North Sydney Boys High School

MATHEMATICS

2019 HSC Course Assessment Task 3
(Trial Examination)
Thursday June 27, 2019.

General Instructions

- Working time –3 hours
  (plus 5 minutes reading time).
- Write using blue or black pen.
  Diagrams may be sketched in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Attempt all questions.

Section I - 10 marks

- Mark your answers on the answer sheet provided.

Section II – 90 marks

- Commence each new question on a new page.
- Show all necessary working in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** ....

Class (please ✓)

☐ Mr Berry ☐ Ms Ziaziaris
☐ Mr Hwang ☐ Mr Zuber
☐ Mr Lin

Question	MC	11	12	13	14	15	16	Total
Marks	/10	/15	/15	/15	/15	/15	/15	/100

Section 1: Multiple Choice– 1 mark each.

Q1. What values for a and b satisfy the following equation:

$$\frac{14}{3-\sqrt{2}} = a + \sqrt{b}$$

(A) $a = 3$ and $b = 2$

(B) $a = 6$ and $b = -2$

(C) $a = 6$ and $b = 2$

(D) $a = 6$ and $b = 8$

Q2. Which of the following is perpendicular to $2x - 4y + 1 = 0$?

(A) $2x + y + 5 = 0$

(B) $2x + 4y - 2 = 0$

(C) $4x - 2y + 3 = 0$

(D) $x - 2y - 1 = 0$

Q3. Which of the following is equal to $\frac{d}{dx}\log_e(\tan x)$?

(A) $\cot x$

(B) $\sec x \cos x$

(C) $\frac{1}{\sec^2 x}$

(D) $\sec^2 x \tan x$

Q4. Which expression is a factorisation of $343 - x^6$?

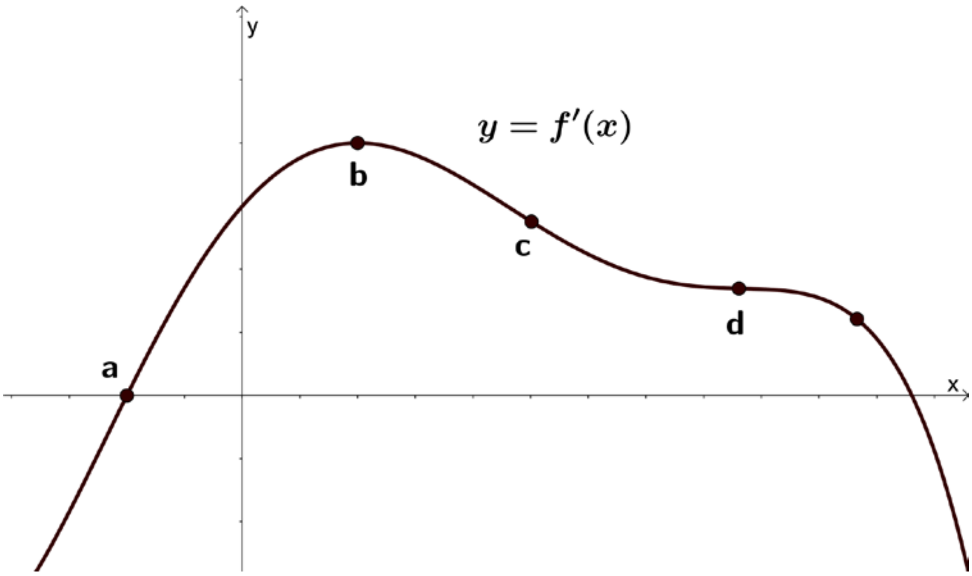
(A) $(7 - x^2)(49 + 7x^2 + x^4)$

(B) $(7 - x^2)(49 + 14x^2 + x^4)$

(C) $(7 - x^2)(49 - 7x^2 + x^4)$

(D) $(7 - x^2)(49 - 14x^2 + x^4)$

Q5. The diagram below shows the graph of the $y = f'(x)$, the derivative of a function:



Which point on this graph indicates an inflexion point on $= f(x)$?

(A) a

(B) b

(C) c

(D) d

Q6. The solution to $8x - 1 > 7x^2$ is

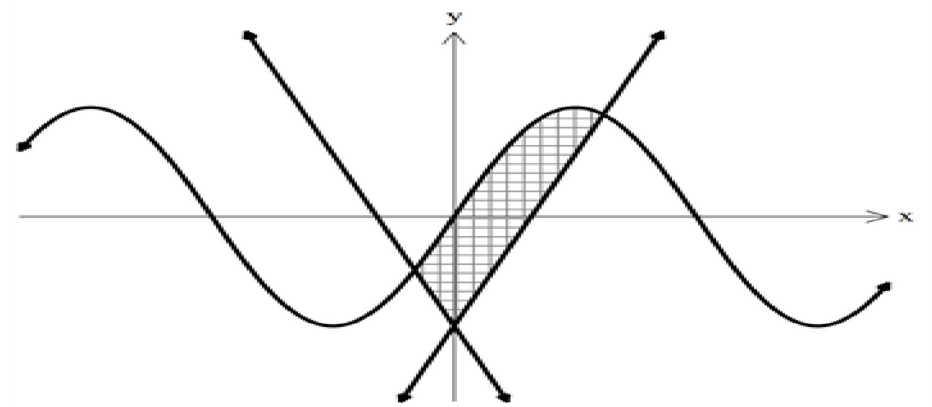
(A) $\frac{1}{7} < x < 1$

(B) $x < \frac{1}{7}, x > 1$

(C) $x < -1, x > -\frac{1}{7}$

(D) $-1 < x < -\frac{1}{7}$

Q7.



Which if the following best describes the shaded region?

.

