NSBHS - 2006

Question 1 (12 marks)

Marks

(a) Evaluate $\frac{12.9}{\sqrt{6.7 \times 3.4}}$ correct to 3 significant figures. **2**

(b) Factorise $1-8y^3$. **2**

(c) Find the value of $\frac{\log_3 8}{\log_3 2}$. **2**

(d) Find a primitive of $5+\sin 2x$ **2**

(e) Find the values of x for which $x^2-6x+5>0$. **2**

(f) Solve the simultaneous equations : **2**

$$2x+y=3$$
$$x-2y=4$$

Question 2 (12 marks) Start question on a new page.

Marks

(a) Differentiate with respect to x:

(i) $(x+1)^7$. **1**

(ii) $x\tan x$ **2**

(iii) $\log_e\left(\frac{x}{x-1}\right)$ **2**

(b) Find:

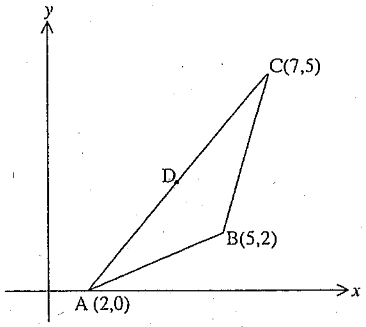
(i) $\int \frac{x}{x^2+6}\,dx$ **2**

(ii) $\int \frac{3}{e^{2x}}\,dx$ **2**

(c) Evaluate $\int_1^e \left(\frac{2}{x}+\frac{x}{2}\right)dx$ leaving your answer in exact form. **3**

Question 3 (12 marks) Start question on a new page. **Marks**

(a) The points A(2,0), B(5,2) and C(7,5) are joined to form a triangle as shown below. D is the midpoint of AC.



(i) Find the length of AC — 1

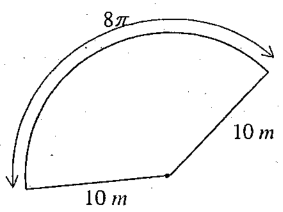
(ii) Find the co-ordinates of D — 1

(iii) Find the slope of DB, and prove that it is perpendicular to AC — 2

(iv) BD is extended to E, so that BD = DE.
Find the co-ordinates of the point E. — 1

(v) Find the area of the quadrilateral ABCE — 2

(b)



The diagram shows a garden bed in the shape of a sector. The arc length is 8π *metres* and the radius is 10 *metres*

(i) Show that the angle of the sector is $\frac{4\pi}{5}$ radians — 1

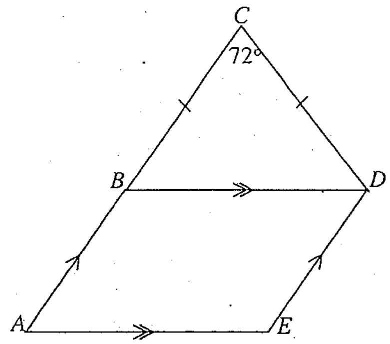
(ii) Calculate the area of this garden bed. — 2

(iii) The garden bed is to be planted with red and yellow tulips. If the tulips can be planted at 15 per square metre, how many tulips can be planted? — 1

(iv) Assuming all tulips flower, what is the expected number of red tulips if the probability of producing a red flower is 0.6? — 1

Question 4 (12 marks) Start question on a new page. **Marks**

(a)



A, B and C are collinear points.
$BD \parallel AE$, $AB \parallel ED$, $BC = \cancel{BD}$ CD
and $\angle BCD = 72°$

Copy this diagram on your answer sheet.

