NORTH SYDNEY GIRLS HIGH SCHOOL

**2007**
**TRIAL HSC EXAMINATION**

# Mathematics

## General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

## Total Marks – 120
Attempt Questions 1–10
All questions are of equal value

At the end of the examination, place your solution booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

**Student Number:**____________________________________**Teacher:**____________________________

**Student Name:**____________________________________

| QUESTION | MARK |
|---|---|
| 1 | /12 |
| 2 | /12 |
| 3 | /12 |
| 4 | /12 |
| 5 | /12 |
| 6 | /12 |
| 7 | /12 |
| 8 | /12 |
| 9 | /12 |
| 10 | /12 |
| TOTAL | /120 |

**Total Marks – 120**
**Attempt Questions 1–10**
**All questions are of equal value**

Begin each question in a SEPARATE writing booklet. Extra writing booklets are available.

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**Question 1** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\dfrac{e^2 - 1\cdot 2^3}{\sqrt{5\cdot 8 + 2\cdot 7}}$ correct to two significant figures. **2**

(b) Differentiate $x^3 - \dfrac{4}{x}$. **2**

(c) Solve $|x+2| \geq 4$. **2**

(d) Find the limiting sum of the series $\dfrac{2}{3} + \dfrac{1}{2} + \dfrac{3}{8} + \ldots.$ **2**

(e) If $\dfrac{4}{\sqrt{3}-1} = a + \sqrt{b}$, find the values of *a* and *b*, where *a* and *b* are integers. **2**

(f) Find the value of *x* correct to 1 decimal place. **2**

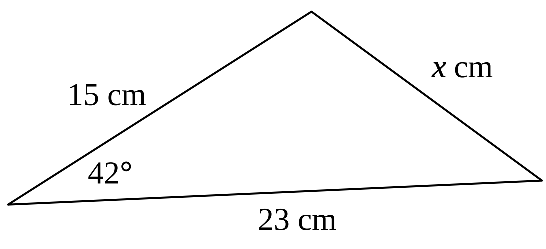



Diagram not to scale

**Question 2** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y = \ln(x-2)$ at the point where $x = 3$. **2**

(b) Differentiate:

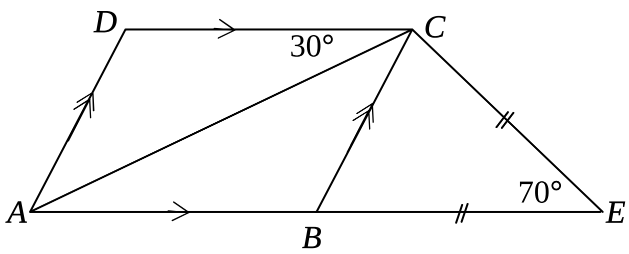
(i) $xe^{4x}$ **2**

(ii) $\sin 3x + \tan x$ **2**

(c) Find $\int \sec^2\left(\frac{x}{2}\right)dx$ **1**

(d) Evaluate $\int_0^2 \left(1 - \frac{2}{e^x}\right) dx$ **2**

(e) In the diagram below, $ABCD$ is a parallelogram. **3**
$\angle ACD = 30°$ and $\angle BEC = 70°$.
Copy or trace the diagram into your writing booklet.
Find the size of $\angle ACB$ giving reasons for your answer.



**Question 3** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $\alpha$ and $\beta$ are the roots of $1 - 4x - 2x^2 = 0$, evaluate

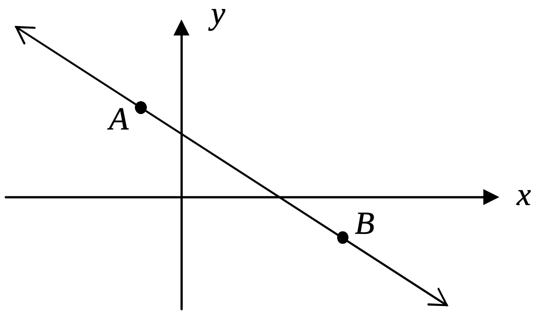
(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $\alpha^2 + \beta^2$ **2**

(b) Solve $2\cos x + 1 = 0$ for $x$ if $0 \le x \le 2\pi$. **2**

(c) In the diagram, the line joining the points $A(-1, 2)$ and $B(3, -1)$ is shown




(i) Show that the equation of the line $AB$ is $3x + 4y - 5 = 0$. **2**

(ii) The line $AB$ makes an angle of $\theta$ with the positive direction of the $x$-axis. Find $\theta$ to the nearest minute. **1**

(iii) From $C(4, 2)$ a perpendicular is drawn to the line $AB$. Find the length of this perpendicular. **2**

(iv) Find the equation of the circle, centre $C$, which has $AB$ as a tangent. **1**

**Question 4** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) The graphs of $y = 4 - x^2$ and $y = 2x + 1$ intersect at the points $A$ and $B$.

