NORTH SYDNEY GIRLS HIGH
2u. TRIAL 1990

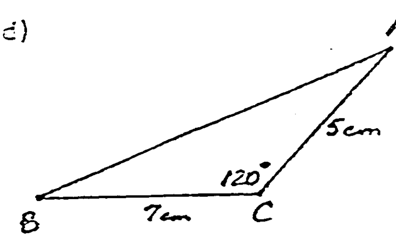
QUESTION 1 (Start a new page)

a) Solve the equation $4(x-2) = x-6$

b) Factorize completely $y^4 - 16$

c) Given that $R = \sqrt{\dfrac{S+P}{S-P}}$ and

$S = 0.032$ and $P = 0.0235$, find R, correct to two decimal places.

d) 

In the given diagram,
BC = 7 cm, AC = 5 cm,
$\angle ACB = 120^0$
Find the exact length of AB.

e) (i) Sketch the curve $(x-1)^2 + (y-2)^2 = 4$

(ii) State whether or not it is a function.

(iii) Write down the domain.

QUESTION 2 (Start a new page)

(a) Evaluate $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

(b) In a container there are 3 red and 5 yellow discs. Three discs are chosen at random. Find the probability that:

(1) all three will be red

(2) all three will be the same colour

(c) Given the points $A(-4, -6)$ and $B(2, 2)$ find:

(i) the gradient of the line joining AB

(ii) the midpoint M of AB

(iii) the equation of the perpendicular bisector of the interval AB

(iv) the perpendicular distance of the point D (4, 0) from the perpendicular bisector found in part (iii)

QUESTION 3 (Start a new page)

(a) Differentiate (i) $3x^2 + \dfrac{3}{x^2}$

(ii) $x \log_e x$

(iii) e^{2-x}

(b) Find the exact value of: $$\int_1^2 (5x - 2)^2 \, dx$$

(c) Find a primitive function of $x\sqrt{x}$

(d) For the parabola $y = (x - 4)^2 - 5$,

write down: (i) the coordinates of its vertex

(ii) the minimum value of the function

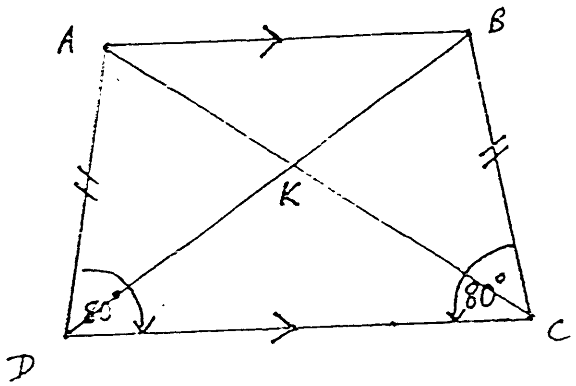
(iii) the coordinates of its focus

QUESTION 4 (Start a new page)

(a) The third term of an arithmetic series is (-4) and the tenth term is (-25).
Find:
(i) the first term

(ii) the common difference

(b)



In the diagram, ABCD is an isosceles trapezium with the angles ADC and BCD equal to 80^0. The diagonals intersect at K.

(i) Copy the sketch onto your answer paper

(ii) Prove $A\hat{C}D = B\hat{D}C$

(iii) Prove Δ ABK and Δ KDC are similar

(iv) Name two isosceles triangles

QUESTION 5 (Start a new page)

(a) Find the equation of the tangent to the curve

$$y = \frac{\log_e x}{x}$$

at the point (1, 0)

(b) The function $f(x)$ is given by $f(x) = 3 + 5x + 2x^2 - 3x^3$

(i) Find the coordinates of the turning points of $f(x)$ and determine whether they are maxima or minima.

(ii) Draw a sketch of $y = f(x)$ in the domain $-\frac{3}{2} \leq x \leq \frac{3}{2}$

(iii) Write down the maximum value of the function in this domain.

QUESTION 6 (Start a new page)

(a) Graph on the number line the values of x for which

$$|4 - 2x| \geq 6$$

(b) (i) Shade on a number plane the region R bounded by the curves

$$y = x^2 + 1 \quad \text{and} \quad y = 9 - x^2$$

(ii) Find the area of this region.