Find the size of $\angle DEA$, giving reasons. — 3

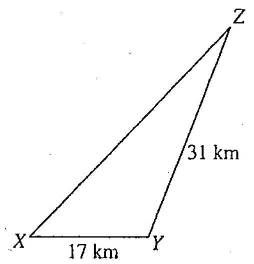
(b) Use Simpson's rule with three function values (i.e. one application) to estimate $\int_1^5 \log_e x \, dx$. — 3

(c) Solve $4^x - 18(2^x) + 32 = 0$ — 3

(d) Solve $2\cos 2x + \sqrt{3} = 0$ for $0 \le x \le 2\pi$ — 3

Question 5 (12 marks) Start question on a new page. **Marks**

(a) In the diagram X, Y and Z represent the locations of three towns. The town Y is due east of X and the bearing of Z from Y is 046°.



(i) Find the size of $\angle XYZ$. 1

(ii) Find the distance XZ to 1 decimal place. 2

(iii) What is the bearing of Y from Z? 1

(b) The root of the equation $x+\frac{1}{x}=7$ are α and β

Find the value of:

(i) $\alpha+\frac{1}{\alpha}$ 1

(ii) $\alpha+\beta$ 1

(c) (i) Show that $\frac{3x+4}{x+1}=\frac{1}{x+1}+3$ 1

Hence:

(ii) Sketch the graph of $y=\frac{3x+4}{x+1}$ showing all the important features. (Do not find stationary points). 2

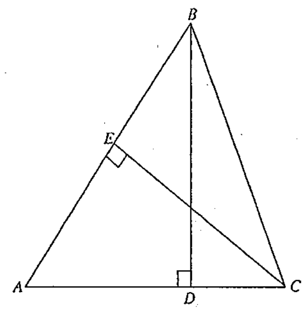
(iii) Find the exact area of the region bounded by the curve $y=\frac{3x+4}{x+1}$, the x and y axes, and the line $x=2$. 3

Question 6 (12 marks) Start question on a new page. **Marks**

(a) Given that $\log_a 3=0.68$ and $\log_a 2=0.42$, find $\log_a 18$ 2

(b) Find the limiting sum of the series $\frac{9}{8}+\frac{3}{4}+\frac{1}{2}$ 2

(c)



The diagram shows BD⊥AC and CE⊥AB

(i) Copy this diagram into your answer booklet and prove $\Delta ECA \parallel\!\mid \Delta DBA$ 3

(ii) If AB = 10 cm, BD = 7cm and AC = 16 cm find the length of CE. 2

(d) The rate of water flowing, R litres per hour, into a pond is given by

$$R=65+4t^{\frac{1}{3}}$$

(i) Calculate the initial flow rate. 1

(ii) If initially there was 15 litres in the pond, find the volume of the water in the pond when 8 hours have elapsed. 2

Question 7 (12 marks) Start question on a new page. **Marks**

(a) Evaluate $\sum_{k=3}^{5} 2^{4-k}$ 1

(b) A particle moves in a straight so that its distance x, in metres, from a fixed point O at time t, in seconds, is given by

$$x = 5t + \log_e(1-2t), \quad 0 \le t \le \frac{1}{2}.$$

(i) Find the initial velocity and acceleration of the particle. 4

(ii) When does the particle come to rest? 2

(c) A parabola has the equation $x^2 = -12y$

(i) Find the co-ordinates of the vertex of the parabola 1

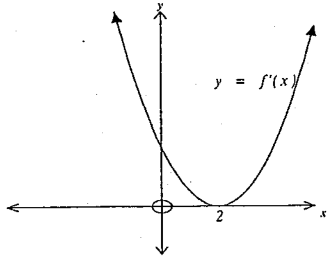
(ii) Write down the focus of the parabola 1

(iii) Find the equation of the tangent of the parabola at the point where $x = 6$. 2

(iv) Find the co-ordinates of Y, the point where the tangent cuts the y-axis 1

Question 8 (12 marks) Start question on a new page. **Marks**

(a) The figure shows the graph of $y = f'(x)$



The curve $y = f(x)$ has a stationary point at (2,0). What is the nature of this stationary point? 1