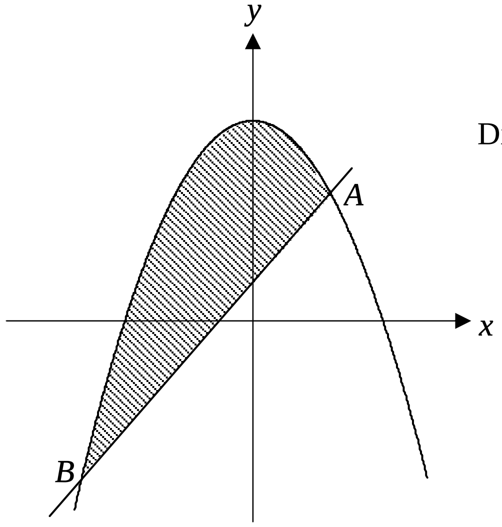



Diagram not to scale

(i) Find the $x$ values of the points $A$ and $B$. **2**

(ii) Find the shaded area bounded by $y = 4 - x^2$ and $y = 2x + 1$. **3**

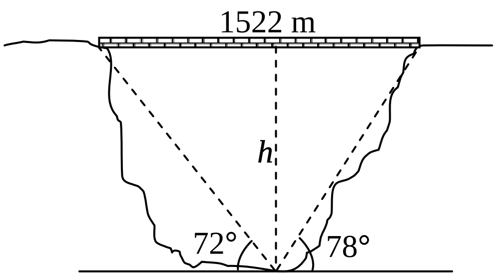
(b) A parabola has the equation $y = x^2 + 6x + 10$

Find: (i) the equation of the axis of symmetry; **1**

(ii) the coordinates of the vertex; **1**

(iii) the focal length. **2**

(c) A bridge is being built across a canyon. The length of the bridge is 1522 metres. From the deepest point in the canyon, the angles of elevation of the ends of the bridge are 78° and 72°. **3**




Show that the depth, $h$ of the canyon is given by

$$h = \frac{1522}{\tan 12° + \tan 18°}$$

**Question 5** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the curve given by $f(x) = 7 + 4x^3 - 3x^4$.

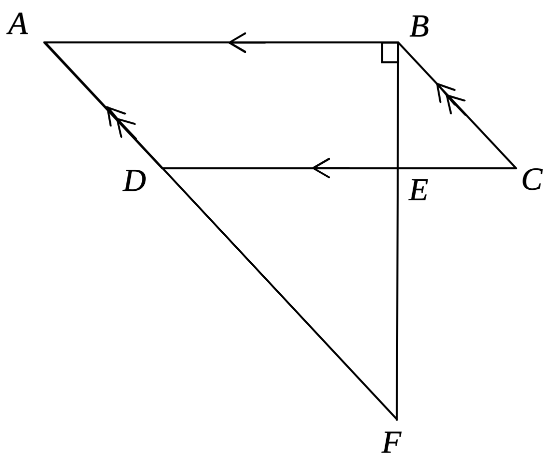
(i) Find the coordinates of the stationary points and determine their nature. **3**

(ii) Find the coordinates of any points of inflexion. **2**

(iii) Sketch the curve for the domain $-1 \le x \le 2$, showing these features. **2**

(iv) What is the minimum value of the function for the domain $-1 \le x \le 2$? **1**

(b) In the diagram below, $ABCD$ is a parallelogram. $FB \perp AB$.




Copy or trace the diagram into your writing booklet.

(i) Prove $\Delta CBE \parallel\!| \Delta AFB$. **3**

(ii) If $CE = 3$ cm, $BC = 7$ cm and $AF = 15$ cm, find $AB$. **1**

**Question 6** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x \, dx$ giving your answer in exact form. **3**

(b) Sketch the graph of $y = 3\cos 2x$ over the domain $-\pi \le x \le \pi$. **3**
Use your graph to find the number of solutions to the equation $3\cos 2x = -1$ over this domain. DO NOT find the solutions.

(c) *AD* and *BC* are arcs of concentric circles with *O* as their centre. *OA* = 8 metres and *OC* = 10 metres. Find the *exact* perimeter of the shaded region. **3**

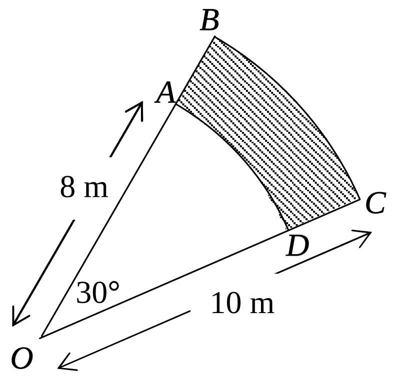
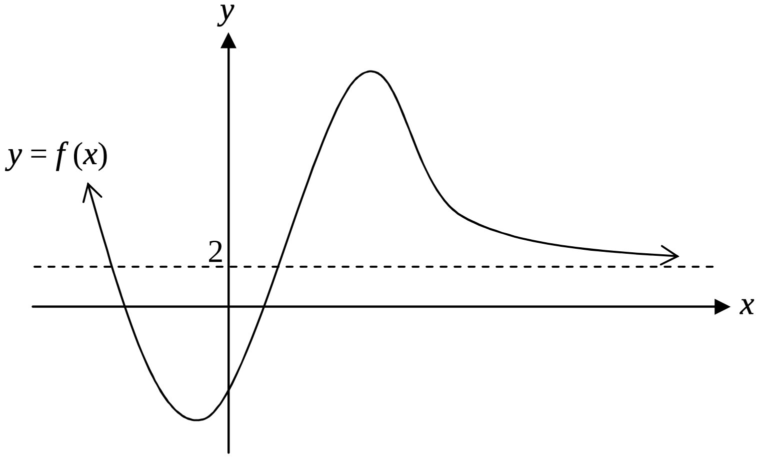



Diagram not to scale

(d) The diagram below shows a sketch of the curve $y = f(x)$. **3**
In your writing booklet, copy or trace the diagram and use it to draw a sketch of the gradient function $y = f'(x)$.



