NORTH SYDNEY GIRLS HIGH SCHOOL

2011
TRIAL HSC EXAMINATION

Mathematics

General Instructions
- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120
Attempt Questions 1–10
All questions are of equal value

At the end of the examination, place your writing booklets in order and put this question paper on top. Submit one bundle. The bundle will be separated before marking commences so that anonymity will be maintained.

Student Number:____________________________ **Teacher:**________________________

Student Name:____________________________

QUESTION	MARK
1	/12
2	/12
3	/12
4	/12
5	/12
6	/12
7	/12
8	/12
9	/12
10	/12
TOTAL	/120

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Total Marks – 120
Attempt Questions 1–10
All questions are of equal value

Begin each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Write down the exact value of $\cos\frac{7\pi}{6}$. **2**

(b) Solve the inequality $|3x+2|<9$. **2**

(c) Write $\frac{4}{1-\sqrt{3}}$ in the form $a+\sqrt{b}$. **2**

(d) Solve simultaneously **3**

$$y = x^2$$
$$y = 2 - x$$

(e) Write $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$ as a fully simplified fraction. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate

(i) $x^2 e^{4x}$ **2**

(ii) $(\ln x + 1)^3$. **2**

(b) Find primitives for each of the following:

(i) $\sqrt{1-4x}$ **2**

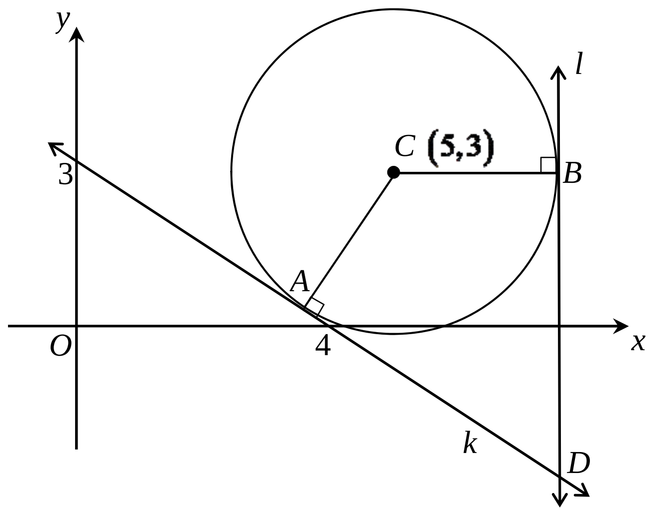
(ii) $e^{\frac{3x}{2}}$. **1**

(c) Evaluate $\int_0^1 \frac{x}{x^2+1}\,dx$. **2**

(d) Find the equation of the tangent to the curve $y = (1-2x)^5$ at the point where $x = 1$. **3**

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

The diagram shows the line k with x- and y-intercepts of 4 and 3 respectively.
The line l is vertical.
The two lines are tangent to a circle with centre $C(5,3)$, and they meet at the point D.
(Note that a tangent is perpendicular to the radius at the point of contact, as shown in the diagram.)

(i) Show that the equation of k is $3x+4y=12$. **2**

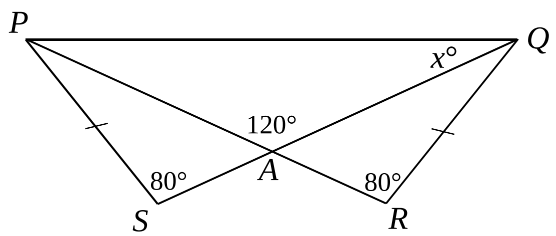
(ii) Use the perpendicular distance formula to find the radius of the circle. **2**

(iii) Hence find the equation of l. **1**

(iv) Find the coordinates of D. **1**

(v) Calculate the area of triangle BCD. **2**

(b)



PR and QS are straight lines intersecting at a point A. Also $PS = QR$, $\angle PAQ = 120°$, $\angle PSA = \angle QRA = 80°$ and $\angle PQA = x°$.

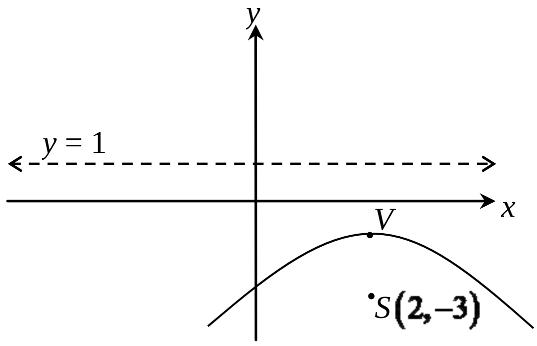
(i) Copy the diagram into your writing booklet.

(ii) Prove that ΔPSA is congruent to ΔQRA. **2**

(iii) Hence find the value of x giving reasons. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

The diagram shows a parabola with focus $S(2,-3)$, directrix $y = 1$, and vertex V.

