NORTH SYDNEY BOYS HIGH SCHOOL

2020 YEAR 12 ASSESSMENT TASK 3

Mathematics Advanced

General Instructions

- Working time – 3 hours
- **Reading time – 10 minutes**
- Write on the lined paper in the booklet provided
- Write Multiple Choice responses on sheet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Use the Reference Sheet

Class Teacher: Please shade the circle.

- ○ Mr Berry
- ○ Mr Hwang
- ○ Mr Ireland
- ○ Ms Lee
- ○ Mr Umakanthan

STUDENT NUMBER: 4……………………………………………..........

(To be used by the exam markers only.)

Questions	1-10	11-17	18-21	22-25	26-29	30-34	35-38	Total
Mark	/10	/15	/16	/15	/15	/15	/14	/100

Section I

Use the multiple-choice answer sheet for Questions 1 – 10

1. The period of $f(x) = 4\cos(\frac{2\pi x}{3}) + 1$ is:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. 3

D. 5

2. If $f(x) = \frac{x^2+1}{4-x^2}$ then the vertical and horizontal asymptotes are respectively:

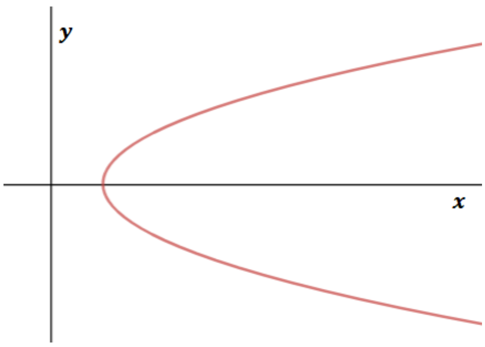
A. $x = -2, \quad x = 2, \quad y = \frac{1}{4}$

B. $x = -2, \quad x = 2, \quad y = -1$

C. $x = -2, \quad x = 2, \quad y = 1$

D. $x = -2, \quad x = 2, \quad y = 0$

3. The graph below shows the relation $y^2 = x - 1$. What type of relation is it?



A. one-to-one

B. many-to-one

C. many-to-many

D. one-to-many

4. The 7th term of an arithmetic sequence is 45 and the 11th term is 77.

Find the first term (*a*) and the common difference (*d*).

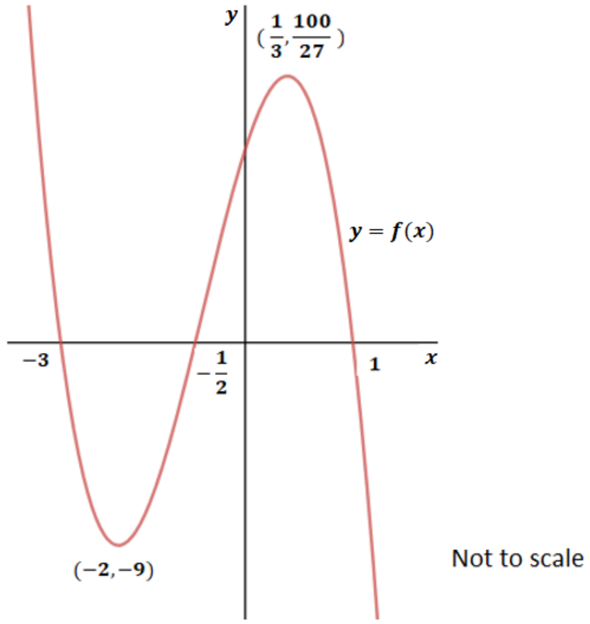
A. $a = -3$ and $d = 8$

B. $a = 3$ and $d = 8$

C. $a = 8$ and $d = -3$

D. $a = 8$ and $d = 3$

5. Part of the graph $y = f(x)$ of the function f is shown:



$f'(x) < 0$ for

A. $x \in (-3, -\frac{1}{2}) \cup (1, \infty)$

B. $x \in (-9, \frac{100}{27})$

C. $x \in (-2, \frac{1}{3})$

D. $x \in (-\infty, -2) \cup (\frac{1}{3}, \infty)$

6. Find the derivative of $e^{x \sin 3x}$.

A. $e^{3x \cos 3x}$

B. $e^{x \sin 3x}(\sin 3x + 3x \cos 3x)$

C. $e^{x \sin 3x}$

D. $e^{x \sin 3x}(\sin 3x - 3x \cos 3x)$

7. The graph of a function f is obtained from the graph of the function g with rule $g(x) = \sqrt{2x - 5}$ by a reflection in the x-axis, followed by a horizontal dilation from the y-axis by a factor of $\frac{1}{2}$. Which one of the following is the rule for the function f?

A. $f(x) = \sqrt{5 - 4x}$

B. $f(x) = -\sqrt{x - 5}$

C. $f(x) = \sqrt{x + 5}$

D. $f(x) = -\sqrt{4x - 5}$

8. The discrete random variable *X* has this probability distribution:

x	0	1	2	3
$P(X=x)$	a	$3\ a$	$5\ a$	$7\ a$

The mean of *X* is:

A. $\frac{1}{16}$

B. 1

C. $\frac{35}{16}$

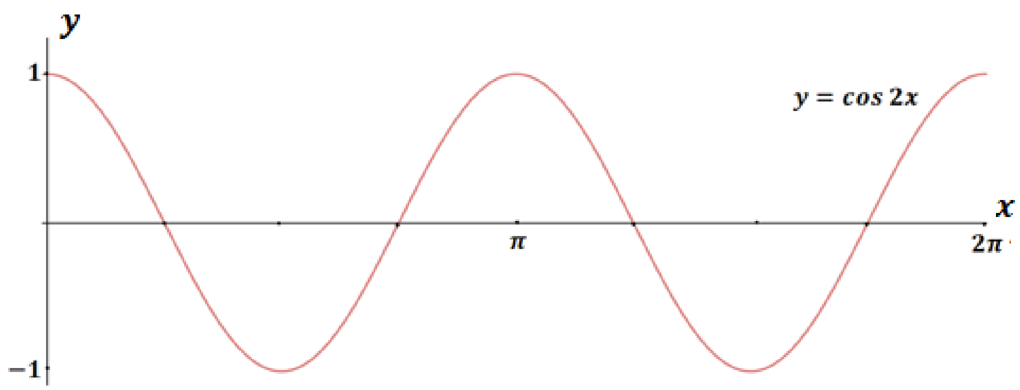
D. $\frac{17}{8}$

9. The difference in intensity of two sources of sound P_1 and P_2 is defined to be $10\ log_{10}(\frac{P_1}{P_2})$ decibels. How much louder is a sound of $112\ dB$ than a sound of $80\ dB$?

A. 32 times
B. 1 585 times
C. 25 times
D. 3·2 times

10. How many solutions does $\mathbf{6\cos 2x = x}$ for $0 \leq x \leq 2\pi$ have ?

(note: graph below shows $y = \cos 2x$)



A. 1
B. 2
C. 3
D. 4

Section II: Short Answer

Instructions

- Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of the booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (1 mark)

Factorise $2x^2 + 5x + 2$ **1**

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Question 12 (2 marks)

Rationalise the denominator: $\frac{2}{3-\sqrt{2}}$ **2**

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Question 13 (3 marks)

Find the following integrals:

(a) $\int \frac{1-2x^5}{x}\,dx$ **2**

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(b) $\int (3x+2)^4 \, dx$ **1**

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Question 14 (3 marks)

Differentiate the following functions:

(a) $y = 3^{5x+2}$ **1**

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(b) $y = \frac{x}{\log_e x}$ **2**

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Question 15 (1 mark)

Write down the domain of $g(x) = \log_e(x+\pi)$ **1**

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Question 16 (3 marks)

Solve $2\log_e x = \log_e(2x+3)$ **3**

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Question 17 (2 marks)

Solve the inequality $x^2 \geq 3x + 18$ **2**

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Question 18 (2 marks)

Find $\int_1^{e^3} \frac{5}{x}\,dx$ **2**

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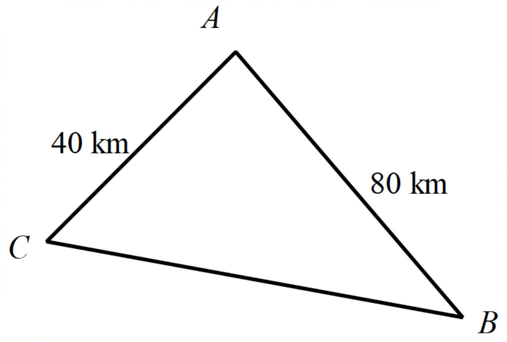
Question 19 (4 marks)

Three towns, A, B and C form a triangle.

Town A is 80 km from Town B and Town C is 40 km from Town A as shown below:



The bearing of Town B from Town A is $130°$. The bearing of Town C from Town A is $240°$

(a) Use this information to find the size of $\angle CAB$, and hence find the area of the triangle formed by the three towns to the nearest square kilometre. **2**

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(b) Using the cosine rule, find the distance between Town B and Town C, to the nearest kilometre. **2**

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Question 20 (3 marks)

(a) Given $f(x) = \sqrt{4 - x^2}$ complete this table of values, correct to 3 decimal places. **1**

x	0	0.5	1	1.5	2
$f(x)$					

(b) Use the Trapezoidal rule, with four sub-intervals, to estimate the value of

$\int_0^2 \sqrt{4 - x^2}\, dx$. **2**

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Question 21 (7 marks)

For the curve: $y = x^3 - 3x^2 - 9x + 4$

(a) Find any stationary points and determine their nature. **3**

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(b) Find any points of inflexion. **2**

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(c) Sketch the curve, showing all main features. **2**

Question 22 (2 marks)

Find the exact value of $\cot\theta$ given that $\cos\theta = 0.6$ and $\sin\theta < 0$. **2**

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Question 23 (3 marks)

A geometric progression has 5^{th} term 9 and 13^{th} term 59 049.