**Question 7** (12 marks) Use a SEPARATE writing booklet. **Marks**

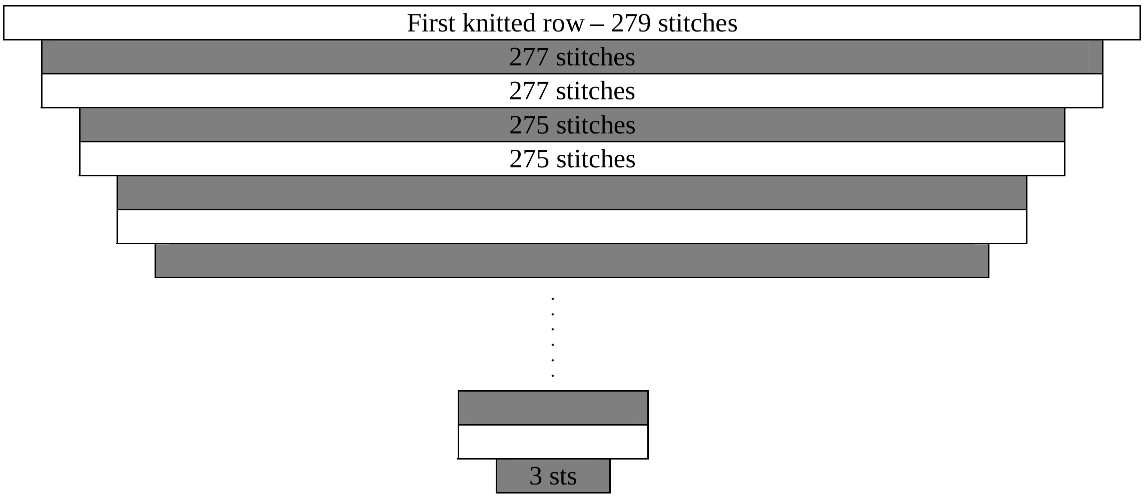
(a) (i) Draw a neat sketch of the function $f(x) = \sqrt{16 - x^2}$ and state the domain. **2**

(ii) Use Simpson's rule with 3 function values to find the area in the first quadrant bound by the curve $y = f(x)$, the $x$-axis and the $y$-axis. Leave your answer in surd form. **2**

(iii) Calculate the exact area of the region. **1**

(iv) Calculate the percentage error in your answer to part (ii) above. **1**

(b) Nikky is knitting a triangular shawl which begins with 279 stitches. The pattern requires that one stitch is decreased at each end of alternate rows until three stitches remain.This is illustrated in the diagram below.




(i) How many rows will Nikky need to knit? **3**

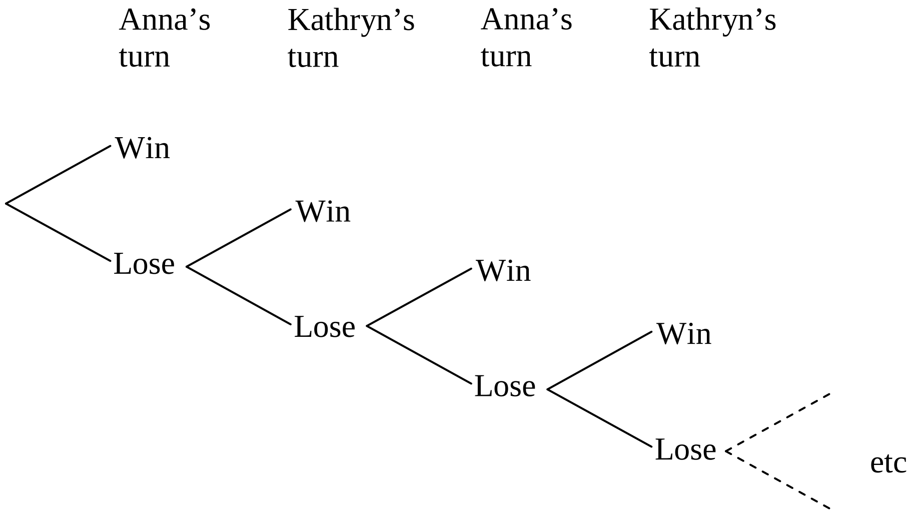
(ii) 1 cm of yarn is used to form each stitch, and 280 cm of yarn is required to begin and end the shawl. Nikky has only 2 balls of yarn, each having a length of 210 m. **3**

Will there be enough yarn to complete the shawl? Support your answer with calculations and reasoning.

**Question 8** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Solve $e^{2x} - 3e^{x} - 4 = 0$ **2**

(b) Anna and Kathryn play a game in which they take turns to throw a die. Anna wins if she throws an even number or a six. Kathryn wins if she throws an odd number or a six. Anna has the first throw and the game continues until someone wins.



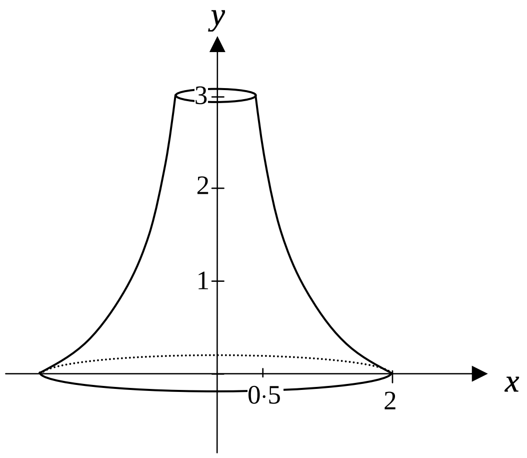

(i) Find the probability that Anna wins on her first throw. **1**

(ii) Show that the probability that Anna wins on her second throw is $\frac{1}{12}$. **1**

(iii) What is the probability that Anna wins on her first, second or third throw? Leave your answer in unsimplified form. **2**

(iv) What is the probability that Anna eventually wins the game? **2**

(c) A vase is formed by rotating the part of the curve $y = \frac{2}{x} - 1$ between $x = 0{\cdot}5$ and $x = 2$ about the $y$ axis. Find the volume of the vase. **4**



**Question 9** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Mother has just been shopping and purchased 3 cans of baked beans and 2 cans of spaghetti. While she is on the phone, Junior removes the labels from all the cans so that they are now indistinguishable. Junior wants baked beans for lunch and Mother decides to open only two cans. She selects the two cans at random.