(i) Write down the coordinates of V. **1**

(ii) Write down the focal length of the parabola. **1**

(iii) Find the equation of the parabola. **1**

(b) The equation $2x^2 - 3x + 5 = 0$ has roots α and β. Find the value of:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

(iii) $(\alpha - \beta)^2$. **2**

(c) A single digit from the digits 1 to 9 is written on each of nine cards, so that each digit is used only once.

Huey holds the cards 1 and 2, Dewey holds 3, 4 and 5, while Louie holds 6, 7, 8 and 9.

A card is chosen by randomly choosing one of Huey, Dewey or Louie and then randomly choosing one of that person's cards.

(i) What is the probability that the 9 card is chosen? **2**

(ii) A two-digit number is to be formed by choosing first the tens digit, and then the units digit. **2**
What is the probability that this number is 92?

(iii) What is the probability that Huey will have no cards left after forming the two-digit number ? **1**

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Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Use Simpson's rule with five function values to approximate $\int_1^5 f(x)\,dx$ **2**

given that

x	1	2	3	4	5
$f(x)$	2	5	7	4	1

Give your answer correct to 3 significant figures.

(b) Consider the function $y = 3\sin \pi x$.

(i) Write down the amplitude. **1**

(ii) Find the period. **1**

(c) Consider the function $f(x) = x^4 - 4x^3 + 1$.

(i) Find the coordinates of any stationary points on the curve $y = f(x)$, and determine their nature. **4**

(ii) Locate any points of inflexion. **2**

(iii) Sketch the curve, showing all the above features. **2**
Do not try to find the x-intercepts.

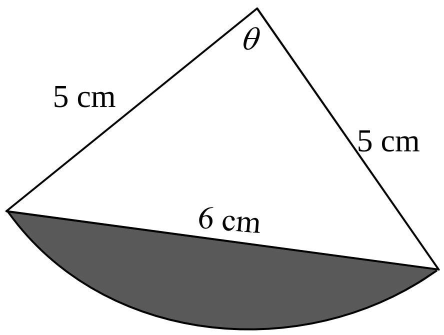
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) The first and thirteenth terms of an arithmetic progression are 7 and 1 respectively.

(i) Find the common difference. **2**

(ii) Find the number of terms required to give a sum of zero. **2**

(b)

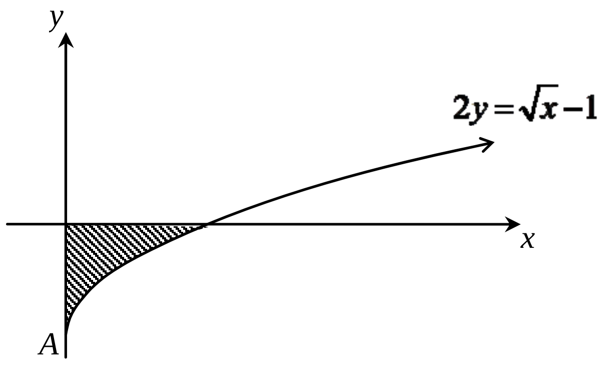


The diagram shows the sector of a circle of radius 5 cm.
A segment is cut off by a chord of length 6 cm.

(i) Calculate the value of θ to the nearest degree. **2**

(ii) Find the area of the shaded segment. **2**

(c)



In the diagram above the region bounded by the curve $2y = \sqrt{x} - 1$, the x-axis and the y-axis has been shaded.

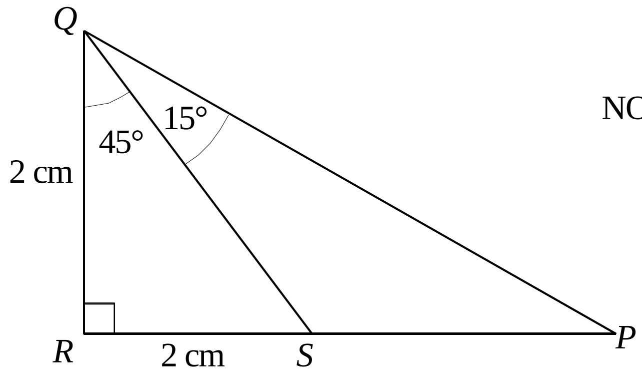
(i) Find the coordinates of A. **1**

(ii) Write an expression for x^2 in terms of y. **1**

(iii) Find the volume of the solid formed when this region is rotated about the y-axis. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT TO SCALE

In the above diagram, $QR = RS = 2$ cm, $\angle RQS = 45°$ and $\angle PQS = 15°$.

(i) Show that $QS = 2\sqrt{2}$ cm. **1**

(ii) Show that $PS = 2\left(\sqrt{3}-1\right)$ cm. **2**

(iii) Use the sine rule to show that $\sin 15° = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$. **3**

(b) The curves $y = \dfrac{1}{x-1}$ and $y = mx+1$ intersect at the points A and B.