(a) Find the first term and the common ratio. **2**

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(b) Find the 19^{th} term. **1**

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Question 24 (5 marks)

The number of bacteria in a culture can be modelled by $B = 120\ 000\ e^{0.4\,t}$

where t is the time in hours after the experiment started.

(a) How many bacteria are there after 6 hours have passed? **1**

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(b) How fast was the culture growing after 6 hours? **1**

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(c) What was the average rate of increase over the first 6 hours? **1**

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(d) How long, in hours and minutes, will it take until the number of bacteria doubles? **2**

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Question 25 (5 marks)

In an experiment, 2 balls are drawn at random and without replacement from

an urn containing 4 red balls and 6 black balls. Let *X* be the number of red balls selected.

(a) Complete the table below: **2**

Outcome	RR	RB	BR	BB	
X	2	1	1	0	
$p(X = x)$	$\frac{2}{15}$				
$x.p(x)$					
X^2					

(b) What is the expected number of red balls drawn? **1**

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(c) What is the variance, *V(X)*, of this distribution? **2**

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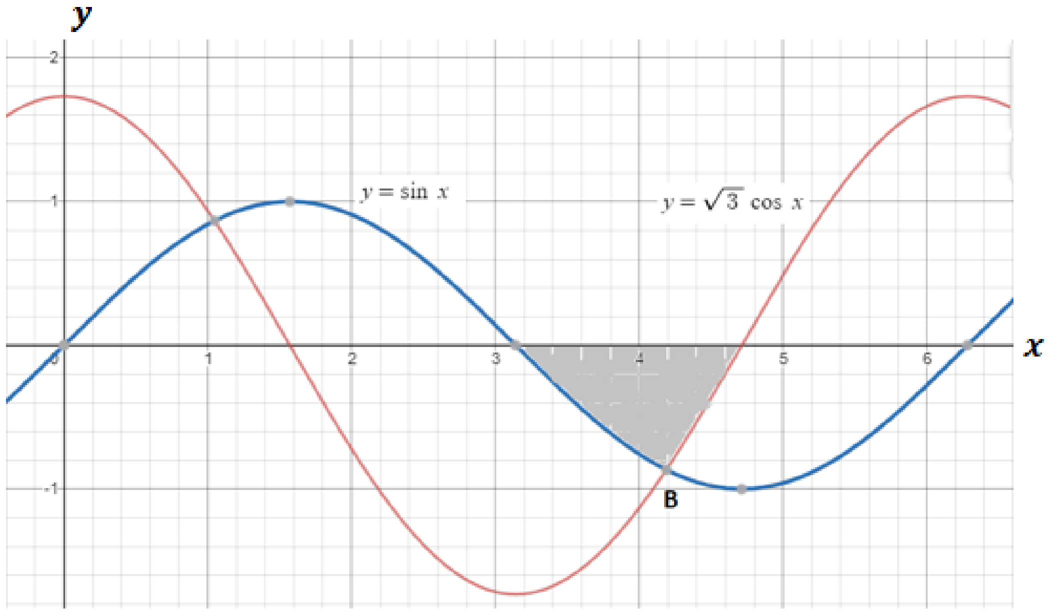
Question 26 (4 marks)



The diagram above shows the graphs of $y = \sin x$ and $y = \sqrt{3}\cos x$, $0 \le x \le 2\pi$.
The second point of intersection is labelled B.

(a) Show, using any appropriate method, that B has coordinates $\left(\frac{4\pi}{3}, \frac{-\sqrt{3}}{2}\right)$ **1**

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(b) Find the exact area of the shaded region. **3**

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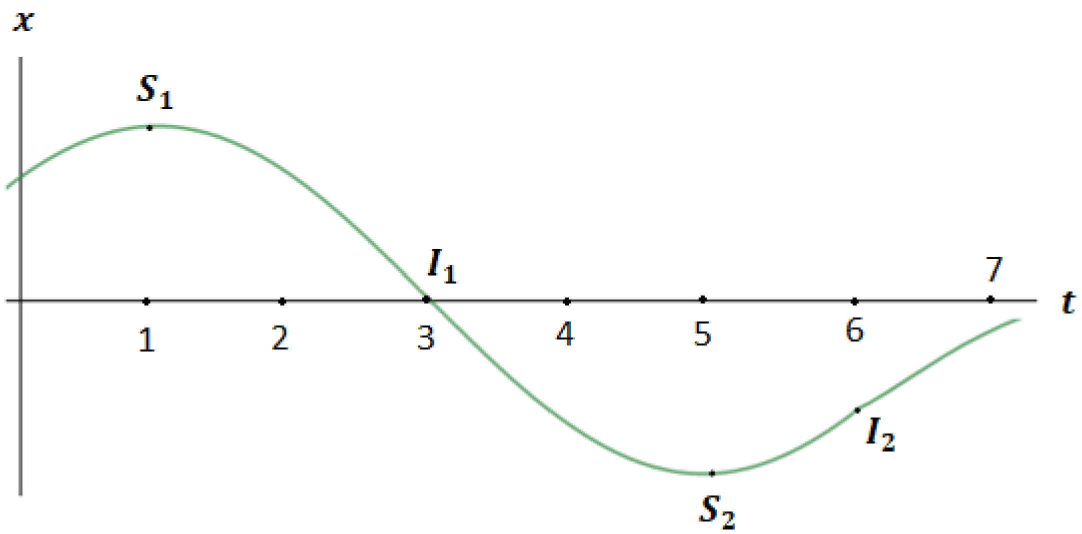
Question 27 (3 marks)



The graph shows the displacement of a particle, moving in a straight line, over the first 7 seconds of its motion. S_1 and S_2 are stationary points, and I_1 and I_2 are inflection points.

State the times, or periods of time, for which:

(a) The particle is stationary. **1**

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(b) The velocity is negative. **1**

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(c) The acceleration is positive. **1**

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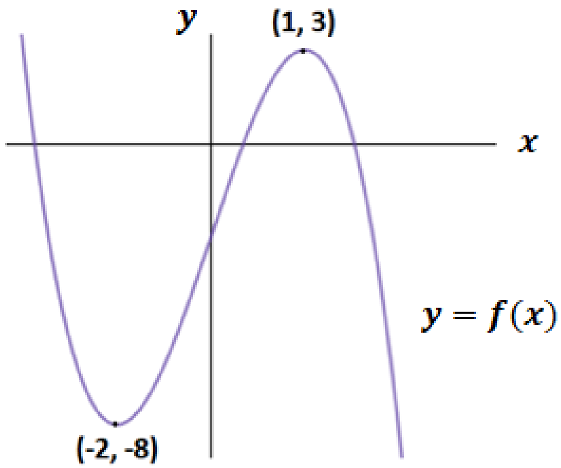
Question 28 (4 marks)

Consider the graph of $y = f(x)$ shown:



(a) Use the space below to sketch the graph of $y = f'(x)$ **2**

(b) Find the area bounded by $y = f'(x)$ and the x-axis. **2**

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Question 29 (4 marks)

For events A and B from a sample space, $P(A|B) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$

(a) Calculate $P(A \cap B)$ **1**

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(b) Calculate $P(\bar{A} \cap B)$ where $\bar{A}$ denotes the complement of A. **1**

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(c) If A and B are independent, calculate $P(A \cup B)$ **2**

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Question 30 (2 marks)

The gradient of a curve is given by $\frac{dy}{dx} = \frac{3x}{x^2+e}$

The curve passes through $(0, 2)$. What is its equation? **2**

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Question 31 (3 marks)

If $f(x) = \sqrt{2-x}$ and $g(x) = \sqrt{x}$, then

(a) Find the rule for the composite function $f \circ g$ **1**

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(b) Find the domain of $f \circ g$ **2**

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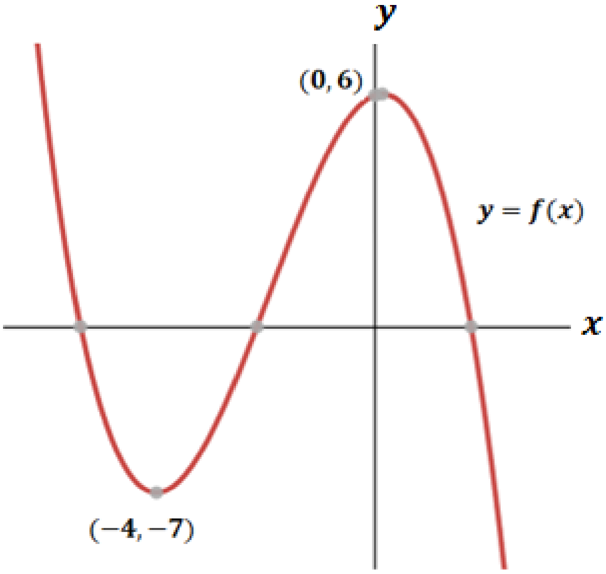
Question 32 (2 marks)