(i) Draw a probability tree to illustrate the situation. **1**

(ii) What is the probability that Mother selects

($\alpha$) two cans of baked beans? **1**

($\beta$) exactly one can of baked beans? **2**

(iii) Mother opens one can and discovers that it is spaghetti. What is the probability that the other can is baked beans? **1**

(b) Alison has planned a holiday which she decides to take in 3 years time. She has estimated that the holiday will cost about \$8 000 and plans to save a fixed amount each month. She invests her savings at the beginning of each month in an account which pays interest at 6% pa compounded monthly.

(i) Let the amount she saves each month be \$$A$ and let $V_n$ be the value of her investment after $n$ months. Show that the value of her investment at the end of 3 months is given by

$$V_3 = A\left(1 \cdot 005 + 1 \cdot 005^2 + 1 \cdot 005^3\right).$$ **1**

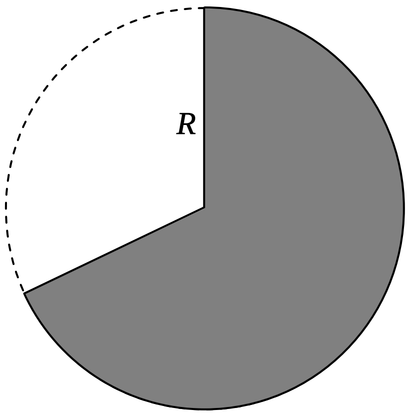
(ii) Find correct to the nearest dollar, the least amount of money that Alison would need to save each month to reach her target. **3**

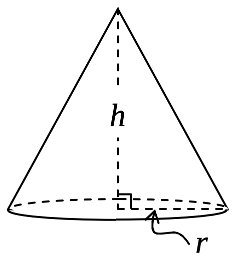
(iii) If, after 2 years of her saving plan, the interest rate rose to 9% pa, how much extra spending money would Alison have if she maintained the amount she was saving as calculated in part (ii) above? **3**

**Question 10** (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the exact value of $a$ if $\int_2^{\sqrt{a}} \frac{x}{x^2 - 1}\,dx = 3$. **3**

(b) From a circular disc of metal whose area is 100 m$^2$, a sector is cut out and used to make a right cone. The radius of the disc is $R$ metres.




(i) Show that the height of the cone is given by $h = \sqrt{\frac{100}{\pi} - r^2}$. **2**

(ii) Show that the volume of the cone is given by

$$V = \frac{r^2\sqrt{100\pi - \pi^2 r^2}}{3}.$$ **1**

(iii) Show that the volume is maximised when $r = \sqrt{\frac{200}{3\pi}}$. **4**

(iv) Given the area of the sector used to make the cone is $A = \pi Rr$ show that the angle in this sector, which gives a maximum volume for the cone, is **2**

$$\theta = \frac{2\pi\sqrt{6}}{3} \text{ radians.}$$

**End of paper**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1:

a) $\dfrac{e^2 - 1.2^3}{\sqrt{5.8+2.7}} = 1.94172\ldots$

$\doteqdot 1.9$ (2 sig fig)

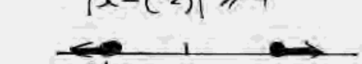
b) $\dfrac{d}{dx}\left(x^3 - \dfrac{4}{x}\right) = \dfrac{d}{dx}\left(x^3 - 4x^{-1}\right)$

$= 3x^2 + 4x^{-2}$

$= 3x^2 + \dfrac{4}{x^2}$

c) $|x+2| \geqslant 4$

$|x-(-2)| \geqslant 4$




$\therefore x \leq -6$ or $x \geqslant 2$

d) $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + \ldots$

$a = \frac{2}{3}$ $\quad \therefore S = \dfrac{a}{1-r}$

$r = \frac{3}{4}$

$= \dfrac{\frac{2}{3}}{1-\frac{3}{4}}$

$= 2\frac{2}{3}$

$\therefore$ the limiting sum is $2\frac{2}{3}$

e) $\dfrac{4}{\sqrt{3}-1} = \dfrac{4(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$

$= \dfrac{4(\sqrt{3}+1)}{3-1}$

$= \dfrac{2\cancel{4}(\sqrt{3}+1)}{\cancel{2}}$

$= 2\sqrt{3}+2$

$= \sqrt{12}+2$

$= a + \sqrt{b}$

$\therefore a = 2, b = 12$

f) $x^2 = 15^2 + 23^2 - 2(15)(23)\cos 42^\circ$

$= 241.230\ldots$

$x = \sqrt{241.230\ldots}$

$= 15.5315\ldots$

$\doteqdot 15.5$ (1 dp)