(A) $x - y - 1 \leq 0$ and $x + y + 1 \leq 0$ and $y \leq \sin x$

(B) $x - y - 1 \leq 0$ and $x + y + 1 \geq 0$ and $y \leq \sin x$

(C) $x - y - 1 \geq 0$ and $x + y + 1 \leq 0$ and $y \leq \sin x$

(D) $x - y - 1 \geq 0$ and $x + y + 1 \geq 0$ and $y \leq \sin x$

Q8. Which of the following is equivalent to $\int_0^1 2^x dx$

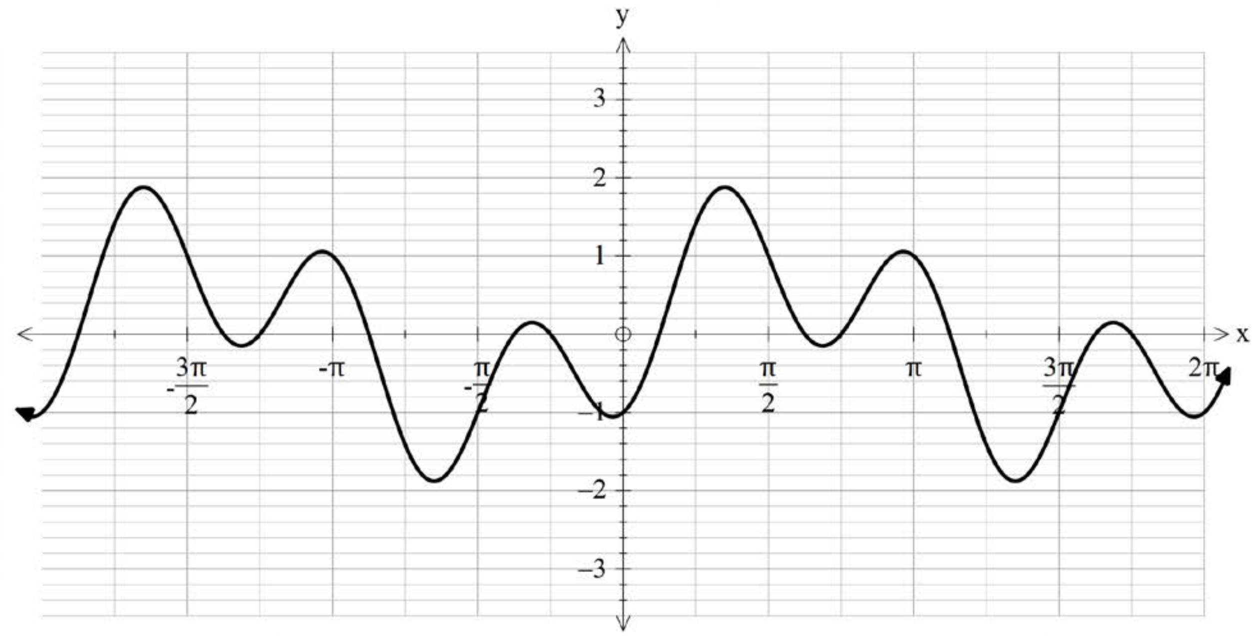
(A) $\frac{1}{\ln 2}$

(B) $\frac{2}{\ln 2}$

(C) $\ln 2$

(D) $2 \ln 2$

Q9. The graph of $y = \sin x - \cos 3x$ is shown:



How many solutions are there for the equation $2 \sin x - 2 \cos 3x = x$

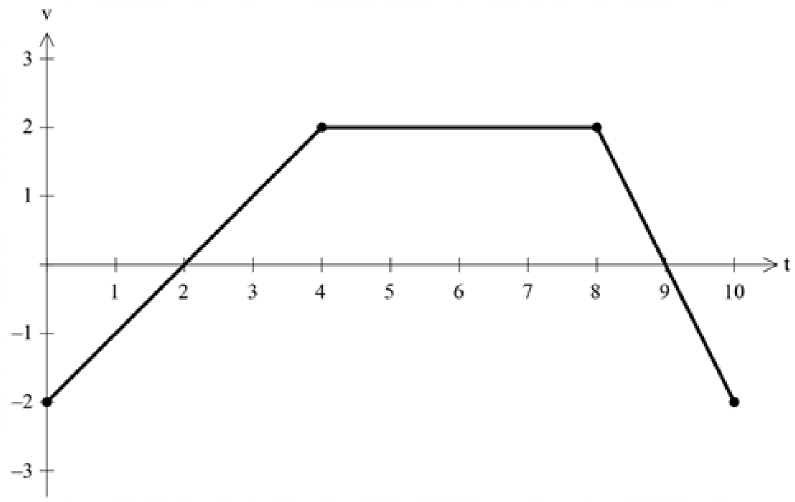
(A) 2

(B) 3

(C) 5

(D) 7

Q10. The following is a velocity-time diagram for a particle moving in one dimension, where time t is in seconds and velocity is in metres/second.



The particle is initially 2 metres to left of the origin.

When does the particle return to the origin?

(A) 2 seconds

(B) 4 seconds

(C) 5 seconds

(D) 6 seconds

End of Section I

Section II – Short Answer 90 marks

Question 11 (15 marks) Commence on a NEW page. Marks

(a) Factorise fully $2x^2y - 3x^2 - 8y + 12$ *2*

(b) State the domain and range of the function $f(x) = \sqrt{16 - x^2}$ *2*

(c) Solve $|\,3x - 1\,| = 4x$ *3*

(d) Differentiate $y = (e^{2x} + \log_e x)^4$ *2*

(e) Find the equation of tangent to the curve $y = 2\cos x$ at $x = \frac{\pi}{3}$. *3*

(f) Solve $\cos 2x = \frac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$. *3*

End of Question 11

Question 12 (15 marks) Commence on a NEW page. Marks

(a) The points $A\,(-8,4),\ B(6,-2),\ C(-4,7)$ are defined in the Cartesian Plane.

i) Show that the equation of the line AB is $6x + 14y - 8 = 0$. *1*

ii) Find the perpendicular distance from line AB to point C. *2*

iii) Hence or otherwise fine the area of ΔABC. *2*

(b) Find:

i) $\displaystyle\int \frac{dx}{(1+5x)^4}$ *1*

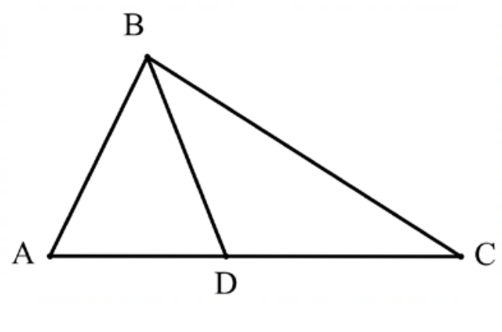
ii) $\displaystyle\int \sqrt{x}\,(4-\sqrt{x}\,)\,dx$ *2*

iii) $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x\,dx$ *2*

(c) Differentiate $y = \log_e \dfrac{5x+1}{(x-3)^2}$ *2*

(d) Triangle ABC is an isosceles triangle with AC = BC.