Given the graph of the function $y = f(x)$ below, with turning points as shown, sketch the transformed function $y = 3f(x+2) - 4$. (x-intercepts not required). **2**



Question 33 (5 marks)

(a) Differentiate $y = \log_e(\cos x)$ with respect to x. **1**

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(b) Sketch $y = \tan x$ for $0 \leq x \leq \frac{\pi}{2}$ **1**

(c) Hence, using parts (a) and (b), find the area bounded by $y = \tan x$, the x-axis, and the line $x = \frac{\pi}{3}$ **3**
(leave answer in simplest exact form)

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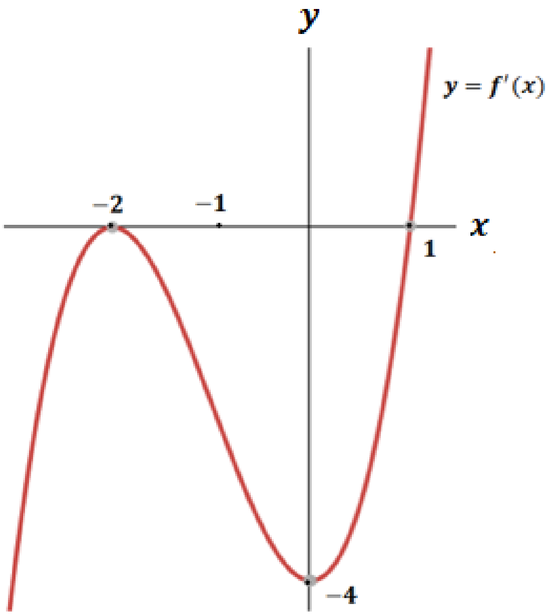
Question 34 (3 marks)

The diagram shows $y = f'(x)$, the graph of the derivative function of $y = f(x)$.



(a) Explain why there is a horizontal point of inflection at $x = -2$ **1**

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(b) Given that $f(0) = 2$, sketch a possible graph of $y = f(x)$. **2**

Question 35 (3 marks)

Find the equation of the normal to $y = x \sin x$ at the point where $x = \frac{\pi}{2}$. **3**

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Question 36 (2 marks)

(a) Sketch $y = |x| - 1$ and $y = 2x + 2$ neatly on the same number plane. **1**

(b) Hence solve the equation $|x| - 2x = 3$ **1**

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Question 37 (4 marks)

One half percent (0.5 %) of a country has a certain viral disease. A test is developed for the disease. The test gives a false positive 3% of the time, and a false negative 2% of the time.

(a) Show that the probability that Andy, a randomly selected person, tests positive is $0 \cdot 03475$ **2**

[*Hint*: in this question, let D be the event that Andy has the disease, and $\overline{D}$ be the event Andy does not have it. Let T be the event that Andy's test comes back positive.]

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(b) Andy just got the bad news that his test came back positive.

Find the probability that Andy actually has the disease. **2**

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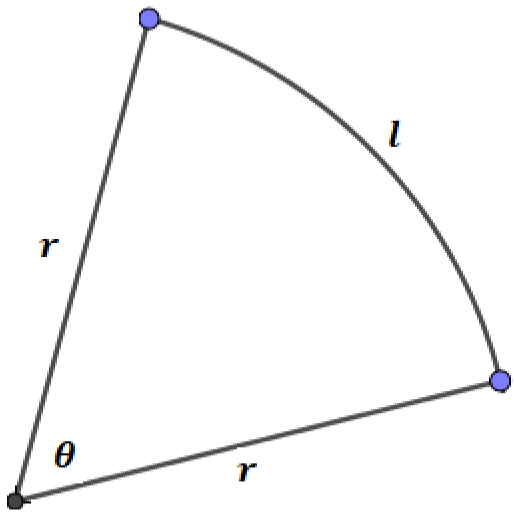
Question 38 (5 marks)

The diagram below shows a sector of a circle of radius r centimetres. The angle at the centre is θ radians, and the perimeter of the whole sector is $8\ cm$.



(a) Show that $r = \frac{8}{2+\theta}$. **1**

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(b) Show that A, the area of the sector in cm^2, is given by

$$A = \frac{32\theta}{(\theta+2)^2}$$ **1**

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(c) If $0 \leq \theta \leq \frac{\pi}{2}$, find the maximum area of the sector, and the value of θ for which this occurs. **3**

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End of examination

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

○ – Mr Ireland	○ – Mr Lin	○ – Mr Berry
○ – Dr Jomaa	○ – Mrs Sarofim	○ – Mr Hwang
○ – Miss Lee	○ – Ms Ziaziaris	○ – Mr Uma

1 – Ⓐ Ⓑ ● Ⓓ

2 – Ⓐ ● Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ ●

4 – ● Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ ●

6 – Ⓐ ● Ⓒ Ⓓ

7 – Ⓐ Ⓑ Ⓒ ●

8 – Ⓐ Ⓑ Ⓒ ●

9 – Ⓐ ● Ⓒ Ⓓ

10 – Ⓐ Ⓑ ● Ⓓ

2020 Mathematics Advanced - Task 3

(1) $T = \dfrac{2\pi}{2\pi/3} = 3$ $\therefore$ (C)

(2) $f(x) = \dfrac{x^2+1}{4-x^2}$ $\therefore$ $x \neq \pm 2$ $\therefore$ (B)

as $x \to \infty$, $y \to -1$

(3) one-to-many, $\therefore$ (D)

(4) $T_7 = a + 6d = 45$

$T_{11} = a + 10d = 77$

$\therefore$ $4d = 32$, $d = 8$

$\therefore$ $a = -3$ $\therefore$ (A)

(5) $f'(x) < 0$ means gradient is negative, $\therefore$ $x < -2$ or ~~x~~ $x > \frac{1}{3}$ $\therefore$ (D)

(6) $\dfrac{d}{dx} e^{x \sin 3x} = (\sin 3x \cdot 1 + x \cdot 3\cos 3x) \cdot e^{x \sin 3x}$

$\therefore$ (B)

(7) $\sqrt{2x-5} \to -\sqrt{2x-5} \to -\sqrt{2\left(\frac{x}{\frac{1}{2}}\right)-5}$

$= -\sqrt{4x-5}$ $\therefore$ (D)

(8) $\sum p = 1 \quad \therefore \quad a + 3a + 5a + 7a = 1$

$\therefore \quad a = \frac{1}{16}$

$\mu = \sum x \cdot p(x) = 0 + 3a + 10a + 21a$

$= 34a$

$= \frac{34}{16} = \frac{17}{8} \quad \therefore$ (D)

(9) $112 \text{ dB} - 80 \text{ dB} = 32 \text{ dB}$

$\therefore \quad \log_{10}\left(\frac{P_1}{P_2}\right) = 3.2 \quad \therefore \quad \frac{P_1}{P_2} = 10^{3.2}$

$\doteq 1585 \quad \therefore$ (B)

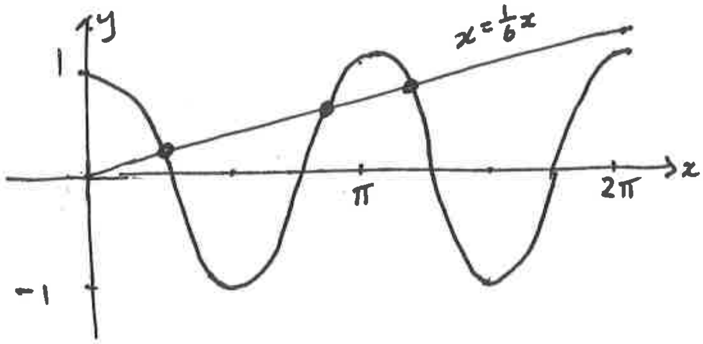
(10) $6 \cos 2x = x$

means $\cos 2x = \frac{1}{6}x$.

If we draw $y = \frac{1}{6}x$, then at

$x = 2\pi, \quad y \doteq 1.047 > \cos 2(2\pi)$

$\therefore$ 3 intersections from $x = 0$ to $x = 2\pi$ $\quad \therefore$ (C)



Section II: Short Answer

Instructions

- Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.
- Your responses should include relevant mathematical reasoning and/or calculations.
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  If you use this space, clearly indicate which question you are answering.