Question 2:

a) $y = \ln(x-2)$

$\dfrac{dy}{dx} = \dfrac{1}{x-2}$

at $x = 3$ : $\dfrac{dy}{dx} = \dfrac{1}{3-2}$

$= 1$

and $\quad y = \ln(3-2)$

$= \ln 1$

$= 0$

$\therefore$ point is $(3, 0)$; $m = 1$

$\therefore$ tangent is

$y - 0 = 1(x-3)$

$y = x - 3$

b) i) $\dfrac{d}{dx}\left(xe^{4x}\right) = x\left(4e^{4x}\right) + e^{4x}(1)$

$= e^{4x}(4x+1)$

ii) $\dfrac{d}{dx}(\sin 3x + \tan x) = 3\cos 3x + \sec^2 x$

c) $\int \sec^2\left(\frac{x}{2}\right) dx = 2\int \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$

$= 2\tan\left(\frac{x}{2}\right) + c$

d) $\displaystyle\int_0^2 \left(1 - \frac{2}{e^x}\right) dx = \int_0^2 \left(1 - 2e^{-x}\right) dx$

$= \left[x + 2e^{-x}\right]_0^2$

$= 2 + 2e^{-2} - (0 + 2e^0)$

$= 2 + 2e^{-2} - 2$

$= 2e^{-2}$

e) $\angle EBC = \angle ECB$ (opposite equal sides in $\triangle EBC$)

$\therefore 70^\circ + 2\angle EBC = 180^\circ$ ($\angle$ sum of $\triangle EBC$)

$\therefore \angle EBC = 55^\circ$

$\angle CAB = 30^\circ$ (alternate, $DC \parallel AB$)

$\therefore 30^\circ + \angle ACB = 55^\circ$ (exterior $\angle$ of $\triangle EBC$)

$\therefore \angle ACB = 25^\circ$

Question 3:

a) $1 - 4x - 2x^2 = 0$

$a = -2$

$b = -4$

$c = 1$

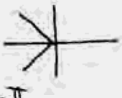
i) $\alpha + \beta = \frac{-b}{a}$
$= \frac{4}{-2}$
$= -2$

ii) $\alpha\beta = \frac{c}{a}$
$= \frac{1}{-2}$
$= -\frac{1}{2}$

iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
$= (-2)^2 - 2(-\frac{1}{2})$
$= 4 + 1$
$= 5$

b) $2\cos x + 1 = 0 \quad ; \; 0 \le x \le 2\pi$

$\cos x = -\frac{1}{2}$



$x = \pi - \frac{\pi}{3}, \pi + \frac{\pi}{3}$
$= \frac{2\pi}{3}, \frac{4\pi}{3}$

c) i) $m_{AB} = \frac{2-(-1)}{-1-3}$
$= \frac{3}{-4}$
$= -\frac{3}{4}$ point is $(-1, 2)$

$\therefore$ AB has equation

$y - 2 = -\frac{3}{4}(x - (-1))$

$4y - 8 = -3x - 3$

$3x + 4y - 5 = 0$ as required

ii) $\tan\theta = m$
$= -\frac{3}{4}$
$\therefore \theta = 180^\circ - 36.869...^\circ$
$= 143^\circ 8'$ (nearest min)

iii) $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ $\quad A = 3$, $B = 4$, $C = -5$, $x_1 = 4$, $y_1 = 2$

$= \frac{|3(4) + 4(2) - 5|}{\sqrt{3^2 + 4^2}}$

$= \frac{15}{5}$
$= 3$

iv) $r = 3$ ; centre $= (4, 2)$

$\therefore$ equation is
$(x-4)^2 + (y-2)^2 = 9$

Question 4:

a) $y = 4 - x^2$ and $y = 2x + 1$

i) $\therefore 4 - x^2 = 2x + 1$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, 1$

$\therefore$ At A, $x = 1$ ; at B, $x = -3$

ii) $A = \int_{-3}^{1} [(4 - x^2) - (2x+1)]\, dx$

$= \int_{-3}^{1} (3 - x^2 - 2x)\, dx$

$\therefore A = \left[3x - \frac{x^3}{3} - x^2\right]_{-3}^{1}$

$= 3 - \frac{1}{3} - 1 - \left(-9 + \frac{27}{3} - 9\right)$

$= 1\frac{2}{3} + 9$

$= 10\frac{2}{3}$

$\therefore$ the area is $10\frac{2}{3}$ unit$^2$

b) $y = x^2 + 6x + 10$

i) Axis of symmetry is $x = -\frac{b}{2a}$

ie $x = -\frac{6}{2}$

$x = -3$

ii) sub $x = -3$ into $y$

$\therefore$ $y = (-3)^2 + 6(-3) + 10$

$= 1$

$\therefore$ vertex is $(-3, 1)$

iii) $y = x^2 + 6x + 10$

$y - 10 + 9 = x^2 + 6x + 9$

$y - 1 = (x+3)^2$

(note that this confirms the vertex as $(-3,1)$)

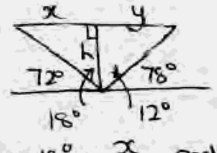
now comparing to $x^2 = 4ay$

$4a = 1$

$a = \frac{1}{4}$

$\therefore$ the focal length is $\frac{1}{4}$

c)