The triangle was constructed so that AB = BD = DC.



i) **<u>Trace or copy the diagram into your workbook.</u>**

ii) Find the value of $\angle BCD$, showing reasoning. *3*

End of Question 12

Question 13 (15 marks) Commence on a NEW page. Marks

(a) For the parabola $12x + y^2 + 6y - 15 = 0$

i) Find the coordinates of the vertex. *2*

ii) Find the coordinates of the focus. *2*

(b) The roots of the equation $x^2 - 2kx + 3k = 0$ differ by four. *3*

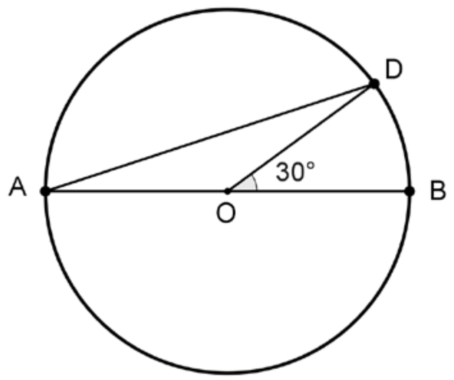
Find the value(s) of k.

(c) A line L passes through the point $Q(4, -2)$ and the intersection of the lines *3*

$$2x + 5y - 1 = 0 \text{ and } x + y - 5 = 0$$

Find the equation of L in general form.

(d) The diagram below shows a circle radius 1 unit, diameter AB and $\angle DOB = 30°$



i) Explain why $\angle DAO = 15°$. *1*

ii) Show that $AD^2 = 2 + \sqrt{3}$. *2*

iii) Hence show that $\sin 15° = \frac{1}{2}\sqrt{2 - \sqrt{3}}$ *2*

End of Question 13

Question 14 (15 marks) Commence on a NEW page. Marks

(a) Given the function

$$f(x) = x^4 - 8x^3$$

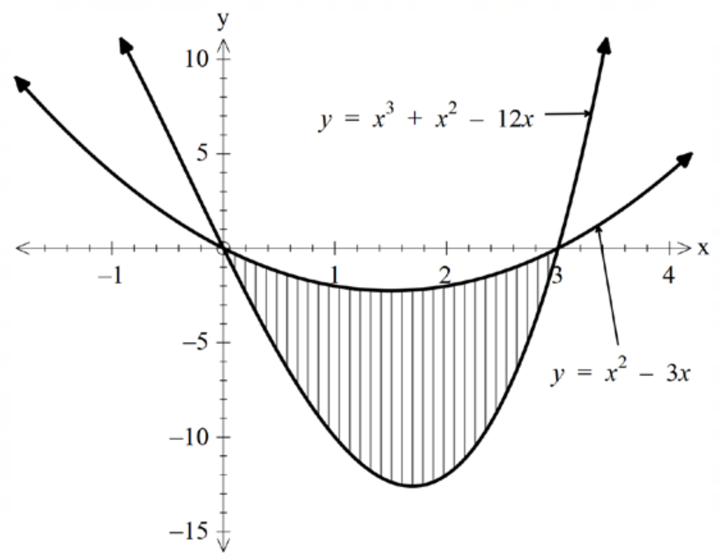
i) Find the stationary points of $y = f(x)$ and determine their nature. *4*

ii) Sketch the graph of $y = f(x)$. *3*

(b) The diagram below shows the graphs of $y = x^2 - 3x$ and $y = x^3 + x^2 - 12x$. *3*

The graphs intersect at $(0,0)$ and $(3,0)$.

Find the area between the two curves.



(c) By considering the sum of a suitable <u>infinite geometric series</u>, express $0.5\dot{4}\dot{3}$ as a fraction in simplest form. *3*

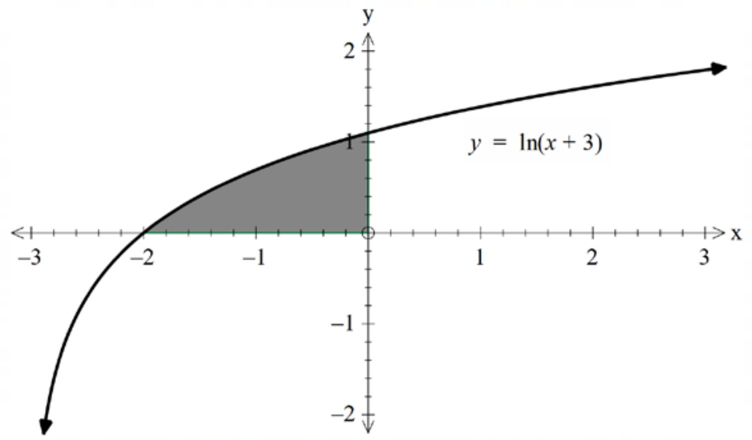
(d) Evaluate $\displaystyle\sum_{k=3}^{100} (-1)^k \, (2k+1)$. *2*

End of Question 14

Question 15 (15 marks) Commence on a NEW page.

(a) The area enclosed by the curve $y = \log_e(x+3)$, the x-axis and the y-axis is rotated about the y-axis. *4*

Find the volume of the resulting solid of revolution, in exact form.



(b) Consider the first quadrant of a circle radius 1: *4*

$$y = \sqrt{1-x^2}, \quad 0 \le x \le 1.$$

Use Simpson's Rule with 5 function values in the range $0 \le x \le 1$ to find an estimate for the area of the quadrant and hence show that

$$\pi \approx \frac{1}{3}(1 + \sqrt{3} + \sqrt{7} + \sqrt{15})$$

(c) Use the fact that *2*

$$\frac{u}{u-1} - \frac{u}{u+1} = \frac{2u}{u^2-1}$$

to find

$$\int \frac{e^x}{e^{2x}-1}\,dx$$

Question 15 continues on the next page

Question 15 (continued)

(d) A particle travels along the *x*-axis with acceleration given by

$$\ddot{x} = -6t$$

where t is the time in seconds. The particle has an initial velocity of 9 m/s.

i) Find the equation of the velocity *1*

ii) When is the particle stationary? *1*

iii) What is the average speed in the first 4 seconds? *3*

End of Question 15

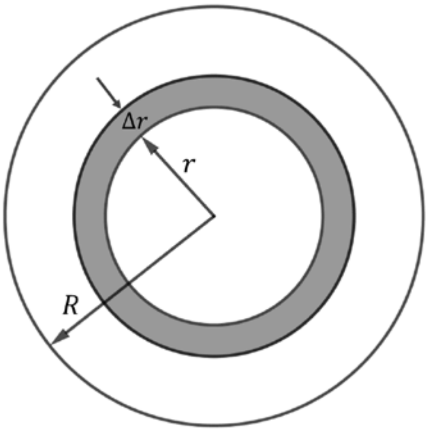
Question 16 (15 marks) Commence on a NEW page.