(i) Show that the x-coordinates of A and B satisfy $mx^2 - (m-1)x - 2 = 0$. **3**

(ii) Hence find the values of m for which $y = mx+1$ is a tangent to the curve $y = \dfrac{1}{x-1}$. Leave your answers in surd form. **3**

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) On being retrenched from his job, Kevin receives a cash payment of \$20 000.

One year later, he receives his first annual payout of \$10 000. He continues to receive annual payouts of \$10 000 every year thereafter.

He places all of this money in his suitcase as he receives it, and spends none.

At the end of every year, just before the next payout, Kevin spends 20% of the money in his suitcase on a holiday.

Let A_n be the amount Kevin has in his suitcase immediately after his n^{th} annual payout.

(i) Show that Kevin has \$26 000 in his suitcase immediately after his first annual payout. **1**

(ii) Show that the money in Kevin's suitcase immediately after his 3rd annual payout is given by **2**

$$A_3 = 20\,000(0.8)^3 + 10\,000\left(1 + 0.8 + 0.8^2\right).$$

(iii) Show that $A_n = 50\,000 - 30\,000\left(0.8^n\right)$. **3**

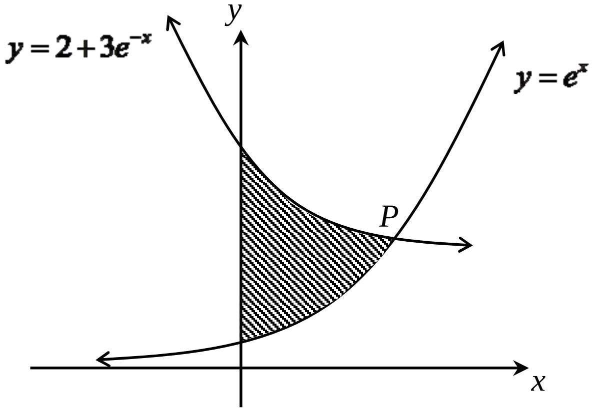
(iv) After how many years will the amount in Kevin's suitcase first exceed \$48 000? **2**

(v) What is the most money Kevin will ever have in his suitcase? **1**

(b) Solve the equation $2\log_e x = \log_e(x+6)$. **3**

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graphs of $y = e^x$ and $y = 2 + 3e^{-x}$ intersecting at the point P.

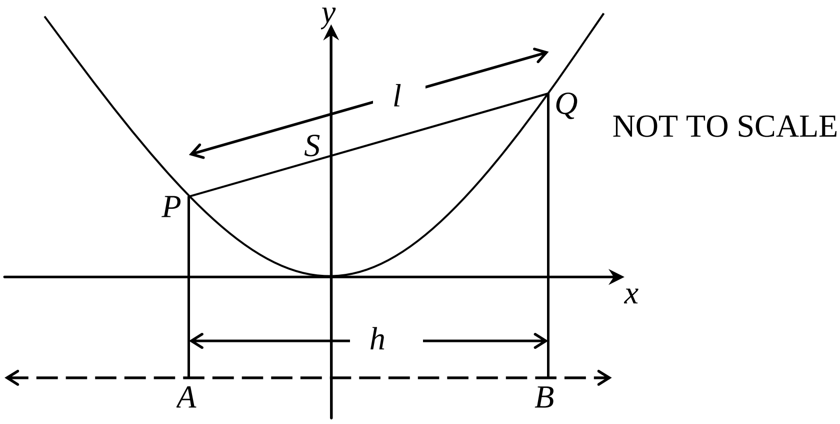
(i) Show that the curves intersect when **1**

$$e^{2x} - 2e^x - 3 = 0 \, .$$

(ii) Hence show that the x-coordinate of the point P is $\ln 3$. **2**

(iii) Hence find the exact area of the shaded region. **3**

(b)



The diagram shows a parabola with vertex at the origin, focus S and directrix AB. **2**

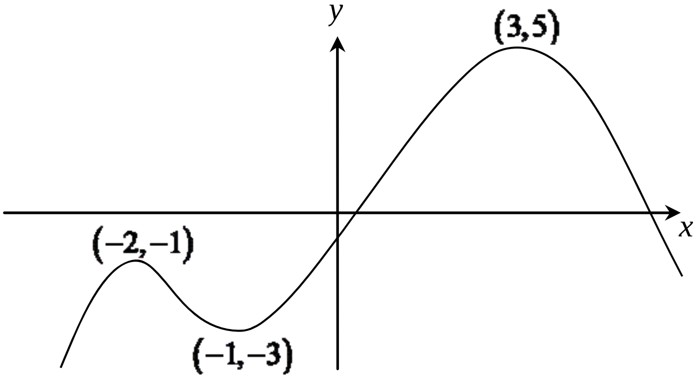
A trapezium has been formed by dropping perpendiculars from the ends of the focal chord PQ to the directrix. The focal chord has length l, and $AB = h$.

Use the locus definition of a parabola to explain why the area of this trapezium is given by $A = \frac{1}{2} lh$.

Question 9 continues on page 13

Question 9 (continued)

(c)



The diagram shows the graph of the function $y = f(x)$, with the coordinates of its turning points shown.