$\tan 18° = \frac{x}{h}$ and $\tan 12° = \frac{y}{h}$

$\therefore$ $x = h\tan 18°$ $\quad y = h\tan 12°$

but $x + y = 1522$

$\therefore$ $h\tan 18° + h\tan 12° = 1522$

$h(\tan 18° + \tan 12°) = 1522$

$\therefore$ $h = \frac{1522}{\tan 18° + \tan 12°}$ as required

Question 5:

a) i) $f(x) = 7 + 4x^3 - 3x^4$

$f'(x) = 12x^2 - 12x^3$

$f''(x) = 24x - 36x^2$

at stat. pts, $f'(x) = 0$

$\therefore 12x^2 - 12x^3 = 0$

$12x^2(1 - x) = 0$

$x = 0, 1$

at $x = 0$: $f(0) = 7$ $\quad$ ie $(0,7)$

$f''(0) = 0$

$\therefore$ possible inflexion

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $f''(x)$ | $-$ | $0$ | $+$ |

$f''(x) = 12x(2 - 3x)$

ie concavity changes

$\therefore (0,7)$ is a point of inflexion

at $x = 1$: $f(1) = 7 + 4 - 3$

$= 8$ $\quad$ ie $(1, 8)$

$f''(1) = 24 - 36$

$< 0$

$\therefore (1,8)$ is a local maximum.

ii) Inflexions occur if $f''(x) = 0$ and the concavity changes

$\therefore 24x - 36x^2 = 0$

$12x(2 - 3x) = 0$

$x = 0, \frac{2}{3}$

at $x = 0$ we have already shown that an inflexion occurs.

at $x = \frac{2}{3}$

| $x$ | $\frac{2}{3}^-$ | $\frac{2}{3}$ | $\frac{2}{3}^+$ |
|---|---|---|---|
| $f''(x)$ | $+$ | $0$ | $-$ |

$\therefore$ concavity changes

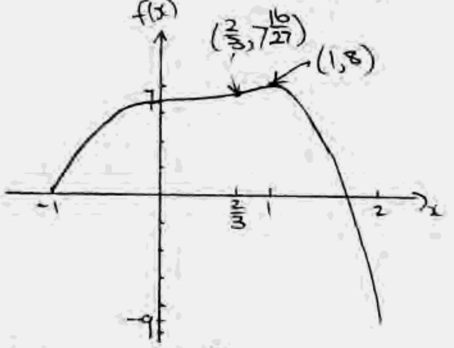
$f\left(\frac{2}{3}\right) = 7 + 4\left(\frac{2}{3}\right)^3 - 3\left(\frac{2}{3}\right)^4$

$= 7\frac{16}{27}$

$\therefore (0,7)$ and $\left(\frac{2}{3}, 7\frac{16}{27}\right)$ are inflexions

iii) $f(-1) = 7 - 4 - 3$
$= 0 \quad \therefore (-1, 0)$

$f(2) = 7 + 32 - 48$
$= -9 \quad \therefore (2, -9)$




iv) min value $= -9$

b) i) In $\triangle CBE$ and $\triangle AFB$
i) $\angle CBE = \angle BFA$
(alternate, $BC \parallel AF$)
ii) $\angle BEC = \angle ABF$
(alternate, $AB \parallel DC$)
$\therefore \triangle CBE \parallel\!\mid \triangle AFB$ (equiangular)

ii) $\frac{CB}{AF} = \frac{BE}{FB} = \frac{CE}{AB}$
(corresponding sds of similar $\triangle$'s)
$\therefore \quad \frac{7}{15} = \frac{3}{AB}$
$AB = \frac{45}{7}$
$AB = 6\frac{3}{7}$

Question 6:

a) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2x \, dx$

$= \left[ \frac{1}{2} \sin 2x \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

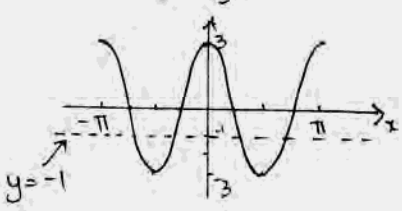
$= \frac{1}{2} \sin \frac{\pi}{2} - \frac{1}{2} \sin \frac{\pi}{3}$

$= \frac{1}{2}(1) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} - \frac{\sqrt{3}}{4}$

$= \frac{2 - \sqrt{3}}{4}$

b) $a = 3$
period $= \frac{2\pi}{2}$
$= \pi$




There are 4 points of intersection
$\therefore$ there are 4 solutions.

c) $30° = \frac{\pi}{6}$

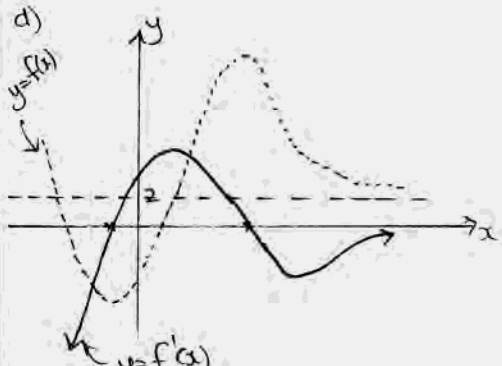
arc $AD = 8\left(\frac{\pi}{6}\right)$
$= \frac{4\pi}{3}$

arc $BC = 10\left(\frac{\pi}{6}\right)$
$= \frac{5\pi}{3}$

$P = \frac{4\pi}{3} + \frac{5\pi}{3} + 2(2)$
$= 4 + \pi$

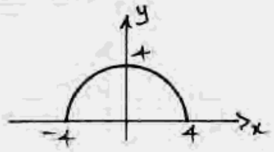
$\therefore$ Perimeter is $(4 + \pi)$ m

d)



<u>Question 7:</u>

a) i)




D: $-4 \le x \le 4$

ii)

