_0^1 \left(1 - \sqrt{y}\right)^2\,dy \\ &= \pi\int_0^1 \left(1 - 2\sqrt{y} + y\right)\,dy \\ &= \pi\int_0^1 \left(1 - 2y^{\frac{1}{2}} + y\right)\,dy \\ &= \pi\left[y - \frac{4}{3}y^{\frac{3}{2}} + \frac{1}{2}y^2\right]_0^1 \\ &= \pi\left(1 - \frac{4}{3} + \frac{1}{2}\right) \\ &= \frac{\pi}{6}\ \text{units}^3 \end{aligned}$$