(c) Find the volume of the solid of revolution formed by rotating the portion of the curve $y = x^{\frac{2}{3}}$ from $0 \leq x \leq 1$ about the y - axis

QUESTION 7 (Start a new page)

(a) If α and β are the roots of the quadratic equation

$$5x^2 - 4x + 2 = 0$$

evaluate

(i) $\alpha + \beta$

(ii) $\alpha\beta$

(iii) $(1 - \alpha)(1 - \beta)$

(iv) $\alpha^2 + \beta^2$

(b) For what values of k does the quadratic equation

$$x^2 + 3 - k(x - 1) = 0$$

have equal roots?

(c) Two students, John and Rebecca, attempt a problem in mathematics. The probability that John will solve the problem is $\frac{2}{3}$, whilst the probability that Rebecca will solve it is $\frac{3}{5}$.

(i) What is the probability that John will solve it but Rebecca does not?

Another student, Sam, attempts the same problem. The probability that Sam will solve the problem is $\frac{7}{10}$.

(ii) What is the probability that at least one of the three students will solve the problem?

QUESTION 8 (Start a new page)

(a) Convert $0.3\dot{8}$ to a rational number in its simplest form.

(b) A sinking ship S, which is 23 nautical miles from a rescue ship R, has a bearing of 034° from R. A lighthouse keeper K observes that S is on a bearing of $275^{\circ}T$ from his position, and R bears $255^{\circ}T$ from his position.

(i) Draw a diagram marking on it the information supplied.

(ii) Find the distance at of the ship S from the lighthouse at K. Give your answer to the nearest nautical mile.

(c) Find the first term and the common difference of an arithmetic series in which

$$S_n = \frac{n^2}{3} + 5n$$

QUESTION 9 (Start a new page)

(a) The minute hand of a watch is 8 mm long. How far does its tip move in 40 minutes?

(b) The function f(x) defined for x in the domain $3 \leqslant x \leqslant 7$ is given by the rule

$$f(x) = \log_e (x^2), \qquad 3 \leqslant x \leqslant 7$$

(i) Draw up a table of values of f(x) correct to one decimal place for $x = 3, 4, 5, 6, 7$

(ii) Use this table to draw a sketch of the graph of $y = f(x)$, for $3 \leqslant x \leqslant 7$

[Do not attempt to find turning points]

(iii) Use Simpson's Rule with five function values to estimate the area between the curve, the x-axis, and the ordinates $x = 3$ and $x = 7$.

(c) Find all real numbers x which satisfy the equation

$$x^4 = 2(4 - x^2)$$

Year 12 Trial 1990

Each question - 12 marks

Question 1.

a) $4(x-2) = x-6$

$4x-8 = x-6$

$3x = 2$

$x = \frac{2}{3}$ (2)

b). $y^4 - 16 = (y^2+4)(y^2-4)$

$= (y^2+4)(y-2)(y+2)$ (2)

c) $R = \sqrt{\dfrac{0.032+0.0235}{0.032-0.0235}}$

$\doteqdot 2.56$ (2)

d). Using cosine rule,

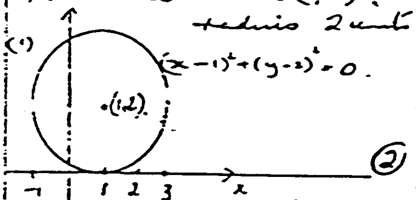
$AB^2 = 5^2+7^2-2.5.7.\cos 150$

$= 25+49-70(-\frac{\sqrt{3}}{2})$

$AB = \sqrt{74+35\sqrt{3}}$ (2)

e) (i) Circle centre $(1,2)$. radius 2 units

$(x-1)^2+(y-2)^2 = 0$.