(b) Consider the curve $y = \frac{1}{x}e^{-x}$:

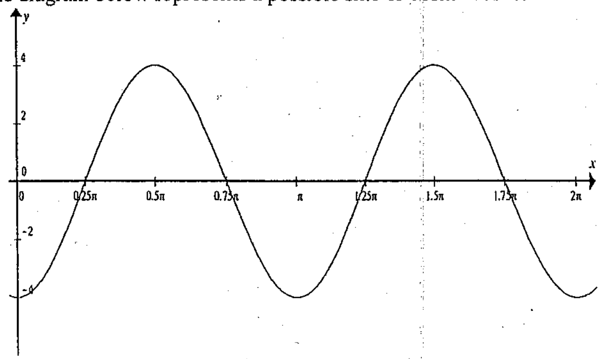
(i) For what values of x is the function defined? 1

(ii) Describe the behaviour of the function as x: 2
(α) approaches zero
(β) increases indefinitely

(iii) Find any stationary points and determine their nature. 3

(iv) Sketch the curve of this function 2

(c) The diagram below represents a possible sine or cosine curve.



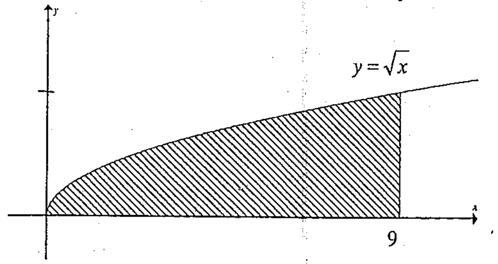
(i) Give the amplitude 1

(ii) Give the period 1

(iii) Write down the possible equation of the curve 1

Question 9 (12 marks) Start question on a new page. **Marks**

(a) Find the volume of the solid formed when the shaded area under the curve $y=\sqrt{x}$, shown below, is rotated around the y**-axis.** 4



(b) (i) Sketch the curve $y=3\sin\frac{\pi x}{2}$ for $-2\le x\le 4$. 1

(ii) Draw on your diagram a line, clearly labelled, which can be used to solve the following equation: 2

$$\sin\frac{\pi x}{2}-\frac{x}{3}=0$$

(iii) Determine the number of solutions to the equation $\sin\frac{\pi x}{2}-\frac{x}{3}=0$ over the domain $-2\le x\le 4$. 1

(c) In a game of chess between two players X and Y, both of approximately equal ability, the player with the *White* pieces, having the first move, has a probability of 0.5 of winning, and the probability that the player with the black pieces for that game winning is 0.3.

(i) What is the probability that the game ends in a draw? 1

(ii) The two players X and Y play each other in chess competition, each player having the *White* pieces once.
In the competition the player who wins the game scores 3 points, the player who loses the game scores 1 point and in a draw each player receives 2 points.
By drawing a tree diagram, or otherwise, find the probability that, as a result of these two games,

(α) X scores 6 points. 1

(β) X scores less than 4 points. 2

Question 10 (12 marks) Start question on a new page. **Marks**

(a) The number N of a certain species is falling according to $N=N_0e^{-0.03t}$ where t is in days and N_0 is the initial number of species present.

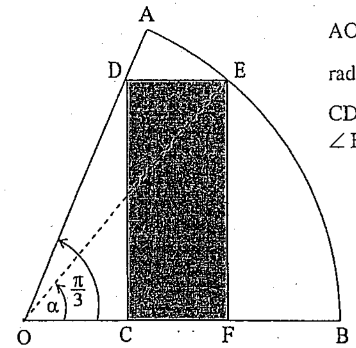
(i) Show that $N=N_0e^{-0.03t}$ is a solution to the differential equation $\frac{dN}{dt}=-0\cdot 03N$. 1

(ii) How long, to the nearest day, will it take for the number of species to halve? 1

(iii) Find, in terms of N_0, the rate of change at the time when the number of species has halved. 1

(iv) Find the number of days, to the nearest whole number, for the number of species to fall to just below 5% of the initial number. 2

(b)



NOT TO SCALE

AOB is a sector of a circle with centre at O and radius r such that $\angle \cancel{OAB}$ AOB $=\frac{\pi}{3}$.
CDEF is a rectangle drawn in the sector and $\angle$ EOF $=\alpha$ as shown in the diagram.