(a) The YouTube channel “5 Minutes Physics” demonstrates how a circle can be dissected into thin strips, then transformed into a simple shape for which we can find the area.

We will now do a similar dissection using mathematics:

Consider a circle of radius R (see diagram below).

Within that circle, consider a thin annulus, r units from the centre, with a width Δr units.



i) Draw a diagram to explain why the area of the annulus in the diagram above, when it is unrolled, can be expressed as: *1*

$$A_{strip} \approx 2\pi r \cdot \Delta r$$

ii) Using no more than three sentences, explain the ideas that allow us to develop the expression in part (i) into the following: *2*

$$A_{circle} = \int_0^R 2\pi r \, dr$$

iii) Hence derive the formula for the area of the circle. *1*

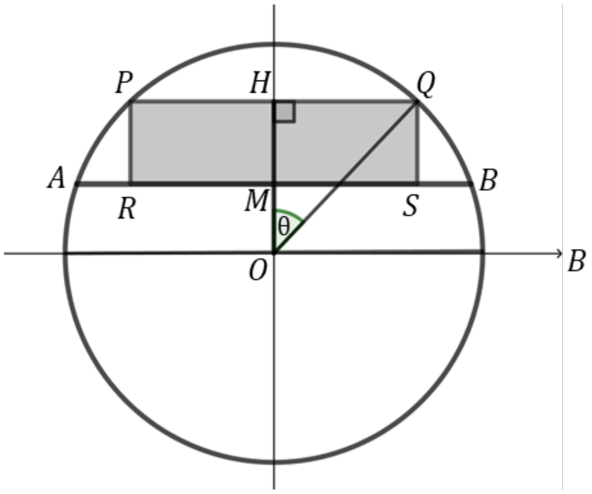
Question 16 continues on the next page

(b) In the diagram below, O is the centre of a circle radius R.

The chord AB has length kR and is parallel to the diameter, with midpoint M.

We now construct a rectangle inside the minor segment AB.

The angle θ is constructed as shown.



i) Show that $OM = R\sqrt{1 - \dfrac{k^2}{4}}$ *2*

ii) Hence show that $p = \sqrt{1 - \dfrac{k^2}{4}}$ will always satisfy $0 < p < 1$. *1*

iii) Show that the area of the rectangle is given by: *2*

$$A = 2R^2 \sin\theta\,(\cos\theta - p) \qquad \text{where} \quad p = \sqrt{1 - \frac{k^2}{4}}$$

iv) Show that $\dfrac{dA}{d\theta} = 2R^2(2\cos^2\theta - p\cos\theta - 1\,)$ *2*

v) For this problem, show that $\dfrac{d^2A}{d\theta^2} < 0$ *2*

vi) Hence show that the area of the rectangle will be maximized when *2*

$$\cos\theta = \frac{1}{4}\left(p + \sqrt{p^2 + 8}\right)$$

END OF THE EXAMINATION.

Multiple Choice

1. (D) $\frac{14}{3-\sqrt{2}} = 6+2\sqrt{2} = 6+\sqrt{8}$

2. (A)

3. ☑ $\frac{\sec^2 x}{\tan x} = \sec x \csc x$

4. (A)

5. (B) Inflexion when concavity changes sign.
Gradient of the curve in this diagram ($f'(x)$) is concavity $f''(x)$.
At Point B, gradient (ie: concavity) changes from +ve to -ve

6. (A)

7. (B)

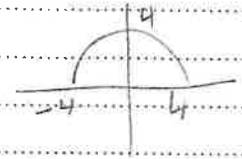
8. (A)

9. (C) Construct the line $y = \frac{x}{2}$ and count intersections

10. (E) Observe we start at $x = -2$m
From $t=0$ to $t=2$ we keep moving left,
$t=2$ to $t=4$ return around at $t=0$
back at $x=-2$m. So at $x=0$ when $t=5$

Q4. a) $2x^2y - 3x^2 - 8y + 12$
$= x^2(2y-3) - 4(2y-3)$
$= (x^2-4)(2y-3)$
$= (x+2)(x-2)(2y-3)$ (2)

b) $f(x) = \sqrt{16-x^2}$



D: $-4 \le x \le 4$
R: $0 \le y \le 4$ (2)

c) $|3x-1| = 4x$

$\therefore 3x-1 = 4x$ or $3x-1 = -4x$
$x = -1$ $\quad$ $7x = 1$
$x = \frac{1}{7}$
But not valid $\quad$ valid ✓
$|-4| \ne -4x$

$\therefore x = \frac{1}{7}$ (3)

$\cos 2x = \frac{\sqrt{3}}{2}$ $\quad 0 \le x \le 2\pi$
$0 \le 2x \le 4\pi$

$2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{4}$

$\therefore x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{4}$ (3)

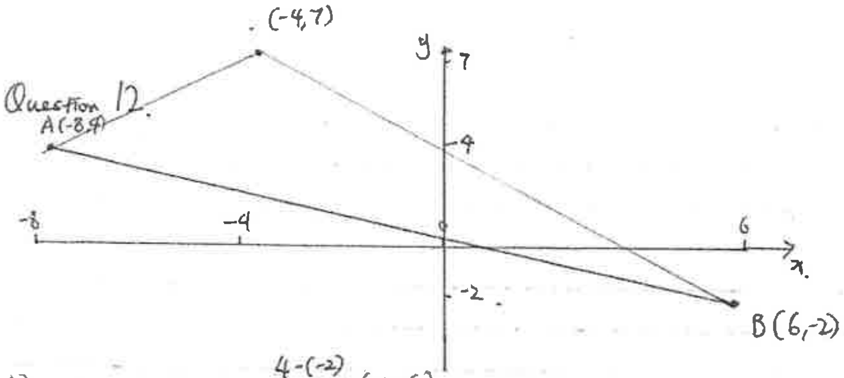
d) $y = (e^{2x} + \log_e x)^4$
$y' = 4(e^{2x} + \log_e x)^3 (2e^{2x} + \frac{1}{x})$ (2)

e) $y = 2\cos x$ $\quad (\frac{\pi}{3}, 1)$
$y' = -2\sin x$
$y'(\frac{\pi}{3}) = -2\sin\frac{\pi}{3}$
$= (-2)\frac{\sqrt{3}}{2} = -\sqrt{3}$ (3)