(2)

(ii) It is not a function

(iii) $\{x : -1 \le x \le 3\}$. (2)

Question 2.

a) $\lim\limits_{x\to3} \dfrac{x^3-27}{x-3} = \dfrac{(x-3)(x^2+3x+9)}{(x-3)}$

$= \lim\limits_{x\to3}(x^2+3x+9)$

$= 9+9+9$

$= 27$ (2)

b). $n(R) = 3$; $n(Y) = 5$; $n(S) = 8$

(i) $P(RRR) = \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6}$

$= \frac{1}{56}$ (2)

(ii) $P(YYY) = \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}$

$= \frac{5}{28}$

$P(\text{same colour}) = P(RRR) + P(YYY)$

$= \frac{1}{56} + \frac{5}{28}$

$= \frac{11}{56}$ (2)

c) (i) $A(-4,-6)$ $B(2,2)$

$m_{AB} = \dfrac{y_2-y_1}{x_2-x_1}$

$= \dfrac{-6-2}{-4-2}$

$= \frac{4}{3}$ (1)

(ii) Midpoint is given by

$\left(\dfrac{-4+2}{2}, \dfrac{-6+2}{2}\right)$

i.e. $(-1,-2)$ (1)

Q2. (cont).

(iii) line $\perp AB$

gradient of line $= -\frac{3}{4}$ $(m_1 \times m_2 = -1)$

Perpendicular bisector of AB through $(-1,-2)$ is given by

$y+2 = -\frac{3}{4}(x+1)$

$4y+8 = -3x-3$

$3x+4y+11 = 0$. (2)

iv. Perpendicular distance is given by

$p = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

$= \dfrac{|12+0+11|}{\sqrt{3^2+4^2}}$

$= \frac{23}{5}$

Perp. distance $= \frac{23}{5}$ units (2)

Question 3.

a) (i) let $y = 3x^2+3x^{-2}$

$\frac{dy}{dx} = 6x - 6x^{-3}$

$= 6x - \frac{6}{x^3}$ (2)

(ii) let $y = x\log_e x$

$\frac{dy}{dx} = \log_e x \cdot 1 + x \cdot \frac{1}{x}$

$= \log_e x + 1$ (2)

(iii) $\frac{d}{dx}\left(e^{2-x}\right) = e^{2-x} \cdot -1$

$= -e^{2-x}$ (2)

b). $\displaystyle\int_1^2 (5x-2)^2\,dx = \left[\frac{(5x-2)^3}{3\times5}\right]_1^2$

$= \frac{8^3}{15} - \frac{3^3}{15}$

$= 32\frac{1}{3}$ (2)

c. $\displaystyle\int x\sqrt{x}\,dx = \int x^{3/2}\,dx$

$= \frac{2}{5}x^{5/2} + C$

$= \frac{2x^2\sqrt{x}}{5} + C$ (1)

d). $y = (x+4)^2 - 5$

$(x+4)^2 = y+5$

(i) Vertex $(-4,-5)$

(ii) Minimum value is (-5).

(iii) $a = \frac{1}{4}$

Focus $(-4, -4\frac{3}{4})$

(3)

Question 4.

a). $T_3 = -4$ $T_{10} = -25$

$a+2d = -4$ ①

$a+9d = -25$ ②

② − ① $7d = -21$

$d = -3$

Sub. in ①

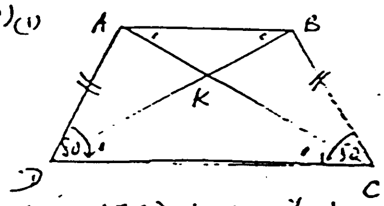
$a - 6 = -4$

$a = 2$

First term is 2

Common difference $= -3$ (4)

b) (i)



Data: ABCD is isosceles trap.

$A\hat{D}C = B\hat{C}D = 50°$

(ii) Aim: To prove $A\hat{C}D = B\hat{D}C$

Proof: In ΔACD, ΔBDC

$\angle ADC = \angle BCD$ (given)

$AD = BC$ (given)

DC is common

$\therefore \Delta ACD \equiv \Delta BDC$

(two sides and included angle).

$\therefore \angle ACD = \angle BDC$

(corresponding angles of congruent triangles).