Question 11 (1 mark)

Factorise $2x^2 + 5x + 2$ **1**

$(2x+1)(x+2)$

Question 12 (2 marks)

Rationalise the denominator: $\frac{2}{3-\sqrt{2}}$ **2**

$= \frac{2}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$ ✓

$= \frac{6 + 2\sqrt{2}}{9-2}$

$= \frac{6 + 2\sqrt{2}}{7}$ ✓

Question 13 (3 marks)

Find the following integrals:

(a) $\int \frac{1-2x^5}{x} dx$ **2**

$= \int \left(\frac{1}{x} - 2x^4\right) dx$

$= \ln x - \frac{2x^5}{5} + C$ ✓✓

(b) $\int (3x+2)^4\, dx$ 1

$= \dfrac{(3x+2)^5}{15} + C$ ✓

Question 14 (3 marks)

Differentiate the following functions:

(a) $y = 3^{5x+2}$ 1

$y' = \ln 3 \cdot 5 \cdot 3^{5x+2}$

i.e. $y' = 5 \ln 3 \cdot 3^{5x+2}$ ✓

(b) $y = \dfrac{x}{\log_e x}$ 2

$y' = \dfrac{\log x \cdot 1 - x \cdot \frac{1}{x}}{(\log x)^2}$ ✓

$\therefore y' = \dfrac{\log_e x - 1}{(\log_e x)^2}$ ✓

Question 15 (1 mark)

Write down the domain of $g(x) = \log_e(x + \pi)$ 1

We need $x + \pi > 0 \quad \therefore \quad x > -\pi$

$\therefore$ $D: x > -\pi$ ✓

Question 16 (3 marks)

Solve $2\log_e x = \log_e(2x+3)$ **3**

$\log_e x^2 = \log_e (2x+3)$

$\therefore\ x^2 = 2x+3$ ✓

$\therefore\ x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\therefore\ x = 3$ or $x = -1$ ✓

But $x \neq -1$ as this can't be subbed into the equation

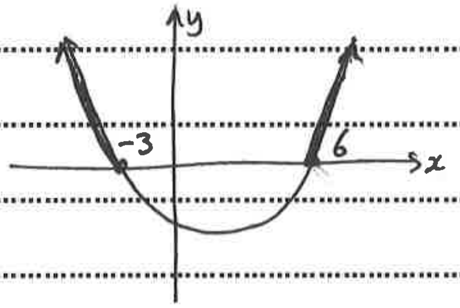
$\therefore\ x = 3$ ✓

Question 17 (2 marks)

Solve the inequality $x^2 \geq 3x + 18$ **2**

$x^2 - 3x - 18 \geq 0$

$(x-6)(x+3) \geq 0$



$\therefore\ x \leq -3$ or $x \geq 6$ ✓✓

Question 18 (2 marks)

Find $\int_1^{e^3} \frac{5}{x}\,dx$ **2**

$= \left[5\ln x\right]_1^{e^3}$ ✓

$= 5\ln e^3 - 5\ln 1$

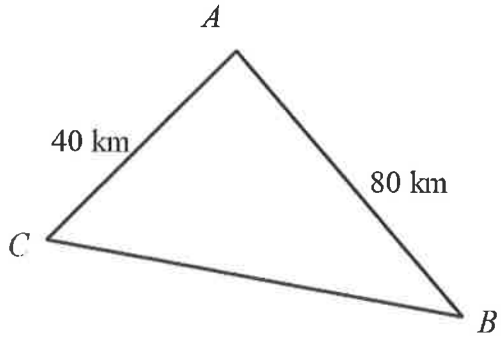
$= 5\times 3 - 5\times 0$

$= 15$ ✓

Question 19 (4 marks)

Three towns, A, B and C form a triangle.

Town A is 80 km from Town B and Town C is 40 km from Town A as shown below:



The bearing of Town B from Town A is 130°. The bearing of Town C from Town A is 240°

(a) Use this information to find the size of $\angle CAB$, and hence find the area of the triangle formed by the three towns to the nearest square kilometre. 2

$\angle CAB = 240^\circ - 130^\circ = 110^\circ$

$\therefore$ Area $\Delta ABC = \frac{1}{2} \cdot 40 \cdot 80 \cdot \sin 110^\circ$

$\doteqdot 1503.508193\ldots$

$= 1504 \text{ km}^2$ (nearest km^2) ✓✓

(b) Using the cosine rule, find the distance between Town B and Town C, to the nearest kilometre. 2

$BC^2 = 40^2 + 80^2 - 2(40)(80)\cos 110^\circ$ ✓

$\doteqdot 10188.92892\ldots$

$\therefore BC \doteqdot 100.940224\ldots$

$\therefore BC = 101$ km (nearest km). ✓

Question 20 (3 marks)

(a) Given $f(x) = \sqrt{4 - x^2}$ complete this table of values, correct to 3 decimal places. 1

x	0	0.5	1	1.5	2
$f(x)$	2	1.936	1.732	1.323	0

✓

(b) Use the Trapezoidal rule, with four sub-intervals, to estimate the value of

$\int_0^2 \sqrt{4 - x^2}\, dx$. 2

$h =$ width sub-interval $= 0.5$

$$\int_0^2 \sqrt{4-x^2}\, dx \doteqdot \frac{0.5}{2}\left[2 + 0 + 2(1.936 + 1.732 + 1.323)\right]$$ ✓

$$= 2.9955$$ ✓

Question 21 (7 marks)

For the curve: $y = x^3 - 3x^2 - 9x + 4$

(a) Find any stationary points and determine their nature. 3

$y' = 3x^2 - 6x - 9$; $y'' = 6x - 6$

$= 3(x^2 - 2x - 3)$ $\therefore y'' = 6(x-1)$ ✓

$\therefore y' = 3(x-3)(x+1)$

For stat. pts, $y' = 0$ $\therefore$ $x = 3$ or $x = -1$

$y = -23$ $y = 9$ ✓

At $(3, -23)$, $y'' = 6(3-1) = 12 > 0$ $\therefore$ local minimum at $(3, -23)$

At $(-1, 9)$, $y'' = 6(-1-1) = -12 < 0$ $\therefore$ local maximum at $(-1, 9)$ ✓

(b) Find any points of inflexion. 2

For inflexions, $y'' = 0$

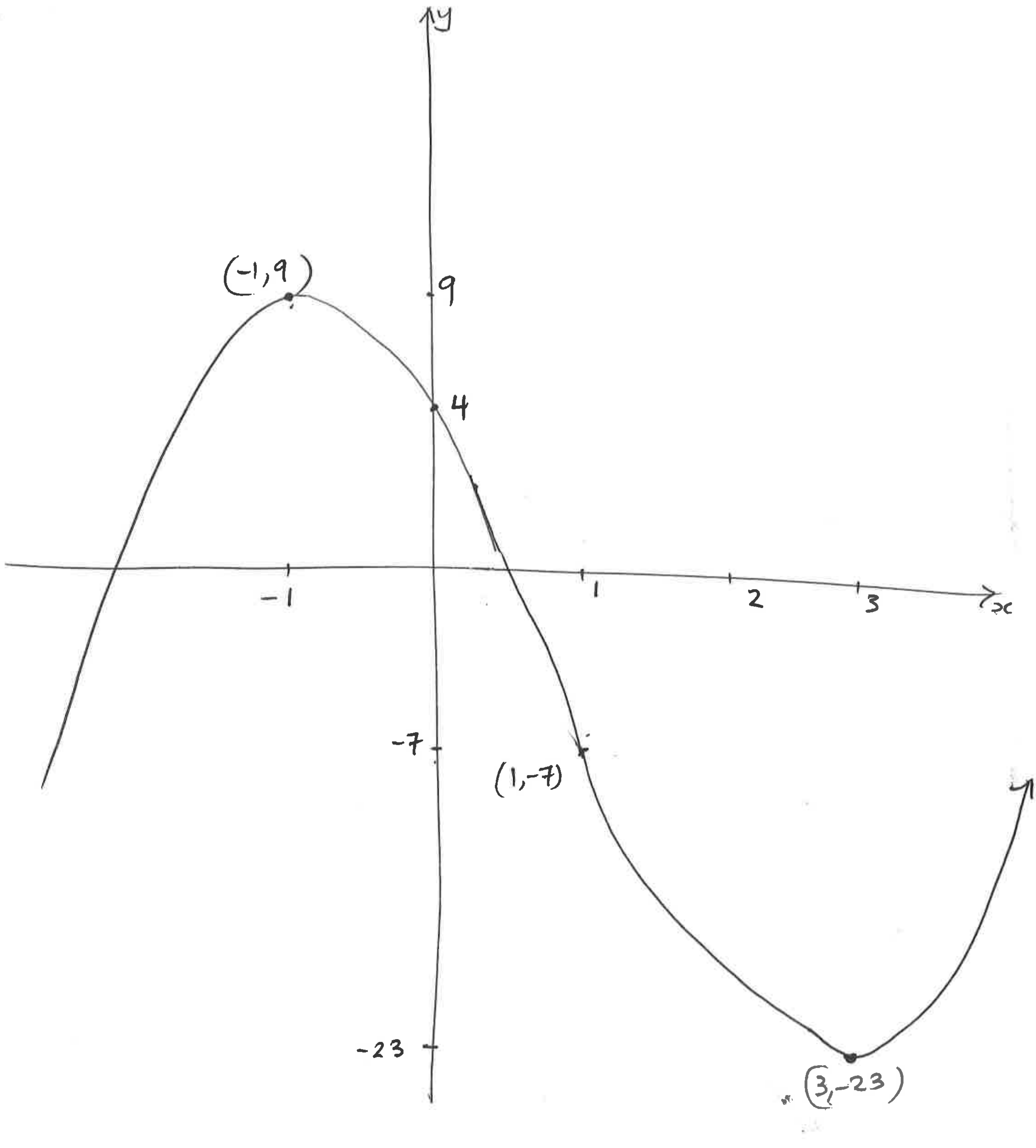
$\therefore$ $6(x-1) = 0$ $\therefore$ $x = 1$, $y = -7$. ✓

Test :

x	0	1	2
y''	-6	0	6

change in concavity, $\therefore$ inflexion at $(1, -7)$ ✓

(c) Sketch the curve, showing all main features. 2

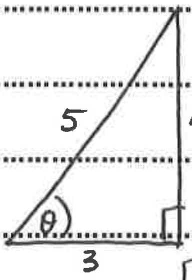


Question 22 (2 marks)

Find the exact value of $\cot\theta$ given that $\cos\theta = 0.6$ and $\sin\theta < 0$. **2**

$\cos\theta = 0{\cdot}6 = \frac{3}{5}$; $\cos\theta > 0$ & $\sin\theta < 0$ $\therefore$ Q4

$\therefore \tan\theta < 0$.