(i) Show that $CF=r\cos\alpha-\frac{r\sin\alpha}{\sqrt{3}}$ 2

(ii) Given that $\frac{1}{2}\sin 2\alpha=\sin\alpha\cos\alpha$, show that the area of rectangle CDEF can be expressed as $A=r^2\left(\frac{1}{2}\sin 2\alpha-\frac{\sqrt{3}}{3}\sin^2\alpha\right)$ 2

(iii) Find the value for α which will produce the rectangle of maximum area. 3

Solution of 2006 2 Unit Trial NSBHS

Question 1

a) $2.70279 \doteqdot 2.70$ (3 sign. figures)

b) $1-8y^3=(1-2y)(1+2y+4y^2)$

c) $\frac{\log_3 8}{\log_3 2} = \frac{3\log_3 2}{\log_3 2}$
$= 3$

d) $\int(\sin 2x+5)\,dx = -\frac{1}{2}\cos 2x + 5x + C$

e) $x^2-6x+5>0$
$(x-5)(x-1)>0$
$x<1$ or $x>5$

f) $2x+y=3$
$x-2y=4$
$x=4+2y$
$\therefore 2(4+2y)+y=3$
$8+4y+y=3$
$\therefore y=-1$
$x=2$

Question 2

a) i) $y=(x+1)^7$
$\frac{dy}{dx}=7(x+1)^6$

ii) $y = x\tan x$
$\frac{dy}{dx} = 1.\tan x + x\sec^2 x$

iii) $y = \log_e\left(\frac{x}{x-1}\right)$
$= \log_e x - \log_e(x-1)$
$\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x-1}$

b) i) $\int \frac{x}{x^2+6}\,dx = \frac{1}{2}\ln(x^2+6)+C$

ii) $\int 3e^{-2x}\,dx = -\frac{3}{2}e^{-2x}+C$

c) $\int_1^e \frac{2}{x}+\frac{x}{2}\,dx = \left[2\ln x+\frac{x^2}{4}\right]_1^e$
$= 2\ln e + \frac{e^2}{4} - \left(0+\frac{1}{4}\right)$
$= \frac{7}{4}+\frac{e^2}{4}$

Question 3

i) $A(2,0)$ $C(7,5)$
$AC = \sqrt{(7-2)^2+(5-0)^2}$
$= \sqrt{50} = 5\sqrt{2}$

ii) $D\left(\frac{9}{2},\frac{5}{2}\right)$

iii) $m_{DB} = \frac{\frac{5}{2}-2}{\frac{9}{2}-5}$
$= -1$
$m_{AC} = \frac{5-0}{7-2}$
$= 1$
$m_{DB}\times m_{AC} = -1$
$\therefore DB \perp AC$

iv) $E(4,3)$

v) ABCE is a kite (diag. bisect at $90°$)
$\therefore$ Area $= \frac{1}{2}\times AC\times EB$.
$EB = \sqrt{(4-5)^2+(3-2)^2}$
$= \sqrt{2}$
$\therefore$ Area $= \frac{1}{2}\, 5\sqrt{2}\times\sqrt{2}$
$= 5$ sq units

b) i) $l = r\theta$
$8\pi = 10\,\theta$
$\therefore \theta = \frac{4\pi}{5}$

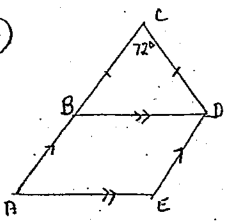
ii) $A = \frac{1}{2}r^2\theta$
$= \frac{1}{2}10^2\times\frac{4\pi}{5}$
$= 40\pi$

iii) No of tulips $= 40\pi\times 15 = 188$
$\doteqdot 1884$

iv) No of red $\doteqdot 0.6\times 1884$
$\doteqdot 1130$

Question 4

a)