(iii) Aim: To prove $\Delta ABK \parallel\!| \Delta KDC$

Proof: In ΔABK, ΔKDC

$\angle ABK = \angle KDC$ (alternate ∠, $AB \parallel DC$)

$\angle KAB = \angle KCD$ (alt. ∠s, $AB \parallel DC$)

$\angle AKB = \angle DKC$ (vert opp)

$\therefore \Delta ABK \parallel\!| \Delta KDC$ (equiangular)

(iv) ΔKAB

ΔKDC

are isosceles triangles (8)

Question 5.

a) $y = \dfrac{\log_e x}{x}$

$\dfrac{dy}{dx} = \dfrac{x \cdot \frac{1}{x} - \log_e x \cdot 1}{x^2}$

$= \dfrac{1 - \log_e x}{x^2}$

At the point $(1,0)$

$\dfrac{dy}{dx} = \dfrac{1-\log 1}{1}$

$= 1$

Equation of tangent is given by

$y - 0 = 1(x-1)$

$y = x - 1$

(4)

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QUESTION 5 (cont).

b) $f(x) = 3 + 5x + 2x^2 - 3x^3$

$f'(x) = 5 + 4x - 9x^2$

$f''(x) = 4 - 18x$

Turning points occur where $f'(x) = 0$.

$5 + 4x - 9x^2 = 0$

$(5 + 9x)(1 - x) = 0$

$x = 1, -\frac{5}{9}$

$f''(x) = 4 - 18x$

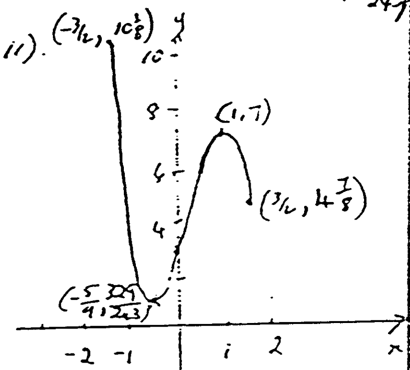
$f''(1) = 4 - 18 < 0$

$\therefore$ maximum at $(1, 7)$

$f''(-\frac{5}{9}) = 4 + 10 > 0$

$\therefore$ minimum at $(-\frac{5}{9}, \frac{329}{243})$

ii)



When $x = 0$, $y = 3$

when $x = -\frac{3}{2}$, $y = 10\frac{1}{8}$

when $x = \frac{3}{2}$, $y = 4\frac{7}{8}$

Maximum value is $10\frac{1}{8}$

(7)

QUESTION 6

a) $|4 - 2x| \geq 6$

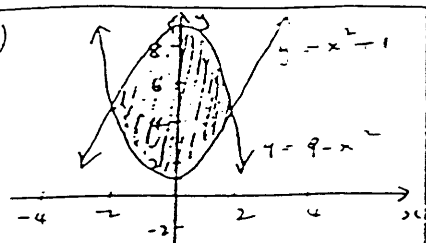
$4 - 2x \geq 6 \qquad 4 - 2x \leq -6$

$-2x \geq 2 \qquad -2x \leq -10$

$x \leq -1 \qquad x \geq 5$

(3)

b)



$\begin{cases} y = x^2 + 1 \\ y = 9 - x^2 \end{cases}$

Points of intersection are given by

$x^2 + 1 = 9 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

Required area is given by

$A = \int_{-2}^{2}(9 - x^2)\,dx - \int_{-2}^{2}(x^2 + 1)\,dx$

$=$

$= 2\int_0^2 (8 - 2x^2)\,dx$

$= 2\left[8x - \frac{2x^3}{3}\right]_0^2$

$= 2\left[16 - \frac{16}{3} - 0\right]$

$= \frac{64}{3}$

Area is $\frac{64}{3}$ sq. units (6)

c) Required volume is given by

$V = \pi\int_0^1 [f(y)]^2\,dy$

$= \pi\int_0^1 y^3\,dy$

$= \pi\left[\frac{y^4}{4}\right]_0^1$

$= \pi\left[\frac{1}{4} - 0\right]$

$= \frac{\pi}{4}$

Volume is $\frac{\pi}{4}$ units^3

(Working: $y = x^{2/3}$

$y^3 = x^2$

When $x = 1$, $y = 0$

when $x = 1$, $y = 1$.)