$\therefore$ tangent: $(y-1) = -\sqrt{3}(x - \frac{\pi}{3})$ ✓✓✓
or $y = -\sqrt{3}x - \frac{\sqrt{3}}{3}\pi + 1$

Question 12.

a)



i) AB : $y+2 = \frac{4-(-2)}{-8-6}(x-6)$

$y+2 = \frac{6}{-14}(x-6)$ ①

$-14y - 28 = 6x - 36$ ✓

$0 = 6x + 14y - 8$ as required.

ii) $\perp d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \frac{|6(-4)+14(7)-8|}{\sqrt{6^2+14^2}}$ ✓ ②

$= \frac{66}{\sqrt{232}}$ ✓ $= \frac{33}{\sqrt{58}}$

iii) Area $\Delta = \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times \sqrt{(6-(-8))^2 + (-2-4)^2} \times \frac{66}{\sqrt{232}}$ ✓ ②

$= \frac{1}{2} \times \sqrt{232} \times \frac{66}{\sqrt{232}}$ ✓ $= 33\ u^2$

b) i) $\int \frac{1}{(1+5x)^4}\,dx = \frac{1}{5}\int \frac{5}{(1+5x)^4}\,dx$

$= \frac{1}{5} \times -\frac{1}{3(1+5x)^3} + C$ ✓ ①

$= \frac{-1}{15(1+5x)^3} + C.$

ii) $\int \sqrt{x}(4-\sqrt{x})\,dx = \int 4x^{\frac{1}{2}} - x\,dx$ ✓ ②

$= 4 \times \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + C$

$= \frac{8}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + C$ ✓

iii) $\int_0^{\frac{\pi}{4}} \sin 2x\,dx = \left[\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{4}}$ ✓ ②.

$= -\frac{\cos\frac{\pi}{2}}{2} - \frac{-\cos 0}{2} = 0 + \frac{1}{2} = \frac{1}{2}$ ✓

c) $y = \ln\left(\frac{5x+1}{(x-3)^2}\right)$ let $u = \frac{5x+1}{(x-3)^2}$

$y = \ln u$

$\frac{du}{dx} = \frac{5(x-3)^2 - 2(x-3)(5x+1)}{(x-3)^4}$

$\frac{dy}{du} = \frac{1}{u} = \frac{(x-1)^2}{5x+1}$

$= \frac{5(x-3) - 2(5x+1)}{(x-3)^3} = \frac{-5x-17}{(x-3)^3}$ ✓

$\frac{dy}{dx} = \frac{(x-1)^2}{5x+1} \times \frac{-5x-17}{(x-1)^3}$

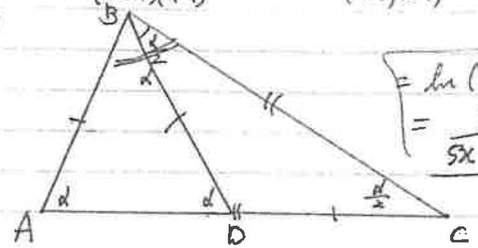
$= \frac{-5x-17}{(5x+1)(x-1)} = -\frac{5x+17}{(5x+1)(x-1)} = -\frac{5x+17}{5x^2+4x-1}$ ✓ ②

or

$= \ln(5x+1) - 2\ln(x-3)$

$= \frac{5}{5x+1} - \frac{2}{x-3}$ ✓ ④

d) i



ii) let $\angle BAD = \alpha$

$\angle BDC = 180 - \alpha$ (straight line) ✓

$\angle DBC = \angle BCD = \frac{180-(180-\alpha)}{2} = \frac{\alpha}{2}$ (base angles of isosceles triangle are equal)

$\angle CBA = \angle BAC = \alpha$ (base angles of isosceles triangle are equal)

$\alpha + \alpha + \frac{\alpha}{2} = 180°$ (angle sum of a triangle equals to 180°) ✓

$\therefore \frac{5\alpha}{2} = 180°$

$5\alpha = 360°$ ✓

$\alpha = \frac{360°}{5} = 72°$

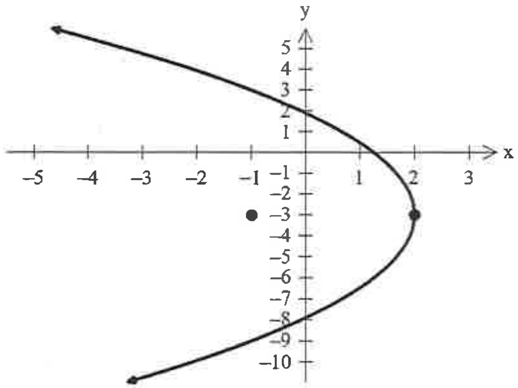
$\therefore \angle BCD = \frac{\alpha}{2} = \frac{72}{2} = 36°$ ③.