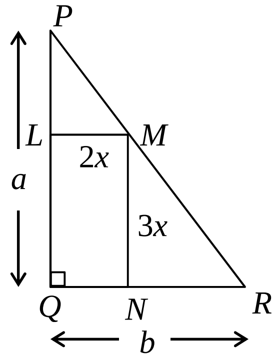
(i) On a number plane, sketch the graph of $y = f'(x)$ where $f'(x)$ is the derivative of $f(x)$. **2**

(ii) Find the area of the region bounded by $y = f'(x)$ and the x-axis. **2**
(Do not attempt to find the equation of either function.)

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet.

(a)



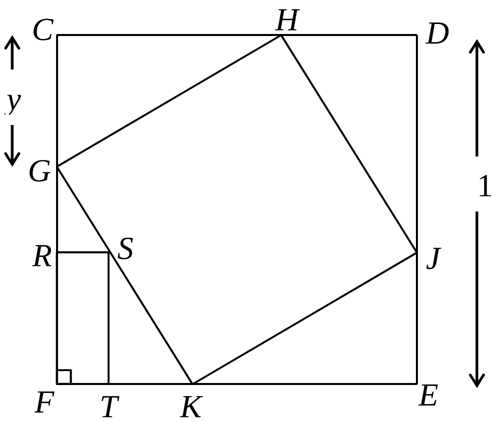
In the diagram, PQR is a right-angled triangle, with $PQ = a$ and $QR = b$. **3**

A rectangle $QLMN$ is inscribed inside the triangle as shown, where $LM = 2x$ and $MN = 3x$.

Given that the triangles PLM and PQR are similar (do not prove this), show that

$$x = \frac{ab}{2a+3b}.$$

(b)



The diagram shows a square $GHJK$ inscribed inside another square $CDEF$. $CDEF$ has a side length of 1 unit. The length of CG is y units.

A rectangle $FRST$ has been inscribed inside the triangle FGK such that $\frac{RS}{ST} = \frac{2}{3}$.

(i) Use part (a) to show that the area of rectangle $FRST$ is given by **3**

$$A = \frac{6y^2(1-y)^2}{(y+2)^2}.$$

(ii) Let $B = \ln A$. Show that $\frac{dB}{dy} = \frac{2(2-4y-y^2)}{y(1-y)(y+2)}$. **3**

(iii) Note that when A has its maximum value, B will also have its maximum value. Show that the maximum possible area of rectangle $FRST$ occurs when **3**

$$y = \sqrt{6} - 2.$$

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2011 Year 12 Mathematics Trial – Solutions

Question 1

(a)

$$\cos\frac{7\pi}{6} = \cos\left(\pi + \frac{\pi}{6}\right)$$
$$= -\cos\frac{\pi}{6}$$
$$= -\boldsymbol{\frac{\sqrt{3}}{2}}$$

(b)

$$|3x+2| < 9$$
$$-9 < 3x + 2 < 9$$
$$-11 < 3x < 7$$
$$-\boldsymbol{\frac{11}{3}} < x < \boldsymbol{\frac{7}{3}}$$

(c)

$$\frac{4}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{4(1+\sqrt{3})}{1-3}$$
$$= -2(1+\sqrt{3})$$
$$= -2 - 2\sqrt{3}$$
$$= \boldsymbol{-2 - \sqrt{12}}$$

(d)

$y = x^2$
$y = 2 - x$
Equating:

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2, 1$$
$$y = 4, 1$$

ie. $\boldsymbol{x = -2, y = 4}$ **OR** $\boldsymbol{x = 1, y = 1}$

(e)

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}} \times \frac{a^2b^2}{a^2b^2} = \frac{ab^2+a^2b}{b^2-a^2}$$
$$= \frac{ab(b+a)}{(b+a)(b-a)}$$
$$= \boldsymbol{\frac{ab}{b-a}}$$

Question 2

(a) (i)

$$\frac{d}{dx}(x^2e^{4x}) = e^{4x} \cdot 2x + x^2 \cdot 4e^{4x}$$
$$= \mathbf{2xe^{4x} + 4x^2e^{4x}}$$
$$= 2x(1+2x)e^{4x}$$

(ii)

$$\frac{d}{dx}(\ln x + 1)^3 = 3(\ln x + 1)^2 \cdot \frac{1}{x}$$
$$= \mathbf{\frac{3}{x}(\ln x + 1)^2}$$

(b) (i)

$$\int \sqrt{1-4x}\,dx = \int (1-4x)^{\frac{1}{2}}\,dx$$
$$= \frac{(1-4x)^{\frac{3}{2}}}{\frac{3}{2}\times -4} + c$$
$$= \mathbf{-\frac{(1-4x)^{\frac{3}{2}}}{6} + c}$$
$$= -\frac{1}{6}\sqrt{(1-4x)^3} + c$$