(3)

QUESTION 7.

a) $5x^2 - 4x + 2 = 0$

i) $\alpha + \beta = -\frac{b}{a}$

$= \frac{4}{5}$

ii) $\alpha\beta = \frac{c}{a}$

$= \frac{2}{5}$

(iii) $(1 - \alpha)(1 - \beta) = 1 - \beta - \alpha + \alpha\beta$

$= 1 - (\alpha + \beta) + \alpha\beta$

$= 1 - \frac{4}{5} + \frac{2}{5}$

$= \frac{3}{5}$

(iv) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (\frac{4}{5})^2 - 2.\frac{2}{5}$

$= -\frac{4}{25}$ (4)

b) $x^2 + 3 - k(x - 1) = 0$

$x^2 - kx + (3 + k) = 0$

$\Delta = b^2 - 4ac$

$= k^2 - 4(3 + k)$

$= k^2 - 12 - 4k$

For equal roots, $\Delta = 0$

$k^2 - 4k - 12 = 0$. (4)

$(k - 6)(k + 2) = 0$

$k = -2, 6$

c) $P(D_S) = \frac{2}{3}$; $P(R_S) = \frac{3}{5}$

$P(S_S) = \frac{7}{10}$.

(i) $P(D_S, R_F) = \frac{2}{3} \times \frac{2}{5}$

$= \frac{4}{15}$

(ii) $P(D_F\ R_F\ S_F) = \frac{1}{3} \times \frac{2}{5} \times \frac{3}{10}$

$= \frac{1}{25}$

(iii) P(at least one solving it)

$= 1 - \frac{1}{25}$

$= \frac{24}{25}$ (3)

QUESTION 8

a). Let $x = 0.388\ldots$

$10x = 3.888\ldots$

$\therefore 9x = 3.5$

$x = \frac{3.5}{9}$

$= \frac{7}{18}$ (2)

(or use series)

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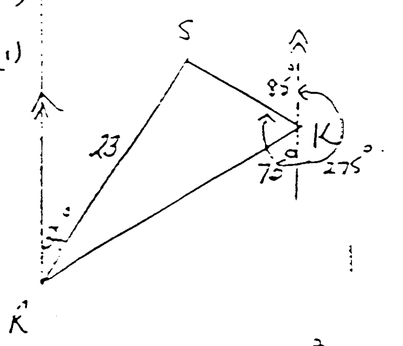
Question 8 (cont)

b) (i)



$255 = 180 + 75°$.

(ii) From diagram, $\angle SKR = 20°$.

$\angle SRK = 180 - (34 + 105)$
$= 41°$

(co-interior angles, north lines parallel)

In ΔSRK, using sine rule

$$\frac{SK}{\sin 41°} = \frac{23}{\sin 20°}$$

$$SK = \frac{23 \sin 41}{\sin 20}$$

$\doteqdot 44$

Distance is approx. 44 n.m. (5)

c) $S_n = \frac{n^2}{3} + 5n$

$T_n = S_n - S_{n-1}$

$= \frac{n^2}{3} + 5n - \frac{(n-1)^2}{3} - 5(n-1)$

$= \frac{n^2}{3} + 5n - \frac{[n^2-2n+1]}{3} - 5n + 5$

$= \frac{n^2 - n^2 + 2n - 1 + 15}{3}$

$= \frac{2n+14}{3}$ (5)

$T_1 = \frac{16}{3}$

$T_2 = \frac{18}{3}$

First term is $\frac{16}{3}$

∴ common difference is $\frac{2}{3}$

$S_1 = \frac{1}{3} + 5 = 5\frac{1}{3}$ (1)

or $S_2 = \frac{4}{3} + 10 = 11\frac{1}{3}$ (1)

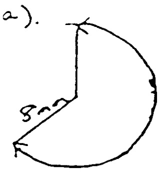
$T_2 = S_2 - S_1$ (1)

$= 6$ (1)

$a = 5\frac{1}{3}$

Question 9

a). $C = \pi d$.