Solutions to Question 13 – 2 Unit Advanced Task 3 2019

(a)

$$12x + y^2 + 6y - 15 = 0$$
$$y^2 + 6y = -12x + 15$$
$$y^2 + 6y + 9 = -12x + 15 + 9$$
$$(y+3)^2 = -12x + 24$$
$$(y+3)^2 = -12(x-2)$$
$$(y+3)^2 = -4(3)(x-2)$$

✓ Putting the equation into the appropriate form



✓ Finding the vertex

✓ Finding the focal length

✓ Finding the focus

(i) Vertex $= (2, -3)$

(ii) Focus $= (-1, -3)$

(b)

Let the roots be $(p-2)$ and $(p+2)$

$$(p+2) + (p-2) = -\frac{b}{a}$$
$$2p = -\frac{-2k}{1}$$
$$2p = 2k$$
$$p = k \ldots ①$$

✓ Creating an equation with the roots

$$(p+2)(p-2) = \frac{c}{a}$$
$$p^2 - 4 = \frac{3k}{1}$$
$$p^2 - 4 = 3k$$
$$k^2 - 4 = 3k \quad \text{from ①}$$
$$k^2 - 3k - 4 = 0$$
$$(k+1)(k-4) = 0$$

✓ Finding an equation in terms of k only

$\therefore k = -1$ or $k = 4$

✓ Solving to find both values of k

(c)

$$(2x + 5y - 1) + k(x + y - 5) = 0$$
$$(2(4) + 5(-2) - 1) + k(4 + (-2) - 5) = 0$$
$$(8 - 10 - 1) + k(4 - 2 - 5) = 0$$
$$-3 - 3k = 0$$
$$3k = -3$$
$$k = -1$$

✓ Writing an equation with the k method

✓ Solving to find k

$$(2x + 5y - 1) + (-1)(x + y - 5) = 0$$
$$2x + 5y - 1 - x - y + 5 = 0$$
$$x + 4y + 4 = 0$$

✓ Finding the equation of the line in general form

Alternative solution

$$2x + 5y - 1 = 0 \ldots ①$$
$$x + y - 5 = 0 \ldots ②$$

$(1) - (2) \times 2$

$$2x + 5y - 1 = 0 \quad -$$
$$2x + 2y - 10 = 0$$

$$3y + 9 = 0$$
$$3y = -9$$
$$y = -3$$

✓ Finding the point of intersection

sub into ②

$$x + (-3) - 5 = 0$$
$$x - 3 - 5 = 0$$
$$x - 8 = 0$$
$$x = 8$$

point of intersection is $(8, -3)$

✓ Using the two point/point gradient formula

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_2}$$
$$\frac{y - (-2)}{x - 4} = \frac{-3 - (-2)}{8 - 4}$$
$$\frac{y + 2}{x - 4} = -\frac{1}{4}$$
$$4(y + 2) = -1(x - 4)$$
$$4y + 8 = -x + 4$$
$$x + 4y + 4 = 0$$

✓ Finding the equation of the line in general form

(d)	(i)	$\angle DAO$, $\angle ADO$ and $\angle DOB$ all refer to the acute angles $\angle DAO + \angle ADO = \angle DOB$ (exterior angle sum of ΔAOD) but as ΔAOD is isosceles $\angle DAO = \angle ADO$ $\therefore$ $2\angle DAO = \angle DOB$ $2\angle DAO = 30°$ $\angle DAO = 15°$	✓ Finding $\angle DAO$ with geometrical reasons stated
	(ii)	$\angle AOD$ and $\angle DOB$ refer to the obtuse and acute angles respectively $\angle AOD = 180° - \angle DOB$ (angles on a straight line are supplementary) $\angle AOD = 180° - 30°$ $= 150°$ $AD^2 = AO^2 + DO^2 - 2 \times AO \times DO \times \cos(\angle AOD)$ $= 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(150°)$ $= 2 - 2 \times \left(\frac{-\sqrt{3}}{2}\right)$ $= 2 - (-\sqrt{3})$ $= 2 + \sqrt{3}$	✓ Using the cosine rule with appropriate sides and angles ✓ Clear substitutions and steps to arrive at the answer
	(iii)	$\angle DAO$ and $\angle AOD$ refer to the acute and obtuse angles respectively $\frac{\sin(\angle DAO)}{DO} = \frac{\sin(\angle AOD)}{AD}$ $\frac{\sin(15°)}{1} = \frac{\sin(150°)}{\sqrt{2+\sqrt{3}}}$ $\sin(15°) = \frac{\left(\frac{1}{2}\right)}{\sqrt{2+\sqrt{3}}}$ $= \frac{1}{2} \times \frac{1}{\sqrt{2+\sqrt{3}}} \times \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2-\sqrt{3}}}$ $= \frac{1}{2} \times \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2^2-(\sqrt{3})^2}}$ $= \frac{1}{2} \times \frac{\sqrt{2-\sqrt{3}}}{\sqrt{4-3}}$ $= \frac{1}{2} \times \frac{\sqrt{2-\sqrt{3}}}{1}$ $= \frac{1}{2}\sqrt{2-\sqrt{3}}$	✓ Using the sine rule with correct values ✓ Clearly show the steps (including rationalising the denominator) leading to the answer

Q14(a) $f(x) = x^4 - 8x^3$

$f'(x) = 4x^3 - 24x^2$

$= 4x^2(x-6)$

$f''(x) = 12x^2 - 48x$

stationary points at $f'(x) = 0$

ie $4x^2(x-6) = 0$

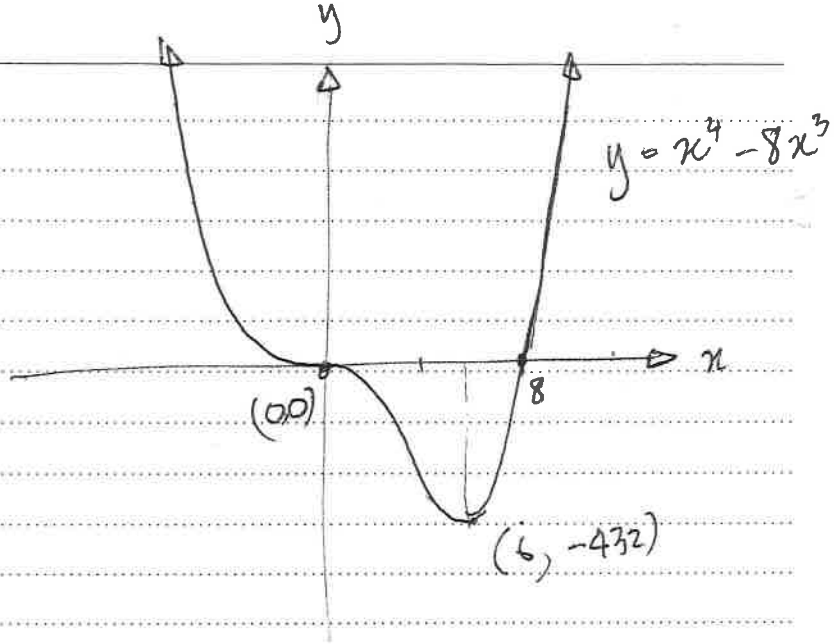
$x = 0,\ x = 6$ $(0,0),\ (6,-432)$

Testing $(0,0)$: $f''(0) = 0$ ← Not a valid test!

x	-1	0	3	6	7
$f'(x)$	28	0	-108	0	196
	\	—	\	—	/

$\therefore$ $(0,0)$ is a horizontal point of inflexion

$\therefore$ $(6,-432)$ is local minimum.