(ii)

$$\int e^{\frac{3x}{2}}\,dx = \frac{e^{\frac{3x}{2}}}{\frac{3}{2}} + c$$
$$= \mathbf{\frac{2}{3}e^{\frac{3x}{2}} + c}$$

(c)

$$\int_0^1 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$$
$$= \frac{1}{2}[\ln(x^2+1)]_0^1$$
$$= \frac{1}{2}(\ln 2 - \ln 1)$$
$$= \mathbf{\frac{1}{2}\ln 2}$$

(d)

$$\frac{d}{dx}(1-2x)^5 = 5(1-2x)^4 \times (-2)$$
$$= -10(1-2x)^4$$

when $x = 1$, $y = -1$ and $m_T = -10$

tangent:

$$y + 1 = -10(x-1)$$
$$y + 1 = -10x + 10$$
$$\mathbf{y = -10x + 9}$$

Question 3

(a) (i) $m_k = -\frac{3}{4}$, y-intercept = 3

k:

$$y = -\frac{3}{4}x + 3$$
$$4y = -3x + 12$$
$$\mathbf{3x + 4y = 12}$$

(ii) k: $3x + 4y - 12 = 0$ $\quad C(5, 3)$

$AC = \frac{|3(5)+4(3)-12|}{\sqrt{3^2+4^2}}$

$= \mathbf{3}$

(iii)

$x = 5 + 3$

$\boldsymbol{x = 8}$

(iv) sub $x = 8$ into equation of k:

$3(8) + 4y = 12$

$y = -12$

$\therefore \boldsymbol{D(8, -3)}$

(v) $BC = 3$, $BD = 3 + 3 = 6$

Area $= \frac{1}{2} \times 6 \times 3$

$= \mathbf{9\ units^2}$

(b) (ii) In Δ's PSA and QRA :

$\angle PSA = \angle QRA = 80°$	(given)
$\angle PAS = \angle QAR$	(vertically opposite)
$PS = QR$	(given)
$\therefore\ \boldsymbol{\Delta PSA \equiv \Delta QRA}$	**(AAS)**

(iii)

$PA = QA$	(corresponding sides of congruent triangles)
ie. ΔPAQ is isosceles	
$\therefore\ \angle APQ = \angle AQP = x$	(base angles of isosceles triangle)
$2x + 120 = 180$	(angle sum of triangle)
$\boldsymbol{x = 30}$	

Question 4

(a) (i) $\boldsymbol{V(2,-1)}$

(ii) $\boldsymbol{a = 2}$

(iii) $\boldsymbol{(x-2)^2 = -8(y+1)}$

(b) (i) $\boldsymbol{\alpha + \beta = \frac{3}{2}}$

(ii) $\boldsymbol{\alpha\beta = \frac{5}{2}}$

(iii)

$$\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta \\ &= \left(\frac{3}{2}\right)^2 - 4\left(\frac{5}{2}\right) \\ &= -\boldsymbol{\frac{31}{4}}\end{aligned}$$

(c) (i) $\frac{1}{3} \times \frac{1}{4} = \boldsymbol{\frac{1}{12}}$

(ii) $\frac{1}{12} \times \frac{1}{3} \times \frac{1}{2} = \boldsymbol{\frac{1}{72}}$

(iii) $\frac{1}{3} \times \frac{1}{3} = \boldsymbol{\frac{1}{9}}$

Question 5

(a) $h = 1$

$$\int_1^5 f(x)\,dx \approx \frac{1}{3}[2 + 1 + 4(5 + 4) + 2(7)]$$
$$= \frac{53}{3}$$
$$= \mathbf{17.7} \ \ (3\text{ s.f.})$$

(b) (i) amplitude = **3**

(ii) period = $\frac{2\pi}{\pi} = \mathbf{2}$

(c) (i)

$$f(x) = x^4 - 4x^3 + 1$$
$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$

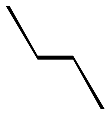
stat points:

$$f'(x) = 0$$
$$4x^3 - 12x^2 = 0$$
$$4x^2(x - 3) = 0$$
$$x = 0, \quad 3$$
$$y = 1, -26$$

$f''(3) = 36 > 0 \quad \Rightarrow$ **minimum turning point at** $(\mathbf{3}, \mathbf{-26})$

$f''(0) = 0$ (test inconclusive – revert to other test)

x	-1	0	1
$f'(x)$	-16	0	-8



∴ **horizontal point of inflexion at** $(\mathbf{0}, \mathbf{1})$

(ii) Possible Points of inflexion:

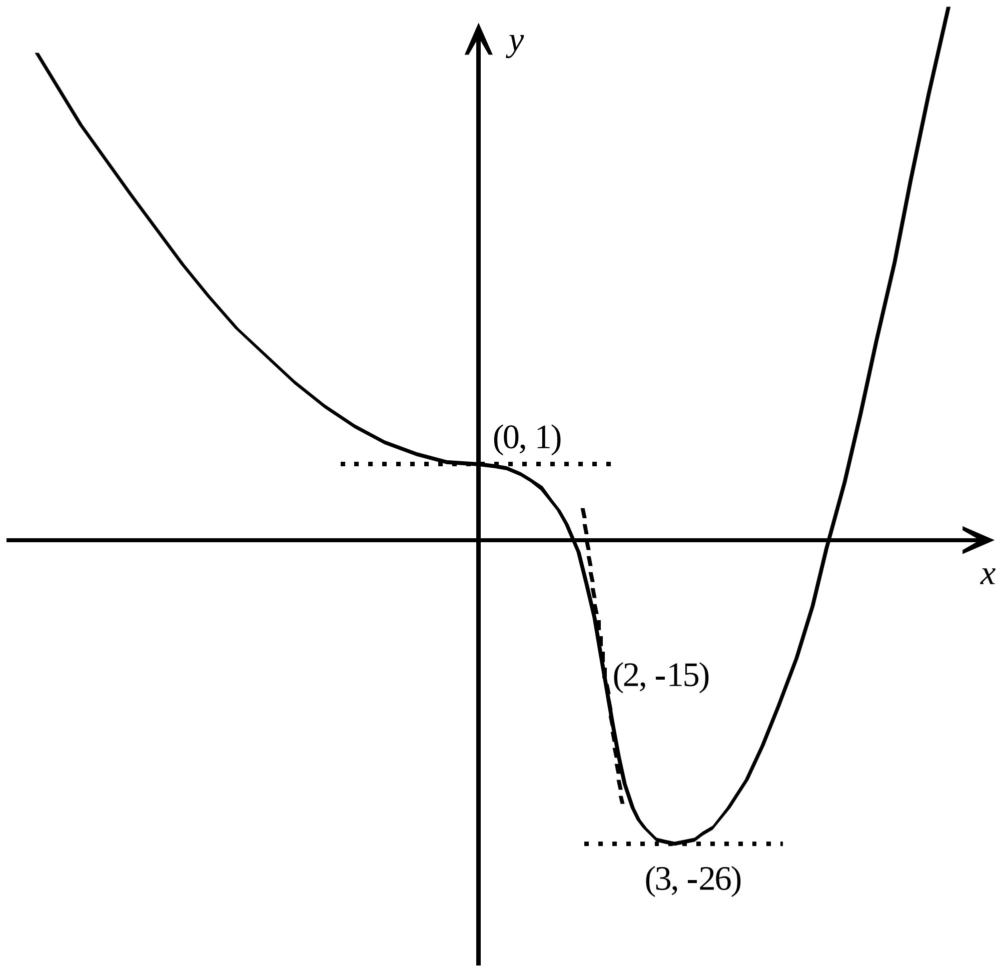
$$f''(x) = 0$$
$$12x^2 - 24x = 0$$
$$12x(x - 2) = 0$$
$$x = 0, \quad 2$$
$$y = 1, \ -15$$

Inflexion at $(0, 1)$ already found in part (i)
Test for inflexion at $= 2$:

x	1	2	3
$f''(x)$	-12	0	36

∴ change in concavity
∴ **points of inflexion at** $(\mathbf{0}, \mathbf{1})$ **and** $(\mathbf{2}, \mathbf{-15})$

(iii)



Question 6

(a) (i)

$$T_n = a + (n-1)d$$
$$1 = 7 + (13-1)d$$
$$12d = -6$$
$$\boldsymbol{d = -\frac{1}{2}}$$

(ii)

$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$0 = \frac{n}{2}\left[14 + (n-1)\left(-\frac{1}{2}\right)\right]$$
$$0 = \frac{n}{4}[28 - (n-1)]$$
$$0 = \frac{n}{4}(29 - n)$$

$n = 0$ (trivial solution) or $n = 29$

Hence, 29 terms are required.

(b) (i)

$$6^2 = 5^2 + 5^2 - 2(5)(5)\cos\theta$$
$$36 = 50 - 50\cos\theta$$
$$50\cos\theta = 14$$
$$\cos\theta = \frac{7}{25}$$
$$\boldsymbol{\theta = 74°} \quad \text{(taking 1st quadrant answer only)}$$

(ii) $\theta = 1.2915$ radians

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$
$$= \frac{1}{2} \times 5^2 \times (1.2915 - \sin 1.2915)$$
$$= \mathbf{4.13\ cm^2} \quad (2\ \text{d.p.})$$

(c) (i) $x = 0 \quad \rightarrow \quad y = -\frac{1}{2} \quad \rightarrow \boldsymbol{A\left(0, -\frac{1}{2}\right)}$

(ii)

$$2y = \sqrt{x} - 1$$
$$\sqrt{x} = (2y+1)$$
$$x = (2y+1)^2$$
$$\boldsymbol{x^2 = (2y+1)^4}$$

(iii)

$$V = \pi\int_{-\frac{1}{2}}^{0}(2y+1)^4\,dx$$
$$= \pi\left[\frac{(2y+1)^5}{5 \times 2}\right]_{-\frac{1}{2}}^{0}$$
$$= \frac{\pi}{10}(1-0)$$
$$= \boldsymbol{\frac{\pi}{10}}\ \mathbf{units^3}$$