$\therefore \tan\theta = -\frac{4}{3}$ $\therefore$ $\cot\theta = -\frac{3}{4}$ ✓✓

[ALT: $\sin\theta = -\sqrt{1-\cos^2\theta}$, as in Q4 $= -\sqrt{1-\frac{9}{25}}$

$\therefore \sin\theta = -\frac{4}{5}$ $\therefore \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{3/5}{-4/5} = -\frac{3}{4}$.]

Question 23 (3 marks)

A geometric progression has 5th term 9 and 13th term 59 049.

(a) Find the first term and the common ratio. **2**

$T_5 = ar^4 = 9$

$T_{13} = ar^{12} = 59049$

$\therefore \frac{ar^{12}}{ar^4} = \frac{59\,049}{9}$ $\therefore r^8 = 6561$

$\therefore r = 3$ or -3 ✓

$\therefore a(81) = 9$ $\therefore a = \frac{1}{9}$ ✓

(b) Find the 19th term. **1**

$T_{19} = ar^{18} = \frac{1}{9}\cdot(3)^{18}$ (or $\frac{1}{9}(-3)^{18}$)

$= 3^{16}$

$= 43\,046\,721$ ✓

Question 24 (5 marks)

The number of bacteria in a culture can be modelled by $B = 120\ 000\ e^{0.4\,t}$

where t is the time in hours after the experiment started.

(a) How many bacteria are there after 6 hours have passed? **1**

$B = 120000\ e^{(0.4)6} = 120\ 000\ e^{2.4}$

$= 1\ 322\ 781$

✓

(b) How fast was the culture growing after 6 hours? **1**

$\frac{dB}{dt} = 0.4 \times 120000\,e^{0.4t} = 48\ 000\ e^{0.4t}$

$\therefore$ at $t=6$, $\frac{dB}{dt} = 48\ 000\,e^{2.4}$

$= 529\ 112$ per hour.

✓

(c) What was the average rate of increase over the first 6 hours? **1**

Starting number at $t=0$ $= 120\ 000\,e^{0} = 120\ 000$.

$\therefore$ average rate $= \frac{1\ 322\ 781 - 120000}{6}$

$= 200\ 463$ per hour

✓

(d) How long, in hours and minutes, will it take until the number of bacteria doubles? **2**

$240\ 000 = 120\ 000\,e^{0.4t}$

$\therefore\ e^{0.4t} = 2$

$\therefore\ 0.4t = \ln 2$

$\therefore\ t = \frac{\ln 2}{0.4} \doteqdot 1.732\ 867\ldots$ hours

$=$ 1 hour 44 mins. (nearest min.)

✓✓

Question 25 (5 marks)

In an experiment, 2 balls are drawn at random and without replacement from an urn containing 4 red balls and 6 black balls. Let X be the number of red balls selected.

(a) Complete the table below: **2**

Outcome	RR	RB	BR	BB	
X	2	1	1	0	
$p(X = x)$	$\frac{2}{15}$	$\frac{4}{10}\cdot\frac{6}{9} = \frac{4}{15}$	$\frac{6}{10}\cdot\frac{4}{9} = \frac{4}{15}$	$\frac{6}{10}\cdot\frac{5}{9} = \frac{1}{3}$	
$x.p(x)$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{4}{15}$	~~4~~ 0	
X^2	4	1	1	0	

✓✓

(b) What is the expected number of red balls drawn? **1**

$E(X) = \sum x.p(x)$

$= \frac{4}{15} + \frac{4}{15} + \frac{4}{15} + 0 = \frac{12}{15}$

$= 0.8$ ✓

(c) What is the variance, $V(X)$, of this distribution? **2**

$E(X^2) = 4\cdot\frac{2}{15} + 1\cdot\frac{4}{15} + 1\cdot\frac{4}{15} + 0\cdot\frac{1}{3} = \frac{16}{15}$

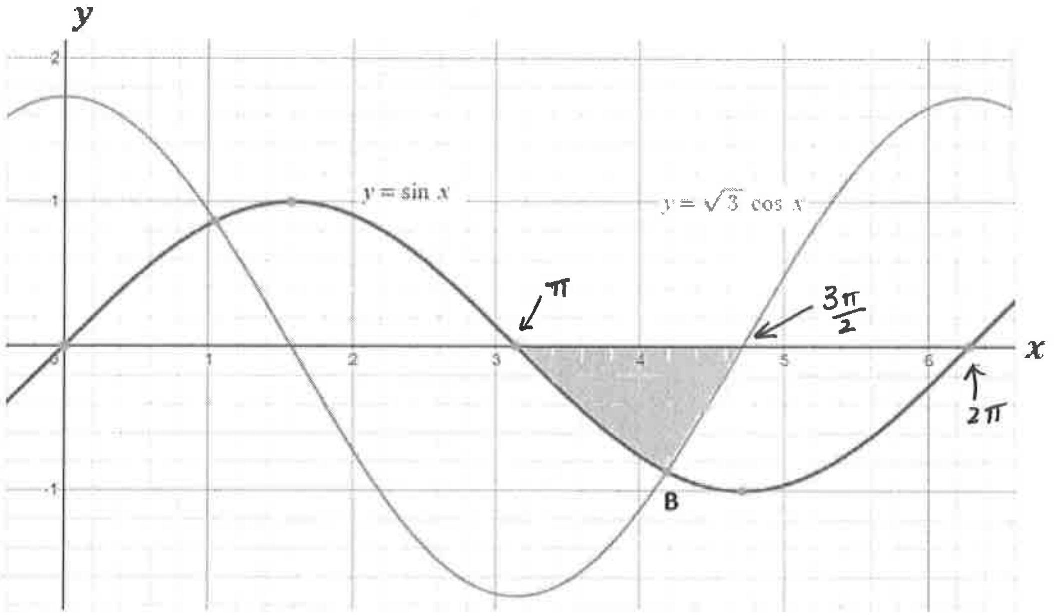
$\therefore V(X) = E(X^2) - (E(X))^2$

$= \frac{16}{15} - (0.8)^2$

$= \frac{32}{75}$

$(= 0.427 \text{ 3d.p.})$ ✓✓

Question 26 (4 marks)



The diagram above shows the graphs of $y = \sin x$ and $y = \sqrt{3}\cos x$, $0 \leq x \leq 2\pi$.
The second point of intersection is labelled B.

(a) Show, using any appropriate method, that B has coordinates $\left(\frac{4\pi}{3}, \frac{-\sqrt{3}}{2}\right)$ **1**

B lies on $\sin x$ and on $\sqrt{3}\cos x$.
Here, $\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$
and $\sqrt{3}\cos\frac{4\pi}{3} = \sqrt{3}\cdot -\cos\frac{\pi}{3}$
$= \sqrt{3}\cdot -\frac{1}{2} = -\frac{\sqrt{3}}{2}$
$\therefore$ curves both go thru $\left(\frac{4\pi}{3}, -\frac{\sqrt{3}}{2}\right)$

ALT: $\sin x = \sqrt{3}\cos x$
$\therefore \tan x = \sqrt{3}$
$\therefore x = \frac{\pi}{3}, \frac{4\pi}{3}$.
But $x > 4$ $\therefore x = \frac{4\pi}{3}$
$\therefore y = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.

✓

(b) Find the exact area of the shaded region. **3**

$$\text{Area} = \left|\int_{\pi}^{\frac{4\pi}{3}} \sin x\,dx\right| + \left|\int_{\frac{4\pi}{3}}^{\frac{3\pi}{2}} \sqrt{3}\cos x\,dx\right|$$ ✓

$$= \left|\Big[-\cos x\Big]_{\pi}^{\frac{4\pi}{3}}\right| + \left|\Big[\sqrt{3}\sin x\Big]_{\frac{4\pi}{3}}^{\frac{3\pi}{2}}\right|$$

$$= \left|-\cos\tfrac{4\pi}{3} - (-\cos\pi)\right| + \left|\sqrt{3}\sin\tfrac{3\pi}{2} - \sqrt{3}\sin\tfrac{4\pi}{3}\right|$$

$$= \left|\tfrac{1}{2} + (-1)\right| + \left|-\sqrt{3} - \sqrt{3}\left(-\tfrac{\sqrt{3}}{2}\right)\right|$$

(as $\sqrt{3} > \frac{3}{2}$)

$$= \tfrac{1}{2} + \left|-\sqrt{3} + \tfrac{3}{2}\right| = \tfrac{1}{2} + \sqrt{3} - \tfrac{3}{2} = \boxed{\sqrt{3} - 1} \text{ units}^2.$$ ✓✓