$\angle CBD = \frac{180°-72°}{2}$ (isos. Δ, base ∠s are =)
$= 54°$
$\angle ABD = 180°-54°$ (st. line)
$= 126°$
$\therefore \angle AED = 126°$ (opp. ∠s of parallelogram)

(b)

x	1	3	5
y	0	1.099	1.609

$y=\log_e x$

$\int_1^5 \log_e x\,dx = \frac{2}{3}\left[0+4\times1.099+1.609\right]$
$\doteqdot 4$

(c) Let $u=2^x$
$\therefore u^2-18u+32=0$
$(u-16)(u-2)=0$
$\therefore u=16$ or $u=2$
i.e. $2^x=16$ or $2^x=2$
$\therefore x=4$ or $x=1$

d) $2\cos 2x+\sqrt{3}=0$
$\cos 2x = -\frac{\sqrt{3}}{2}$
$2x = \pi-\frac{\pi}{6}, \pi+\frac{\pi}{6}, 3\pi-\frac{\pi}{6}, 3\pi+\frac{\pi}{6}$
$x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

QUESTION 5

i) $\angle XYZ = 46° + 90°$
$= 136°$ ✓

ii) $XZ^2 = 17^2 + 31^2 - 2 \times 17 \times 31 \times \cos 136°$ ✓
$XZ \doteqdot 44.8$ ✓

iii) $180° + 46° = 226°$ ✓

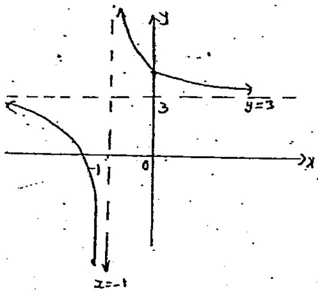
$x + \frac{1}{x} = 7$ ✓

i) $\alpha + \frac{1}{\alpha} = 7$

ii) $x^2 + 1 = 7x$
$x^2 - 7x + 1 = 0$
$\alpha + \beta = \frac{-b}{a}$ ✓
$= 7$

i) $\frac{1}{x+1} + 3 = \frac{1 + 3(x+1)}{x+1}$ ✓
$= \frac{3x+4}{x+1}$

ii)



✓ for asymptotes
✓ graph

iii) $A = \int_0^2 \frac{1}{x+1} + 3 \, dx$ ✓
$= \left[\ln(x+1) + 3x\right]_0^2$ ✓
$= \ln 3 + 6 - (0 + 0)$ ✓
$= \ln 3 + 6$ sq units

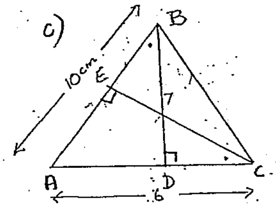
QUESTION 6

a) $\log_a 18 = \log_a(2 \times 3^2)$
$= \log_a 2 + 2\log_a 3$ ✓
$= 0.42 + 2 \times 0.68$ ✓
$= 1.78$

b) $a = \frac{9}{8}$, $r = \frac{3}{4} \div \frac{9}{8}$ ✓
$= \frac{2}{3}$

$S = \frac{a}{1-r}$
$= \frac{\frac{9}{8}}{1 - \frac{2}{3}}$ ✓
$= \frac{27}{8}$

c)