[Optional: identify (4, -256) as another inflexion point - but not required]

14 b) $A = \int_0^3 \left((x^2 - 3x) - (x^3 + x^2 - 12x)\right) dx$

$= \int_0^3 (-3x - x^3 + 12x)\, dx$

$= \int_0^3 (-x^3 + 9x)\, dx$

$= \left[\frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3 = \left(\frac{9(9)}{2} - \frac{81}{4}\right) - (0 - 0)$

$= \frac{81}{4}\ u^2$

14 (c) $0.5\dot{4}\dot{3} = 0.5 + 0.043 + 0.00043 + \dots + \dots$

$= \frac{1}{2} + \frac{43}{1000} + \frac{43}{1000}\left(\frac{1}{100}\right) + \frac{43}{1000}\left(\frac{1}{100}\right)^2 + \dots$

GP $a = \frac{43}{1000}$, $r = \frac{1}{100}$

$= \frac{1}{2} + \frac{\frac{43}{1000}}{1 - \frac{1}{100}}$ $\left(S_\infty = \frac{a}{1-r}\ |r| < 1\right)$

$= \frac{1}{2} + \frac{43}{1000} \times \frac{100}{99}$

$= \frac{269}{495}$

(d) $\sum_{k=3}^{100} (-1)^k (2k+1) = (-1)(2(3)+1) + (-1)^2(2(4)+1)$

$+ (-1)(2(5)+1) + (-1)^2(2(6)+1)$

$+ \dots (-1)(2(99)+1) + (-1)^2(2(100)+1)$

$n = 100 - 3 + 1 = 96$

$= -7 + 9$
$- 11 + 13$
$+ \dots - 199 + 200$

$= 2 + 2 + 2 + \dots + 2$ (96/2 pairs)

$= \frac{96}{2} \times 2 = 96$

Question 15

a) $y = \ln(x+3)$ $\quad e^y = x+3$, $\quad x = e^y + 3$. ✓

$V = \pi \int_0^{\ln 3} (e^y - 3)^2 \, dy$ ✓

$= \pi \int_0^{\ln 3} e^{2y} - 6e^y + 9 \, dy$ ✓ (4)

$= \pi \left[\frac{e^{2y}}{2} - 6e^y + 9y \right]_0^{\ln 3}$

$= \pi\left[\left(\frac{1}{2}e^{\ln 3^2} - 6e^{\ln 3} + 9\ln 3\right) - \left(\frac{e^0}{2} - 6e^0 + 0\right)\right]$ ✓

$= \pi\left(\frac{9}{2} - 18 + 9\ln 3 - \left(\frac{1}{2} - 6\right)\right)$

$= \pi(9\ln 3 - 8)\, u^3$

b). $y = \sqrt{1-x^2}$ $\quad 0 \leq x \leq 1$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y	1	$\frac{\sqrt{15}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{7}}{4}$	0

✓ $h = \frac{1}{4}$

$A \doteq \frac{1}{12}\left[1 + 0 + 4\left(\frac{\sqrt{15}}{4} + \frac{\sqrt{7}}{4}\right) + 2\left(\frac{\sqrt{3}}{2}\right)\right]$ ✓ (4)

$= \frac{1}{12}(1 + \sqrt{15} + \sqrt{7} + \sqrt{3})\, u^2$. ✓

$A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi \, u^2$.

$\frac{\pi}{4} \approx \frac{1}{12}(1 + \frac{\sqrt{3}}{3} + \sqrt{7} + \sqrt{15})$ ✓

$\therefore \pi \approx \frac{1}{3}(1 + \sqrt{3} + \sqrt{7} + \sqrt{15})$

c) $\frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{e^{2x} - 1} = 2 \times \frac{e^x}{e^{2x}-1}$ (given) ✓

$\therefore \int \frac{e^x}{e^{2x}-1} dx = \frac{1}{2} \int \frac{2e^x}{e^{2x}-1} dx$ $\quad$ $x \neq 0$ $\quad$ (2)

$= \frac{1}{2} \int \frac{e^x}{e^x - 1} - \frac{e^x}{e^x+1} \, dx$

$= \frac{1}{2}\left[\ln(e^x - 1) - \ln(e^x + 1)\right] + c$ ✓

$= \frac{\ln(e^x - 1)}{2} - \frac{\ln(e^x+1)}{2} + c$ $\quad$ $x \neq 0$

Question 15

d) $\ddot{x} = -6t$ $\quad \dot{x}(0) = 9$ m/s.

i) $\dot{x} = \int \ddot{x} \, dt$

$= -3t^2 + C$ ✓ (1)

$9 = -3(0) + C$ $\quad \therefore C = 9$.

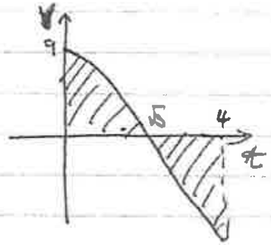
$\therefore \dot{x} = -3t^2 + 9$

ii) when $\dot{x} = 0$.

$9 - 3t^2 = 0$ (1)

$t = \sqrt{3}$ s $(t > 0)$ ✓

iii)



$\int_0^{\sqrt{3}} 9 - 3t^2 \, dt + \int_{\sqrt{3}}^4 9 - 3t^2 \, dt$

= distance.

$= \left|\left[9t - t^3\right]_0^{\sqrt{3}}\right| + \left|\left[9t - t^3\right]_{\sqrt{3}}^4\right|$

$= (9\sqrt{3} - 3\sqrt{3}) + |(36 - 64) - 6\sqrt{3}|$ ✓

$= 6\sqrt{3} + |-28 - 6\sqrt{3}|$

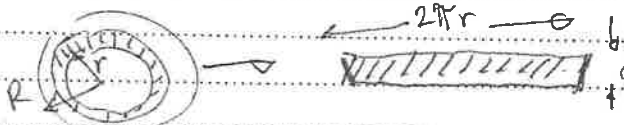
$= 12\sqrt{3} + 28$ (3)

average speed $= \frac{12\sqrt{3} + 28}{4} = 3\sqrt{3} + 7$ m/s. ✓

Q16(a)

(1)

(i)



Area of strip ≈ rectangle Δr high, circumference $2\pi r$ width.