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $f(x)$ | 4 | $\sqrt{12}$ | 0 |

$f(x) = \sqrt{16 - x^2}$

$A \doteqdot \frac{4-0}{6}\left[4 + 4\sqrt{12} + 0\right]$

$= \frac{2}{3}\left[4 + 8\sqrt{3}\right]$

$\therefore$ Area is about $\frac{2}{3}(4 + 8\sqrt{3})$ unit$^2$

iii) $A = \frac{1}{4}\pi r^2$ where $r = 4$

$= \frac{1}{4}\pi(4)^2$

$= 4\pi$

$\therefore$ Area is $4\pi$ unit$^2$

iv) % error $= \dfrac{4\pi - \frac{2}{3}(4+8\sqrt{3})}{4\pi} \times 100\%$

$= 5.268\ldots\%$

$\doteqdot 5.3\%$ 2 sig fig.

b) Sequence of rows

$= 279, 277, 277, 275, 275, \ldots, 5, 5, 3$

$= (279, 277, 275, \ldots, 5), (277, 275, \ldots, 3)$

i) Note that each sequence has $d = -2$ and, an equivalent number of terms.

Consider $279, 277, 275, \ldots, 5$

$d = -2$ $\quad T_n = a + (n-1)d$

$n = ?$ $\quad 5 = 279 + (n-1)(-2)$

$a = 279$ $\quad 5 = 279 - 2n + 2$

$T_n = 5$ $\quad 2n = 276$

$n = 138$

$\therefore$ Total no of rows $= 2 \times 138$

$= 276$

ii) Yarn used

$= 279 + 277 + 275 + \ldots + 5$

$+ 277 + 275 + \ldots + 3 + 280$

$= \frac{138}{2}\left[2(279) + 137(-2)\right]$

$+ \frac{138}{2}\left[2(277) + 137(-2)\right] + 280$

$= 39196$

Yarn available $= 2 \times 21000$

$= 42000$

$\therefore$ there is enough yarn as yarn used < yarn available.

<u>Question 8:</u>

a) $e^{2x} - 3e^x - 4 = 0$

$(e^x - 4)(e^x + 1) = 0$

$\therefore e^x = 4$ or $e^x = -1$

but $e^x > 0$ for all $x$

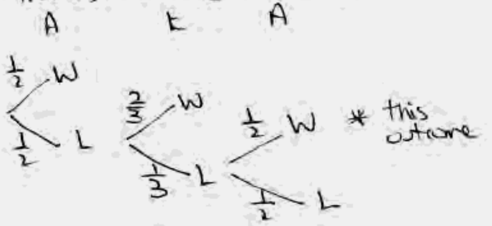
$\therefore e^x = 4$ is the only possibility

$\therefore x = \ln 4$.

b) i) Anna wins with a 2, 4 or 6

$\therefore P(\text{Anna wins}) = \frac{1}{2}$

ii) For Anna to win on the second throw, $P(\text{Kathryn wins}) = \frac{2}{3}$




$\therefore P(\text{Anna wins on second throw})$

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2}$

$= \frac{1}{12}$

iii) P(Anna wins on her 1st, 2nd or 3rd throw)

$= \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{2}$

$= \frac{43}{72}$

iv) P(Anna wins)

= P(wins 1st throw) + P(wins 2nd) + P(wins 3rd) + ...

$= \frac{1}{2} + \frac{1}{12} + \frac{1}{72} + \ldots$

$= \frac{a}{1-r}$ where $a = \frac{1}{2}$, $r = \frac{1}{6}$

$= \frac{\frac{1}{2}}{1-\frac{1}{6}}$

$= \frac{3}{5}$

c) $y = \frac{2}{x} - 1$

$y + 1 = \frac{2}{x}$

$x = \frac{2}{y+1}$

$V = \pi \int_0^3 x^2 \, dy$

$= \pi \int_0^3 \left(\frac{2}{y+1}\right)^2 dy$

$= 4\pi \int_0^3 (y+1)^{-2} \, dy$

$= 4\pi \left[-(y+1)^{-1}\right]_0^3$

$= 4\pi \left[-(4)^{-1} - (-1)^{-1}\right]$

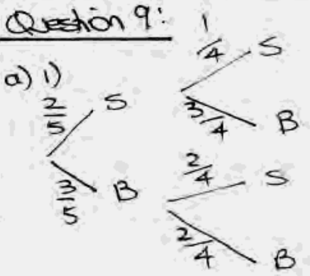
$= 4\pi \left[-\frac{1}{4} + 1\right]$

$= 3\pi$

$\therefore$ Volume is $3\pi$ unit$^3$.

Question 9:

a) i)




ii) α) $P(BB) = \frac{3}{5} \times \frac{2}{4}$

$= \frac{3}{10}$

β) P(exactly 1 B)

$= \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4}$

$= \frac{3}{10} + \frac{3}{10}$

$= \frac{3}{5}$

iii) P(at least 1 spaghetti)

$= 1 - P(\text{both beans})$

$= 1 - \frac{3}{5} \times \frac{2}{4}$

$= \frac{7}{10}$

$\therefore P(SB) = \frac{2}{5} \times \frac{3}{4} + \frac{3}{5} \times \frac{2}{4}$

$= \frac{3}{5}$

$\therefore P(\text{other is Beans}) = \frac{\frac{3}{5}}{\frac{7}{10}}$

$= \frac{6}{7}$

b) i) 6% pa = 0.5% per month

$V_1 = A(1.005)$

$V_2 = V_1(1.005) + A(1.005)$

$= A(1.005)^2 + A(1.005)$

$V_3 = V_2(1.005) + A(1.005)$

$= A(1.005)^3 + A(1.005)^2 + A(1.005)$

$= A(1.005 + 1.005^2 + 1.005^3)$

as required.