✓

(i) $\angle A$ is common
$\angle AEC = \angle BDA = 90°$ (given) ✓
$\therefore \triangle ECA \,|||\, \triangle DBA$ equiangular ✓

ii) $\frac{CE}{BD} = \frac{AC}{AB}$ (corres sides of similar Δs) ✓
$\frac{CE}{7} = \frac{6}{10}$ ✓
$CE = 4.2$ cm

d) $R = 65 + 4t^{1/3}$

i) $t = 0$, $R = 65$ L/hr ✓

ii) $V = \int 65 + 4t^{1/3} \, dt$
$= 65t + 4 \times \frac{3}{4} t^{4/3} + C$
$= 65t + 3t^{4/3} + C$ ✓
$t = 0$, $V = 15$
$\therefore C = 15$

$\therefore V = 65t + 3t^{4/3} + 15$
When $t = 8$,
$V = 65 \times 8 + 3 \times 8^{4/3} + 15$
$= 583$ litres ✓

QUESTION 7

a) $\sum_{k=3}^{5} 2^{4-k} = 2 + 2^0 + 2^{-1}$ ✓
$= 3\frac{1}{2}$

b) i) $x = 5t + \log_e(1 - 2t)$
$\frac{dx}{dt} = 5 + \frac{-2}{1-2t}$ ✓
$= 5 - 2(1-2t)^{-1}$ —(1) ✓
$\frac{d^2x}{dt^2} = 2(1-2t)^{-2} \cdot {-2}$ ✓
$= \frac{-4}{(1-2t)^2}$ —(2) ✓

$t = 0$, $v = 3$ m/s
$a = -4$ m/s²

ii) $v = 0$
$\therefore 5 - \frac{2}{1-2t} = 0$ ✓
$5(1-2t) - 2 = 0$
$3 - 10t = 0$
$t = \frac{3}{10}$ sec ✓

c) i) $V(0,0)$ ✓

ii) $-12 = 4a$
$a = -3$
$\therefore F(0,-3)$ ✓

iii) $y = -\frac{x^2}{12}$
$\frac{dy}{dx} = -\frac{x}{6}$ ✓
When $x = 6$, $m_T = -1$
$y = -3$
$y - y_1 = m(x - x_1)$

iv) $x = 0$
$y = 3$
$\therefore A(0,3)$ ✓

QUESTION 8

a) Horizontal point of inflexion ✓

b) i) all real x, $x \neq 0$ ✓

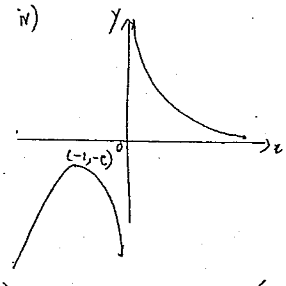
ii) α) $x \to 0$, $y \to \infty$ ✓
β) $x \to \infty$, $y \to 0$ ✓

iii) $y = \frac{e^{-x}}{x}$
$\frac{dy}{dx} = \frac{-e^{-x}x - e^{-x}}{x^2}$ ✓
$\frac{dy}{dx} = 0$, $-e^{-x}x - e^{-x} = 0$
$-e^{-x}(x+1) = 0$
$\therefore x = -1$ ✓
$y = \frac{e}{-1} = -e$

x	-1^-	-1	-1^+
y'	+	0	−

$\therefore$ MAX $(-1, -e)$ ✓

iv)



✓
✓

c) i) 4 ✓
ii) π ✓
iii) $y = -4\cos 2x$ ✓

QUESTION 9

a) $V = \pi r^2 h - \int_0^3 \pi x^2 \, dy$ ✓

$y = \sqrt{x}$

$y^4 = x^2$

$\therefore V = \pi \times 9^2 \times 3 - \int_0^3 \pi y^4 \, dy$

$= 243\pi - \pi \left[\frac{y^5}{5}\right]_0^3$ ✓✓

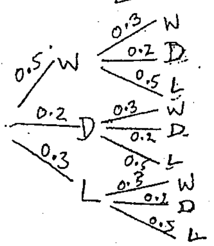
$= 243\pi - \frac{243\pi}{5}$

$= \frac{972\pi}{5}$ ✓

c) i Suppose X has white first

Let W = X wins

L = X loses.