Angle of movement $= \frac{2}{3} \times 360$
$= 240°$.

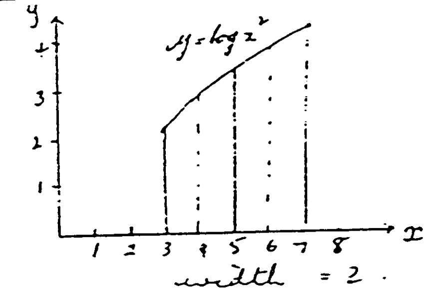
(i) $P = \frac{2}{3}\pi d$
$= \frac{2}{3} \times \pi \times 16$
$\doteqdot 33.5$

or (ii) $l = r\theta$
$= 8 \times \frac{4\pi}{3}$
$\doteqdot 33.5$ (2)

Length $\doteqdot 33.5$ mm.

b). $f(x) = \log_e(x^2)$.

x	3	4	5	6	7
$\ln x^2$	2.2	2.8	3.2	3.6	3.9



width = 2.

Area is given by

$$A = \frac{b-a}{6}\left\{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right\}$$

Required area is given by

$A \doteqdot \frac{2}{6}\{f(3) + 4f(4) + 2f(5) + 4f(6) + f(7)\}$

$= \frac{1}{3}\{2.2 + 4\times2.8 + 2\times3.2 + 4\times3.6 + 3.9\}$

$= 12.7$.

Area is 12.7 units² (6)

(c) $x^4 = 2(4 - x^2)$

$x^4 + 2x^2 - 8 = 0$.

Let $M = x^2$, equation becomes

$M^2 + 2M - 8 = 0$

$(M+4)(M-2) = 0$

$M = -4$ or 2

∴ $x^2 = -4$ no solution

$x^2 = 2$

$x = \pm\sqrt{2}$

Solution: $x = \pm\sqrt{2}$. (4)

Question 10

a) $3^{10} + 3^9 + 3^8 + \ldots$

series is geometric

$a = 3^{10}$, $r = \frac{1}{3}$

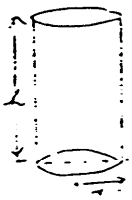
$S_\infty = \frac{a}{1-r}$
$= \frac{3^{10}}{1 - 1/3}$
$= \frac{3^{11}}{2}$

$S_{10} = \frac{a(1-r^n)}{1-r}$
$= \frac{3^{10}(1-(\frac{1}{3})^{10})}{1 - 1/3}$
$= \frac{3}{2} \cdot 3^{10}(1 - \frac{1}{3^{10}})$

(variations) $= \frac{3^{11}}{2} - \frac{3}{2}$

$S_\infty - S_{10} = \frac{3^{11}}{2} - (\frac{3^{11}}{2} - \frac{3}{2})$
$= 1\frac{1}{2}$ (5)

b). Given $S_A = 300\pi$ (cm²)



Can is open at the top.

$S_A = \pi r^2 + 2\pi r h$

$\therefore \pi r^2 + 2\pi r h = 300\pi$

$r^2 + 2rh = 300$

$h = \frac{300 - r^2}{2r}$

(i) $V = \pi r^2 h$
$= \pi r^2 \cdot \frac{(300 - r^2)}{2r}$
$= 150\pi r - \frac{\pi r^3}{2}$.

(ii) $\frac{dV}{dr} = 150\pi - \frac{3\pi r^2}{2}$

$\frac{d^2V}{dr^2} = -3\pi r$

For a maximum volume, $\frac{dV}{dr} = 0$ and $\frac{d^2V}{dr^2} < 0$.

$150\pi - \frac{3\pi r^2}{2} = 0$.

$\frac{3\pi r^2}{2} = 150\pi$.

$r^2 = 100$

$r = \pm 10$.

Since $r > 0$, $r = 10$.

$\frac{d^2V}{dr^2} = -30\pi < 0$

∴ max. volume when $r = 10$.

$h = \frac{300 - 100}{20}$
$= 10$.

Maximum volume is given when radius = 10cm and height is 10