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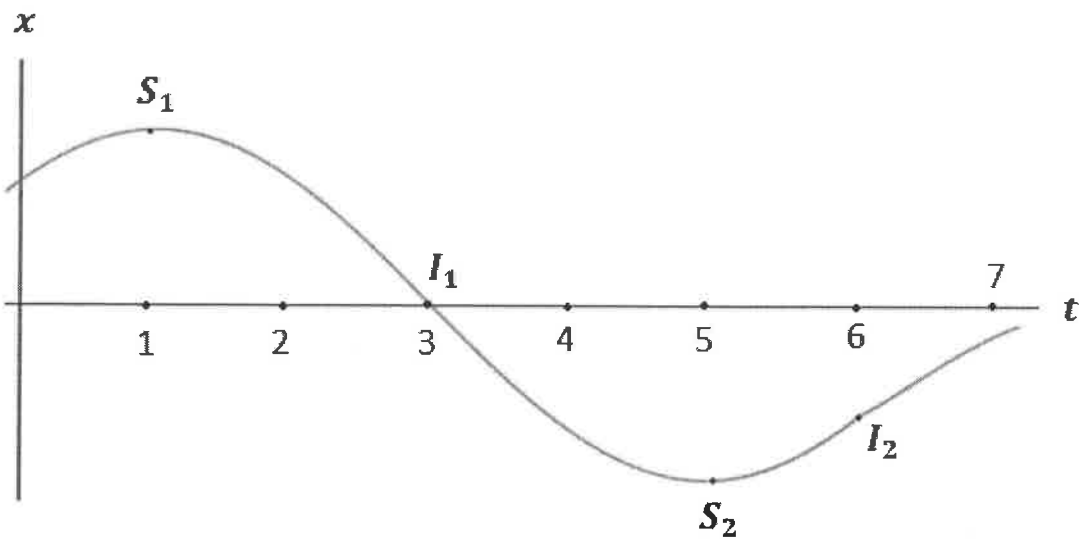
Question 27 (3 marks)



The graph shows the displacement of a particle, moving in a straight line, over the first 7 seconds of its motion. S_1 and S_2 are stationary points, and I_1 and I_2 are inflection points.

State the times, or periods of time, for which:

(a) The particle is stationary. **1**

At $t=1$ and $t=5$ ✓

(b) The velocity is negative. **1**

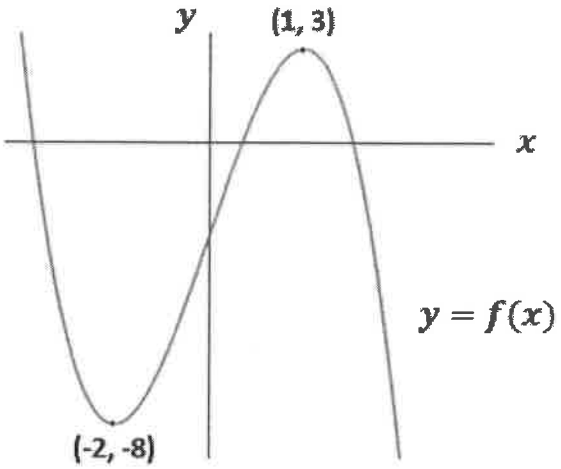
$v=\frac{dx}{dt}$ ∴ $1<t<5$ ✓

(c) The acceleration is positive. **1**

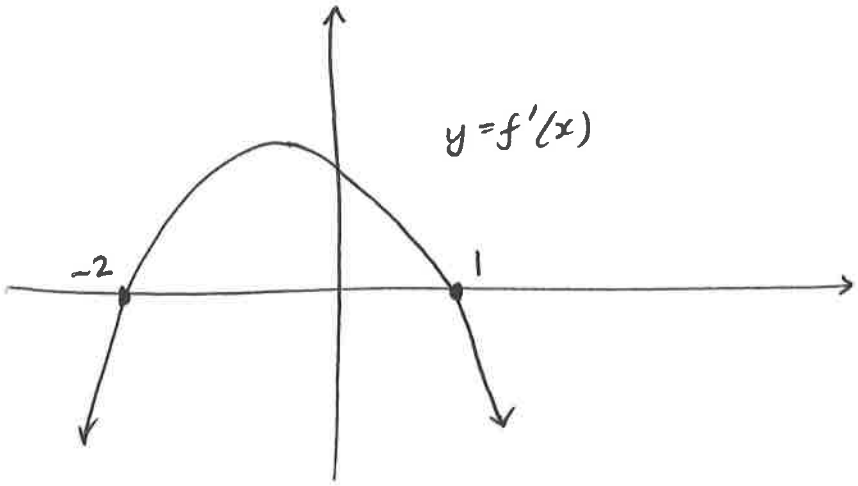
$a=\frac{d^2x}{dt^2}$ ∴ concave up ∴ $3<t<6$ ✓

Question 28 (4 marks)

Consider the graph of $y = f(x)$ shown:



(a) Use the space below to sketch the graph of $y = f'(x)$ **2**



✓ x-ints.
✓ shape.

(b) Find the area bounded by $y = f'(x)$ and the x-axis. **2**

$$\text{Area} = \int_{-2}^{1} f'(x)\,dx$$

$$= \left[f(x) \right]_{-2}^{1}$$

$$= 3 - (-8) = 11u^2$$

✓✓

Question 29 (4 marks)

For events A and B from a sample space, $P(A|B) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$

(a) Calculate $P(A \cap B)$ **1**

$P(A \cap B) = P(A|B) \cdot P(B)$

$= \frac{3}{4} \cdot \frac{1}{3}$

$= \frac{1}{4}$ ✓

(b) Calculate $P(\bar{A} \cap B)$ where $\bar{A}$ denotes the complement of A. **1**

$P(\bar{A} \cap B) = P(\bar{A}|B) \cdot P(B)$

$= \frac{1}{4} \cdot \frac{1}{3}$

$= \frac{1}{12}$ ✓

[ALT: $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$]

(c) If A and B are independent, calculate $P(A \cup B)$ **2**

"Independent" means $P(A) = P(A|B) = \frac{3}{4}$ ✓

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{4} + \frac{1}{3} - \frac{1}{4}$

$= \frac{5}{6}$ ✓

Question 30 (2 marks)

The gradient of a curve is given by $\frac{dy}{dx} = \frac{3x}{x^2+e}$

The curve passes through $(0, 2)$. What is its equation? **2**

$\frac{dy}{dx} = \frac{3x}{x^2+e} = \frac{3}{2} \cdot \frac{2x}{x^2+e}$

$\therefore\ y = \frac{3}{2} \ln(x^2+e) + C$ ✓

At $x=0,\ y=2$ $\quad \therefore\ 2 = \frac{3}{2} \ln(0^2+e) + C$

$2 = \frac{3}{2} \ln e + C$

$= \frac{3}{2} \cdot 1 + C \qquad \therefore\ C = \frac{1}{2}$

$\therefore\ y = \frac{3}{2} \log_e(x^2+e) + \frac{1}{2}$ ✓

Question 31 (3 marks)

If $f(x) = \sqrt{2-x}$ and $g(x) = \sqrt{x}$, then

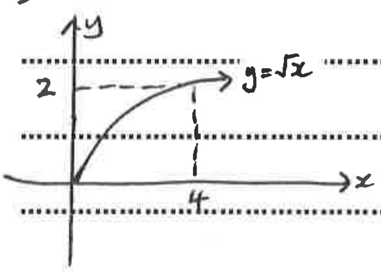
(a) Find the rule for the composite function $f \circ g$ **1**

$f \circ g(x) = \sqrt{2-\sqrt{x}}$ ✓

(b) Find the domain of $f \circ g$ **2**

We need $x \geq 0$ (for $\sqrt{x}$) and also $2-\sqrt{x} \geq 0$.

ie $x \geq 0$ and $\sqrt{x} \leq 2$ $\quad \therefore\ x \leq 4$.