i) $P(D) = 1 - 0.5 - 0.3$

$= 0.2$ ✓

ii) $P(6) = P(WW)$

$= 0.5 \times 0.3$

$= 0.15$ ✓

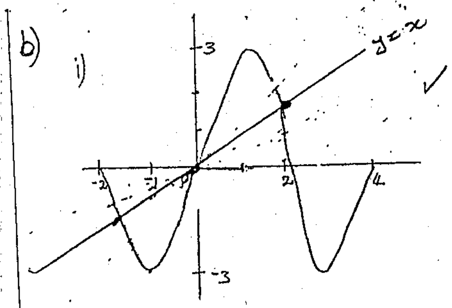
iii) $P(<4) = P(DL) + P(LD) + P(LL)$ ✓

$= 0.2 \times 0.5 + 0.3 \times 0.2 + 0.3 \times 0.5$

$= 0.10 + 0.06 + 0.15$

$= 0.31$ ✓

b) i)



✓

ii) $3\sin\frac{\pi x}{2} = x$ ✓✓

iii) 3 solutions ✓

10) a) i) $N = N_0 e^{-0.03t}$

$\frac{dN}{dt} = -0.03 N_0 e^{-0.03t}$

$= -0.03 N$ ✓

ii) $N = \frac{1}{2} N_0$

$\therefore \frac{1}{2} N_0 = N_0 e^{-0.03t}$

$\ln\left(\frac{1}{2}\right) = -0.03t$

$t = -\frac{\ln\left(\frac{1}{2}\right)}{0.03} = \frac{\ln 2}{0.03}$ ✓

$\doteqdot 23$ days

iii) $\frac{dN}{dt} = -0.03 \times \frac{1}{2} N_0$

$= -0.015 N_0$ ✓

iv) $N < 0.05 N_0$

$\therefore 0.05 N_0 > N_0 e^{-0.03t}$ ✓

$t > \frac{\ln 0.05}{-0.03} \doteqdot 99.9$

i.e. $t = 100$ days ✓

b) i) $CF = OF - OC$

$\frac{OF}{OE} = \cos\alpha$

$\therefore OF = r\cos\alpha$ ✓

$\frac{OC}{DC} = \cot\frac{\pi}{3}$

$OC = \frac{1}{\sqrt{3}} DC$

$DC = EF = r\sin\alpha$ ✓

$\therefore OC = \frac{r\sin\alpha}{\sqrt{3}}$

$CF = r\cos\alpha - \frac{r\sin\alpha}{\sqrt{3}}$ (given)

b) ii) $A = CF \times EF$

$= \left(r\cos\alpha - \frac{r\sin\alpha}{\sqrt{3}}\right) \times r\sin\alpha$ ✓

$= r^2\left(\cos\alpha\sin\alpha - \frac{\sin^2\alpha}{\sqrt{3}}\right)$

$= r^2\left(\frac{1}{2}\sin 2\alpha - \frac{\sqrt{3}\sin^2\alpha}{3}\right)$ ✓

iii) $\frac{dA}{d\alpha} = r^2\left(\cos 2\alpha - \frac{2\sqrt{3}}{3}\sin\alpha\cos\alpha\right.$

$\frac{dA}{d\alpha} = 0$ $\quad \cos 2\alpha - \frac{2\sqrt{3}}{3}\frac{\sin 2\alpha}{2} = 0$

$\therefore \cos 2\alpha = \frac{\sqrt{3}}{3}\sin 2\alpha$ ✓

$\therefore \tan 2\alpha = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{3} = \sqrt{3}$

$2\alpha = \frac{\pi}{3}$

$\therefore \alpha = \frac{\pi}{6}$ ✓

Test for max.

α	$\pi/12$	$\pi/6$	$\pi/3$
$\frac{dA}{d\alpha}$	+	0	−