(ii) Area whole circle: add the strips from $r=0$ to $r=R$

$$A \approx \sum_{r=0}^{r=R} (2\pi r)\,\Delta r$$

(2) to get exact shrink to very small (infinitesimal / limit) $\Delta r \to dr$

$$\therefore A = \int_0^R 2\pi r\, dr$$

(iii) $$\therefore A = \int_0^R 2\pi r\, dr$$

(1)

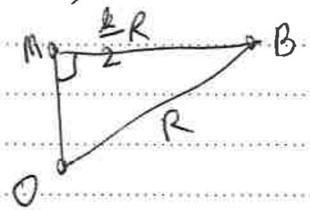
$$= \left[\pi r^2\right]_0^R$$

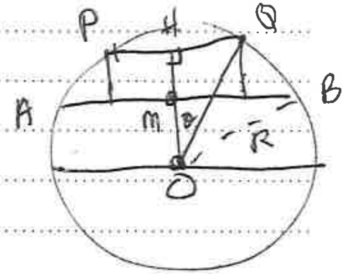
$$= \pi R^2 - 0$$

$$= \pi R^2 \ (u^2)$$

Q16(b) *DRAW DIAGRAMS*

(i)





By definition, $MB = \frac{k}{2}R$

By Pythagoras $R^2 = \left(\frac{k}{2}R\right)^2 - OM^2$

$$\therefore OM = \sqrt{R^2 - \left(\frac{kR}{2}\right)^2} \quad (OM > 0)$$

$$= R\sqrt{1 - \frac{k^2}{4}}$$

(ii) By definition $OM = pR$

from diagram $0 < OM < R$

$\therefore 0 < pR < R$

$\therefore 0 < p < 1$

or you could use algebra (more work, less insight!)

$0 \leq k \leq 2$ (from diagram)

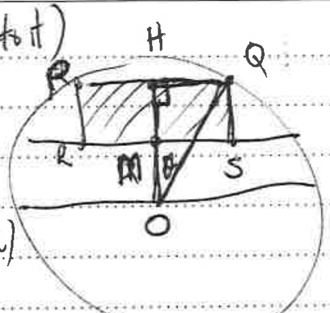
$0 \leq k^2 \leq 4$

$0 \leq \frac{k^2}{4} \leq 1$

$\therefore 0 \leq 1 - \frac{k^2}{4} \leq 1$

$\therefore 0 \leq \sqrt{1 - \frac{k^2}{4}} \leq 1$

(iii) DRAW DIAGRAM (or refer to it)



$HO = R\cos\theta$

$HM = HO - OM$

$\therefore HM = R\cos\theta - pR$ (from ii)

$\therefore$ width $PQ = 2(R\sin\theta)$

$\therefore$ Area $= (2R\sin\theta)(R\cos\theta - pR)$

$= 2R^2\sin\theta(\cos\theta - p)$ as required.

(iv) $\frac{dA}{d\theta} = 2R^2\left(\cos\theta(\cos\theta - p) + \sin\theta(-\sin\theta)\right)$

$= 2R^2(\cos^2\theta - p\cos\theta - \sin^2\theta)$

$= 2R^2(\cos^2\theta - p\cos\theta - (1-\cos^2\theta))$

$= 2R^2(2\cos^2\theta - p\cos\theta - 1)$

(v) $\frac{d^2A}{d\theta^2} = 2R^2(-4\sin\theta\cos\theta + p\sin\theta)$

$= 2R^2\sin\theta(p - 4\cos\theta)$

Consider $2R^2\sin\theta(p - 4\cos\theta)$

$\sin\theta > 0$ for $0 < \theta < \frac{\pi}{2}$

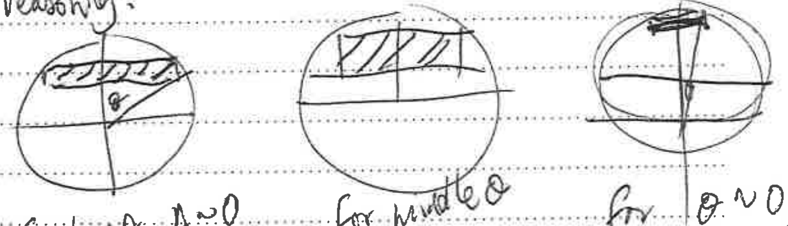
From the area of the rectangle: $\cos\theta - p > 0$

$\therefore p - \cos\theta < 0$

$\therefore p - 4\cos\theta < 0$

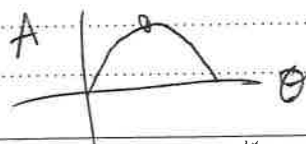
so $\frac{d^2A}{d\theta^2} = 2R^2$ is $(+)(-) < 0$ for $0 < \theta < \frac{\pi}{2}$

Alternate reasoning:



for large θ, A ~ 0   for middle θ A > 0   for θ ~ 0, A ~ 0

So area must do this:



ie there is a maximum.

[should also say A(θ) is continuous]

(v) Solve the equation $\frac{dA}{d\theta}=0$

$$2R^2(2\cos^2\theta - p\cos\theta - 1) = 0$$

ie $\cos\theta = \frac{+p \pm \sqrt{p^2+8}}{4}$ (quadratic in disguise)

$\cos\theta$ must be positive $(0 \le \theta \le \frac{\pi}{2})$

But $p - \sqrt{p^2+8}$ is negative ; (must show why)

(a) $0 \le p \le 1$

$\therefore 1 - \sqrt{1^2+8} < 0$

$0 - \sqrt{8} < 0$

or

$p^2 < p^2 + 8$

$\sqrt{p^2} < \sqrt{p^2+8}$ for $p>0$

$p < \sqrt{p^2+8}$

only positive result is:

$$\cos\theta = \frac{1}{4}\left(p + \sqrt{p^2+8}\right)$$