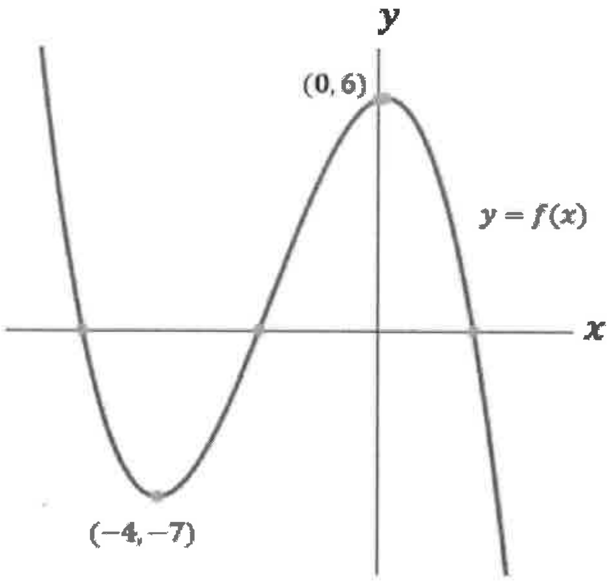


$\therefore$ D: $0 \leq x \leq 4$

(i.e. $[0, 4]$) ✓✓

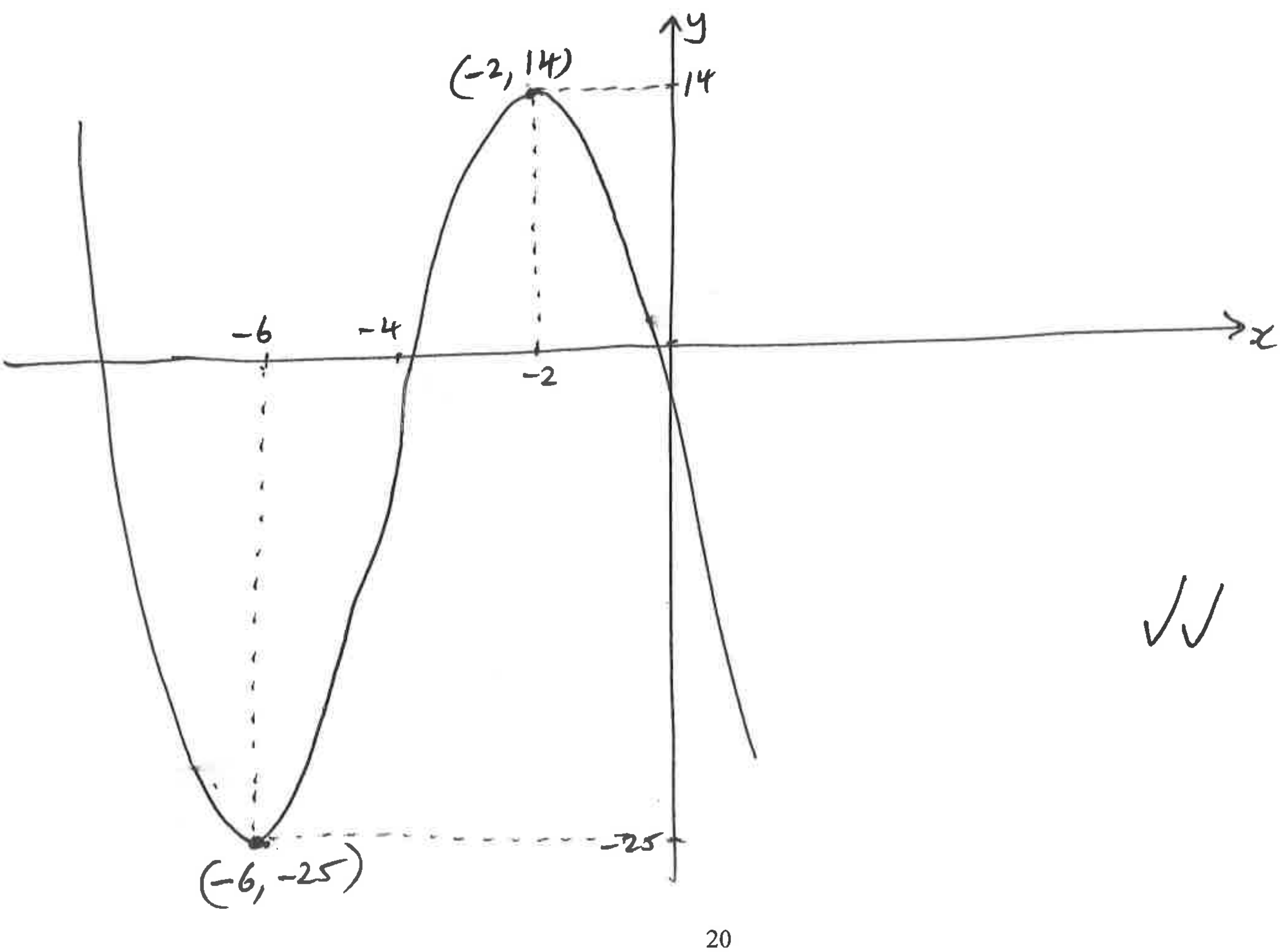
Question 32 (2 marks)

Given the graph of the function $y = f(x)$ below, with turning points as shown, sketch the transformed function $y = 3f(x+2) - 4$. (x-intercepts not required). **2**



(0,6) transforms to $x = -2$, $y = 3(6) - 4 = 14$ $\therefore (-2, 14)$

(−4, −7) transforms to $x = -6$, $y = 3(-7) - 4 = -25$ $\therefore (-6, -25)$.



✓✓

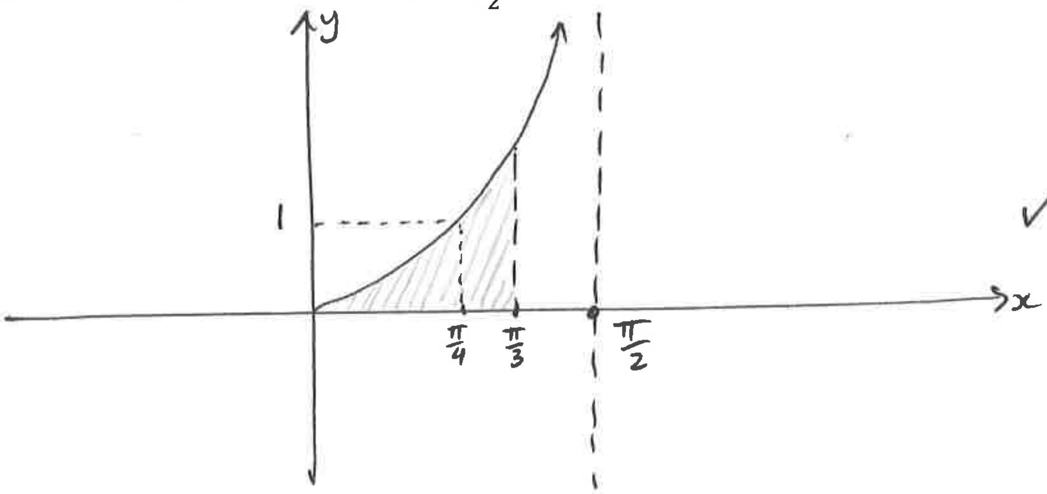
Question 33 (5 marks)

(a) Differentiate $y = \log_e(\cos x)$ with respect to x. **1**

$y' = -\sin x \cdot \frac{1}{\cos x}$

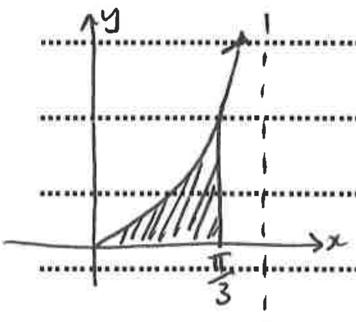
$= -\tan x$ ✓

(b) Sketch $y = \tan x$ for $0 \le x \le \frac{\pi}{2}$ **1**



✓

(c) Hence, using parts (a) and (b), find the area bounded by $y = \tan x$, the x-axis, and the line $x = \frac{\pi}{3}$ **3**
(leave answer in simplest exact form)



$A = \int_0^{\pi/3} \tan x \, dx$

$= -\int_0^{\pi/3} -\tan x \, dx$

$= -\left[\log_e(\cos x)\right]_0^{\pi/3}$ ✓

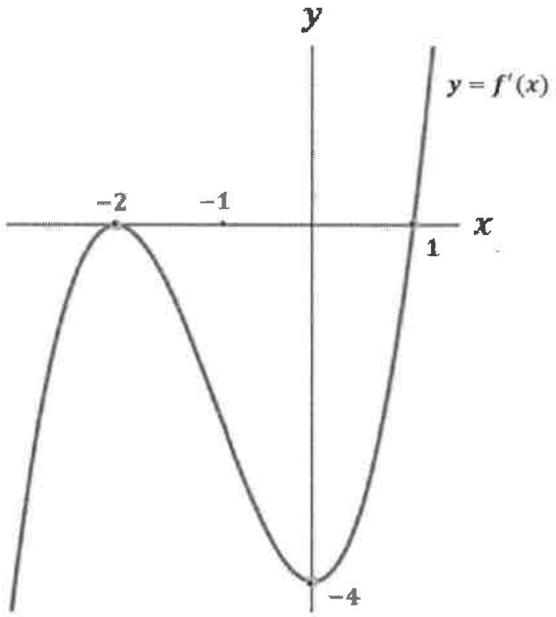
$= -\left[\log_e\left(\frac{1}{2}\right) - \log_e 1\right]$ ✓

$= -\left(\log_e 1 - \log_e 2 - \log_e 1\right)$

$= \log_e 2 \text{ units}^2$ ✓

Question 34 (3 marks)

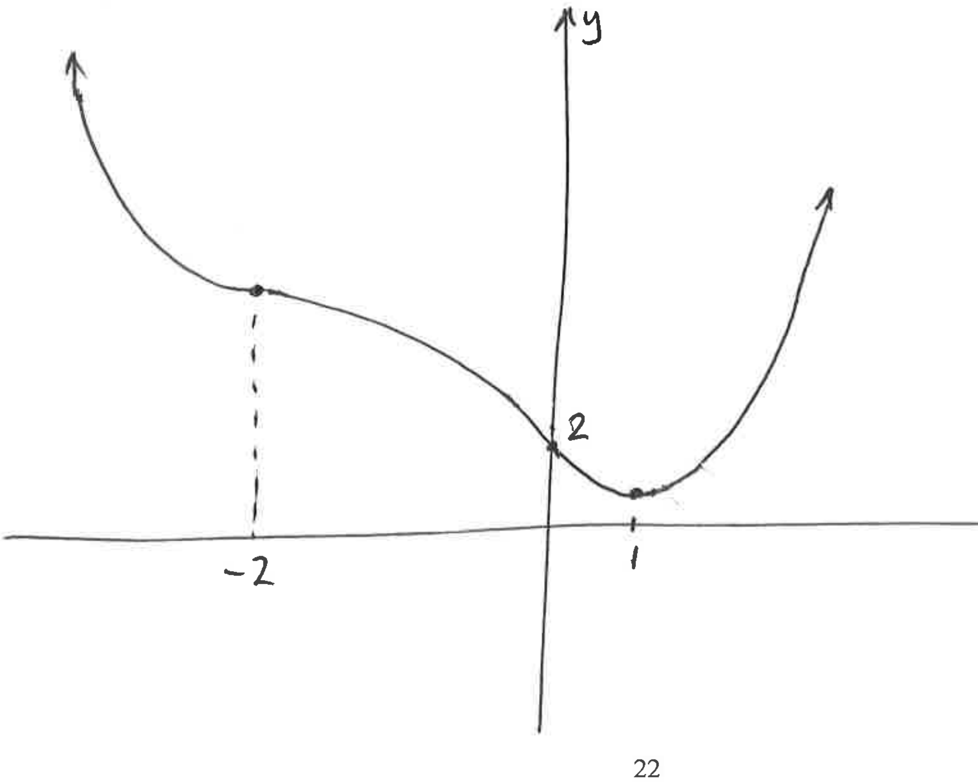
The diagram shows $y = f'(x)$, the graph of the derivative function of $y = f(x)$.