ii) $V_{36} = A(1.005 + 1.005^2 + 1.005^3 + \ldots + 1.005^{36})$

$= A\left[\frac{a(r^n-1)}{r-1}\right]$ where $a = 1.005$, $r = 1.005$, $n = 36$

but $V_{36} > 8000$

$\therefore A\left[\frac{1.005(1.005^{36}-1)}{1.005-1}\right] > 8000$

$A > \frac{8000(0.005)}{1.005(1.005^{36}-1)}$

$A > 202.363\ldots$

$\therefore$ she should invest at least \$203

iii) $V_{24} = A(1.005 + 1.005^2 + 1.005^3 + \ldots + 1.005^{24})$

using the new rate...

$V_{25} = V_{24}(1.0075) + A(1.0075)$

$V_{26} = V_{25}(1.0075) + A(1.0075)$

$= V_{24}(1.0075)^2 + A(1.0075)^2 + A(1.0075)$

$\therefore V_{36} = V_{24}(1.0075)^{12} + A(1.0075 + 1.0075^2 + \ldots + 1.0075^{12})$

$= \frac{A.\,1.005(1.005^{24}-1)(1.0075)^{12}}{0.005} + \frac{A.\,1.0075(1.0075^{12}-1)}{0.0075}$

using the pattern established in (ii).

Question 10:

a) $\int_2^{\sqrt{a}} \frac{x}{x^2-1}\,dx = 3$

$\therefore \frac{1}{2}\int_2^{\sqrt{a}} \frac{2x}{x^2-1}\,dx = 3$

$\frac{1}{2}\left[\ln(x^2-1)\right]_2^{\sqrt{a}} = 3$

$\frac{1}{2}\left[\ln(a-1) - \ln 3\right] = 3$

$\ln(a-1) - \ln 3 = 6$

$\ln\left(\frac{a-1}{3}\right) = 6$

$\frac{a-1}{3} = e^6$

$a - 1 = 3e^6$

$a = 3e^6 + 1$

(Part b) continues over leaf)

$\therefore$ Added value

$= A\left[\frac{1.005(1.005^{24}-1)(1.0075)^{12}}{0.005} + \frac{1.0075(1.0075^{12}-1)}{0.0075} - \frac{1.005(1.005^{36}-1)}{0.005}\right]$

using $A = 203$

$= 208.144$

$\therefore$ she has about \$208 extra.

b) i) $R^2 = h^2 + r^2$ by Pythagoras' Thm

$\therefore h^2 = R^2 - r^2$

but $\pi R^2 = 100$

$\therefore R^2 = \frac{100}{\pi}$

$\therefore h^2 = \frac{100}{\pi} - r^2$

$h = \pm\sqrt{\frac{100}{\pi} - r^2}$ but $h > 0$

$\therefore h = \sqrt{\frac{100}{\pi} - r^2}$ as required

ii) $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi r^2 \sqrt{\frac{100}{\pi} - r^2}$

$= \frac{r^2}{3}\sqrt{\pi^2\left(\frac{100}{\pi} - r^2\right)}$

$\therefore V = \dfrac{r^2\sqrt{100\pi - r^2\pi^2}}{3}$ as required.

iii) $V = \frac{r^2}{3} \cdot (100\pi - r^2\pi^2)^{\frac{1}{2}}$

$\therefore \frac{dV}{dr} = \frac{r^2}{3} \cdot \frac{1}{2}(100\pi - r^2\pi^2)^{-\frac{1}{2}}(-2r\pi^2)$

$+ (100\pi - r^2\pi^2)^{\frac{1}{2}} \cdot \frac{2r}{3}$

$= \frac{r}{3}(100\pi - r^2\pi^2)^{-\frac{1}{2}}\left[-r^2\pi^2 + 200\pi - 2r^2\pi^2\right]$

$= \dfrac{r(200\pi - 3\pi^2 r^2)}{3\sqrt{100\pi - r^2\pi^2}}$

for a max/min

$\frac{dV}{dr} = 0 \therefore r = 0$ or $200\pi - 3\pi^2 r^2 = 0$

ie $r = 0$ or $r^2 = \frac{200}{3\pi}$ but $r > 0$

$\therefore r = \sqrt{\frac{200}{3\pi}}$

| $r$ | $\sqrt{\frac{200}{3\pi}}^-$ | $\sqrt{\frac{200}{3\pi}}$ | $\sqrt{\frac{200}{3\pi}}^+$ |
|---|---|---|---|
| $\frac{dV}{dr}$ | $+$ | $0$ | $-$ |

$\therefore$ at $r = \sqrt{\frac{200}{3\pi}}$ a max. occurs

iv) $A = \pi R r$

but $A = \frac{1}{2}R^2\theta$

$\therefore \frac{1}{2}R^2\theta = \pi R r$

$\theta = \frac{2\pi R r}{R^2}$

$= \frac{2\pi r}{R}$

but $R = \sqrt{\frac{100}{\pi}}$ as $R > 0$

$\therefore \theta = 2\pi \cdot \sqrt{\frac{200}{3\pi}} \cdot \sqrt{\frac{\pi}{100}}$

$= 2\pi\sqrt{\frac{2}{3}}$

$= \frac{2\pi\sqrt{6}}{3}$ as required.

~ the end ~