Question 7

(a) (i)

$$QS^2 = 2^2 + 2^2$$
$$= 8$$
$$\boldsymbol{QS = 2\sqrt{2}}\ \textbf{cm}$$

(ii)

$$\frac{PR}{QR} = \tan 60°$$
$$\frac{PR}{2} = \sqrt{3}$$
$$PR = 2\sqrt{3}$$
$$\therefore PS = PR - RS$$
$$= 2\sqrt{3} - 2$$
$$= \boldsymbol{2\left(\sqrt{3} - 1\right)}\ \textbf{cm}$$

(iii) $\angle QPS = 180 - 60 - 90 = 30°$ (angle sum of ΔQPR)

$$\frac{\sin 15°}{2(\sqrt{3}-1)} = \frac{\sin 30°}{2\sqrt{2}}$$
$$\sin 15° = \cancel{2}\left(\sqrt{3} - 1\right) \times \frac{1}{\cancel{2}} \times \frac{1}{2\sqrt{2}}$$
$$= \boldsymbol{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

(b) (i)

$$\frac{1}{x-1} = mx + 1$$
$$1 = (x - 1)(mx + 1)$$
$$1 = mx^2 + x - mx - 1$$
$$\boldsymbol{mx^2 - (m - 1)x - 2 = 0}$$

(ii) If the line is a tangent to the curve, the above equation has only one solution:

$$\Delta = 0$$
$$(m - 1)^2 + 8m = 0$$
$$m^2 - 2m + 1 + 8m = 0$$
$$m^2 + 6m + 1 = 0$$
$$m = \frac{-6 \pm \sqrt{6^2 - 4(1)(1)}}{2(1)}$$
$$m = \frac{-6 \pm \sqrt{32}}{2}$$
$$\boldsymbol{m = -3 \pm 2\sqrt{2}}$$

Question 8

(a) (i) $A_1 = 20\,000 \times 0.8 + 10\,000 = \$\mathbf{26\,000}$

(ii)

$$\begin{aligned}A_2 &= A_1 \times 0.8 + 10\,000\\ &= [20\,000(0.8) + 10\,000] \times 0.8 + 10\,000\\ &= 20\,000(0.8)^2 + 10\,000(0.8) + 10\,000\\ A_3 &= A_2 \times 0.8 + 10\,000\\ &= [20\,000(0.8)^2 + 10\,000(0.8) + 10\,000] \times 0.8 + 10\,000\\ &= 20\,000(0.8)^3 + 10\,000(0.8)^2 + 10\,000(0.8) + 10\,000\\ \boldsymbol{A_3} &= \mathbf{20\,000(0.8)^3 + 10\,000(1 + 0.8 + 0.8^2)}\end{aligned}$$

(iii)

$$\begin{aligned}A_n &= 20\,000(0.8)^n + 10\,000(1 + 0.8 + 0.8^2 + \cdots + 0.8^{n-1})\\ &= 20\,000(0.8)^n + 10\,000\left(\frac{1-0.8^n}{1-0.8}\right)\\ &= 20\,000(0.8)^n + 50\,000(1 - 0.8^n)\\ &= 20\,000(0.8)^n + 50\,000 - 50\,000(0.8)^n\\ &= \mathbf{50\,000 - 30\,000(0.8)^n}\end{aligned}$$

(iv)

$$\begin{aligned}A_n &> 50\,000\\ 50\,000 - 30\,000(0.8)^n &> 48\,000\\ 30\,000(0.8)^n &< 2\,000\\ (0.8)^n &< \frac{1}{15}\\ \log(0.8)^n &< \log\left(\frac{1}{15}\right)\\ n\log(0.8) &< -\log 15\\ n &> -\frac{\log 15}{\log(0.8)} \quad \text{(switching inequality since } \log(0.8) \text{ is negative)}\\ n &> 12.14\end{aligned}$$

Hence 13 years are required.

(v) as $n \to \infty$, $(0.8)^n \to 0$, so $A_n \to \$\mathbf{50\,000}$

(b)

$$\begin{aligned}2\log_e x &= \log_e(x+6)\\ \log_e x^2 &= \log_e(x+6)\\ x^2 &= x + 6\\ x^2 - x - 6 &= 0\\ (x-3)(x+2) &= 0\\ x &= 3,\ -2\\ \boldsymbol{x} &= \mathbf{3} \ \ (-2 \text{ doesnt satisfy original equation})\end{aligned}$$

Question 9

(a) (i)

$$2 + 3e^{-x} = e^{x}$$
$$(\times e^{x}): \quad 2e^{x} + 3 = e^{2x}$$
$$\boldsymbol{e^{2x} - 2e^{x} - 3 = 0}$$