(a) Explain why there is a horizontal point of inflection at $x = -2$ **1**

At $x = -2$, $f'(x) = 0$ ∴ it's a stationary point.
Also, $f''(x) > 0$ on left of $x = -2$, & $f''(x) < 0$ on right
∴ change in concavity ∴ (horizontal) point of inflection.
(ALT: $f'(x)$ is 0 at $x = -2$ but negative on both sides of -2
i.e. ↘ — ↘ ∴ inflection.)

(b) Given that $f(0) = 2$, sketch a possible graph of $y = f(x)$. **2**



- horiz. inflexion at $x = -2$
- minimum turning pt at $x = 1$
- y-int = 2
- shape.

Question 35 (3 marks)

Find the equation of the normal to $y = x\sin x$ at the point where $x = \frac{\pi}{2}$. **3**

$y = x\sin x \quad \therefore y' = \sin x \cdot 1 + x \cdot \cos x$

i.e. $y' = \sin x + x\cos x$ ✓

At $x = \frac{\pi}{2}$, $y = \frac{\pi}{2} \cdot \sin\frac{\pi}{2} = \frac{\pi}{2}$

and, $y' = \sin\frac{\pi}{2} + \frac{\pi}{2} \cdot \cos\frac{\pi}{2} = 1 \quad \therefore m_N = -1$ ✓

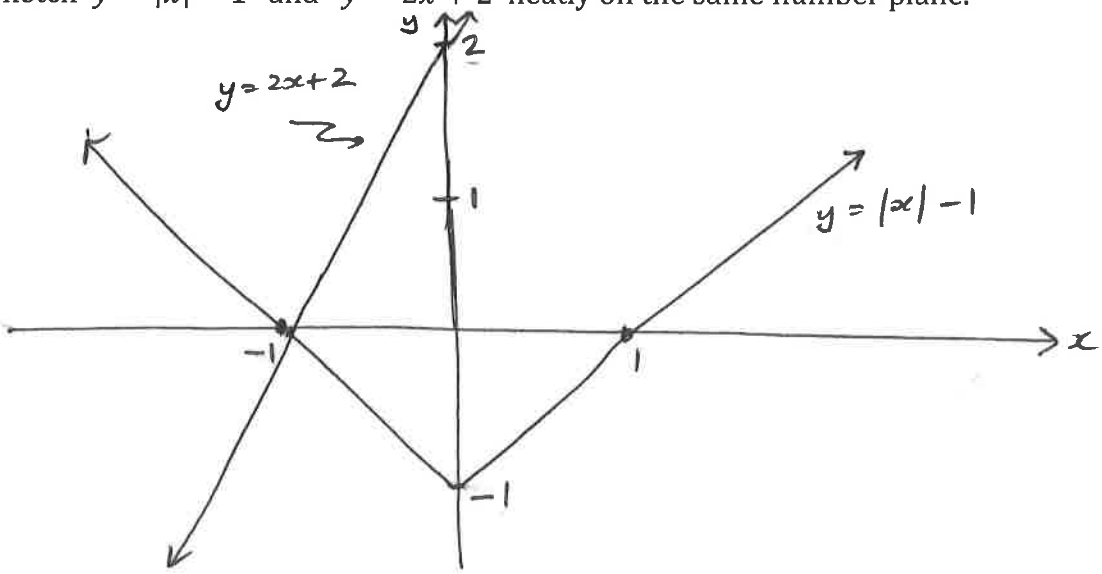
$\therefore$ normal is $y - \frac{\pi}{2} = -1\left(x - \frac{\pi}{2}\right)$

$= -x + \frac{\pi}{2}$

$\therefore y = -x + \pi$ (i.e. $x + y - \pi = 0$) ✓

Question 36 (2 marks)

(a) Sketch $y = |x| - 1$ and $y = 2x + 2$ neatly on the same number plane. **1**



✓

(b) Hence solve the equation $|x| - 2x = 3$ **1**

If $|x| - 2x = 3$ then $|x| = 2x + 3$

$\therefore |x| - 1 = 2x + 2$

So solution will be where graphs intersect in part (a) ✓

$\therefore x = -1$

Question 37 (4 marks)

One half percent (0.5 %) of a country has a certain viral disease. A test is developed for the disease. The test gives a false positive 3% of the time, and a false negative 2% of the time.

(a) Show that the probability that Andy, a randomly selected person, tests positive is $0 \cdot 03475$ **2**

[*Hint*: in this question, let D be the event that Andy has the disease, and $\overline{D}$ be the event Andy does not have it. Let T be the event that Andy's test comes back positive.]

$$P(T) = P(T/D) \cdot P(D) + P(T/\bar{D}) \cdot P(\bar{D})$$

$$= (0{\cdot}98)(0{\cdot}005) + (0{\cdot}03)(0{\cdot}995)$$

$$= 0{\cdot}03475$$

✓✓

(b) Andy just got the bad news that his test came back positive.

Find the probability that Andy actually has the disease. **2**

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{P(T \cap D)}{P(T)}$$

$$= \frac{P(T/D) \cdot P(D)}{P(T)}$$

$$= \frac{(0{\cdot}98)(0{\cdot}005)}{0{\cdot}03475}$$

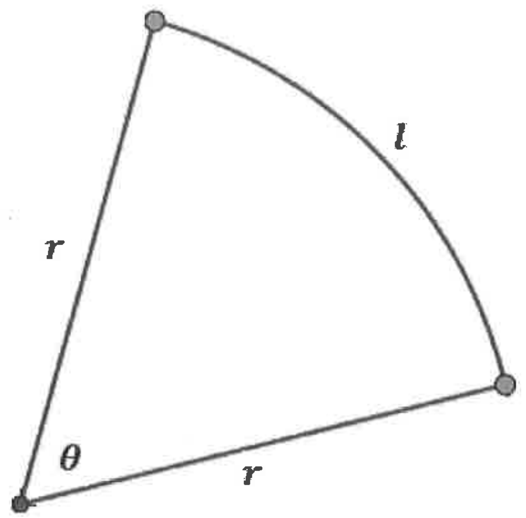
$$= 0{\cdot}141\ldots$$

(i.e. $\doteqdot 14\%$)

✓✓

Question 38 (5 marks)

The diagram below shows a sector of a circle of radius r centimetres. The angle at the centre is θ radians, and the perimeter of the whole sector is $8\ cm$.



(a) Show that $r = \frac{8}{2+\theta}$. **1**

$l = r\theta$

$\therefore\ r + r + r\theta = 8$

$2r + r\theta = 8$

$r(2+\theta) = 8 \qquad \therefore\ r = \frac{8}{2+\theta}$, as required. ✓

(b) Show that A, the area of the sector in cm^2, is given by

$$A = \frac{32\theta}{(\theta+2)^2}$$ **1**

$A = \frac{1}{2}r^2\theta = \frac{1}{2}\left(\frac{8}{2+\theta}\right)^2 \cdot \theta = \frac{1}{2} \cdot \frac{64\theta}{(2+\theta)^2}$

$\therefore\ A = \frac{32\theta}{(2+\theta)^2}$, as required. ✓

(c) If $0 \leq \theta \leq \frac{\pi}{2}$, find the maximum area of the sector, and the value of θ for which this occurs. **3**

$$\frac{dA}{d\theta} = \frac{(\theta+2)^2 \cdot 32 - (32\theta)\cdot 2(\theta+2)}{(\theta+2)^4}$$

✓

For max. A, $\frac{dA}{d\theta} = 0 \quad \therefore \quad 32(\theta+2)^2 - 64\theta(\theta+2) = 0$

$\therefore \quad 32(\theta+2)\left[\theta+2 - 2\theta\right] = 0$

$\therefore \quad 32(\theta+2)(2-\theta) = 0$

$\therefore \quad \theta = -2 \quad$ or $\quad \theta = 2$

But $0 \leq \theta \leq \pi/2$, so these both lie outside the possible range of θ.

So we need to test the endpoints, 0 and $\pi/2$:-

At $\theta = 0$, $A = 0$

At $\theta = \pi/2$, $A = \dfrac{32(\pi/2)}{(\pi/2 + 2)^2} = \dfrac{16\pi}{\left(\frac{\pi+4}{2}\right)^2}$

$\therefore A = \dfrac{64\pi}{(\pi+4)^2}$

$\doteqdot 3.9422 \text{ cm}^2$

Thus max. area $= \dfrac{64\pi}{(\pi+4)^2} \text{ cm}^2 \doteqdot 3.9422 \text{ cm}^2$, obtained when $\theta = \pi/2$ radians.

✓✓

End of examination