(ii)

$$e^{2x} - 2e^{x} - 3 = 0$$
$$(e^{x} - 3)(e^{x} + 1) = 0$$
$$e^{x} = 3 \quad \text{or} \quad e^{x} = -1$$
$$\boldsymbol{x = \ln 3} \qquad \text{(no solution)}$$

(iii)

$$A = \int_{0}^{\ln 3} (2 + 3e^{-x} - e^{x})\, dx$$
$$= [2x - 3e^{-x} - e^{x}]_{0}^{\ln 3}$$
$$= \left(2\ln 3 - 3\left(\tfrac{1}{3}\right) - 3\right) - (0 - 3 - 1)$$
$$= (\mathbf{2\ln 3})\ \mathbf{units^2}$$

(b)

Area $= \frac{AB}{2}(AP + BQ)$

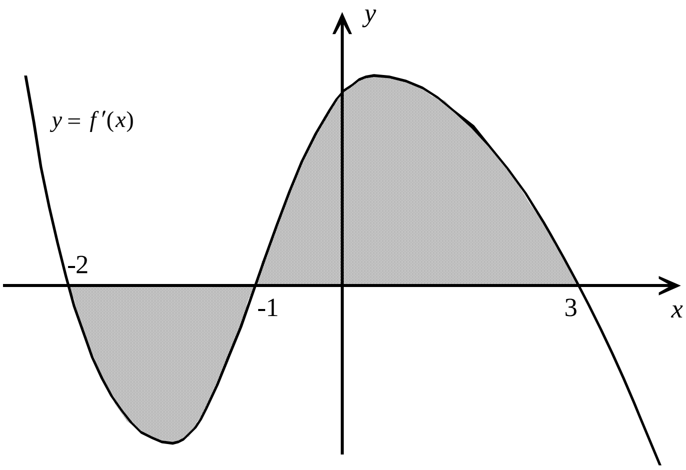
but $AP = SP$ and $BQ = SQ$ (focus definition of parabola)

$\therefore\ A = \frac{h}{2}(SP + SQ)$

$= \frac{h}{2} \times l$

$\boldsymbol{A = \frac{1}{2} lh}$

(c) (i)



(ii)

Area $= \int_{-1}^{-2} f'(x)\, dx + \int_{-1}^{3} f'(x)\, dx$

$= [\,f(x)\,]_{-1}^{-2} + [\,f(x)\,]_{-1}^{3}$

$= f(-2) - f(-1) + f(3) - f(-1)$

$= -1 - (-3) + 5 - (-3)$

$= \mathbf{10\ units^2}$

Question 10

(a) $PL = a - 3x,$

$$\frac{PL}{PQ} = \frac{LM}{QR}$$
$$\frac{a-3x}{a} = \frac{2x}{b}$$
$$b(a-3x) = 2ax$$
$$ab - 3bx = ab$$
$$2ax + 3bx = ab$$
$$x(2a+3b) = ab$$
$$\boldsymbol{x = \frac{ab}{2a+3b}}$$

(b) (i) Note: all triangles are similar

$a = GF = 1 - y$

$b = FK = y$

$$x = \frac{ab}{2a+3b}$$
$$= \frac{(1-y)\cdot y}{2(1-y)+3y}$$
$$= \frac{y(1-y)}{y+2}$$

Area, $A = 2x \cdot 3x$

$$= 6x^2$$
$$= 6 \cdot \left[\frac{y(1-y)}{y+2}\right]^2$$
$$\boldsymbol{A = \frac{6y^2(1-y)^2}{(y+2)^2}}$$

(ii)

$$B = \ln\frac{6y^2(1-y)^2}{(y+2)^2}$$
$$= \ln 6 + 2\ln y + 2\ln(1-y) - 2\ln(y+2)$$
$$\frac{dB}{dy} = \frac{2}{y} - \frac{2}{1-y} - \frac{2}{y+2}$$
$$= \frac{2(1-y)(y+2) - 2y(y+2) - 2y(1-y)}{y(1-y)(y+2)}$$
$$= \frac{2(y+2-y^2-2y-y^2-2y-y+y^2)}{y(1-y)(y+2)}$$
$$= \boldsymbol{\frac{2(2-4y-y^2)}{y(1-y)(y+2)}}$$

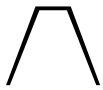
(iii) max/min when $\frac{dB}{dy} = 0$

$$2 - 4y - y^2 = 0$$
$$y^2 + 4y - 2 = 0$$
$$y^2 + 4y + 4 = 6$$
$$(y+2)^2 = 6$$
$$y + 2 = \pm\sqrt{6}$$
$$y = -2 \pm \sqrt{6}$$
$$y = \sqrt{6} - 2 \ \ \text{(as } y \text{ cannot be negative)}$$

Note: (1) the domain for y is $0 < y < 1$ (see diagram):

(2) $\sqrt{6} - 2 \approx 0.45$

y	0.4	$\sqrt{6} - 2$	0.5
$\frac{dB}{dy}$	$+0.83$	0	-0.8



$\therefore$ **maximum area occurs when $\boldsymbol{y = \sqrt{6} - 2}$**