NORTH SYDNEY GIRLS HIGH SCHOOL

2015

TRIAL HSC EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.

Total Marks – 100

Section I Pages 2 - 6

10 marks

- Attempt Questions 1 -10
- Allow about 15 minutes for this section

Section II Pages 7 - 19

90 Marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:______________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

QUESTION	MARKS
1-10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1- 10.

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1 For which values of k is the expression $x^2+(2-k)x+k(2-k)$ positive definite?

(A) $-2<k<2$

(B) $k>2,\ k<-2$

(C) $k>2,\ k<\dfrac{2}{5}$

(D) $\dfrac{2}{5}<k<2$

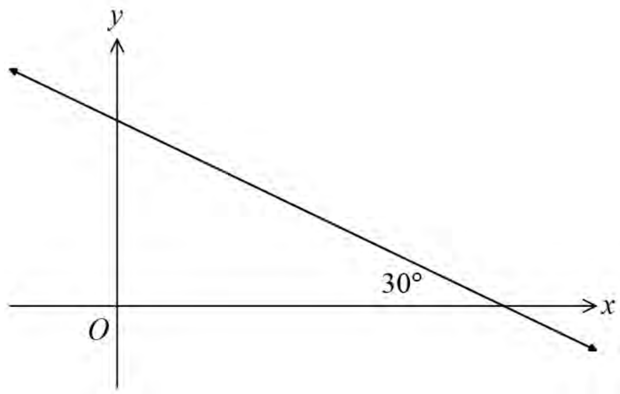
2 What is the gradient of the line?

(A) $\sqrt{3}$

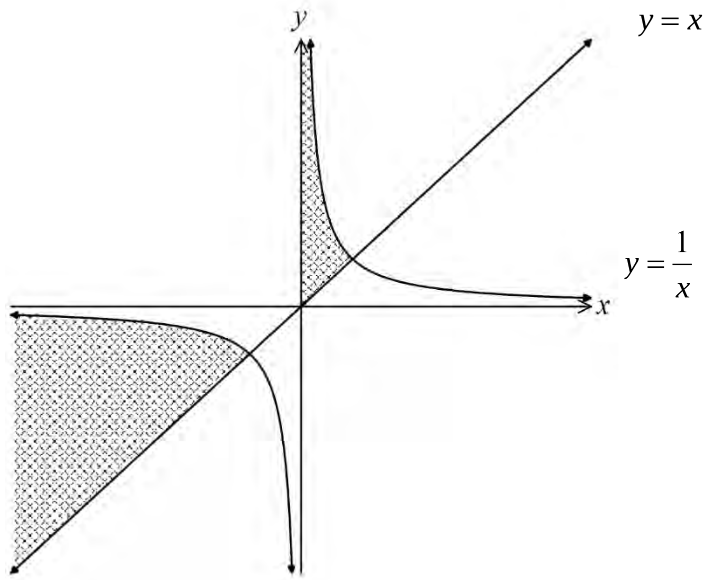
(B) $-\sqrt{3}$

(C) $\dfrac{1}{\sqrt{3}}$

(D) $-\dfrac{1}{\sqrt{3}}$



3 Which of the following inequalities best describe the shaded region below?



(A) $y \le \frac{1}{x}$ and $y \le x$

(B) $y \le \frac{1}{x}$ and $y \ge x$

(C) $y \ge \frac{1}{x}$ and $y \le x$

(D) $y \ge \frac{1}{x}$ and $y \ge x$

4 Consider the series $\log_a 54 + \log_a 18 + \log_a 6 + ...$

Which of the following statements best describes this series?

(A) A geometric series with common ratio of -3.

(B) A geometric series with common ratio of $-\log_a 3$.

(C) An arithmetic series with common difference of -3.

(D) An arithmetic series with common difference of $-\log_a 3$.

5 How many solutions of the equation $(\sin x+1)(\tan^2 x-3)=0$ lie between 0 and π?

(A) 1

(B) 2

(C) 3

(D) 4

6 For what values of x is the curve $f(x)=x^4-2x^3$ concave down?

(A) $0<x<\frac{3}{2}$

(B) $0\le x\le\frac{3}{2}$

(C) $0<x<1$

(D) $0\le x\le 1$

7 In the diagram below, *ABCD* is a square. *X*, *Y* and *Z* are points on sides *AB*, *BC* and *CD* respectively such that $XB=YC=ZD$.

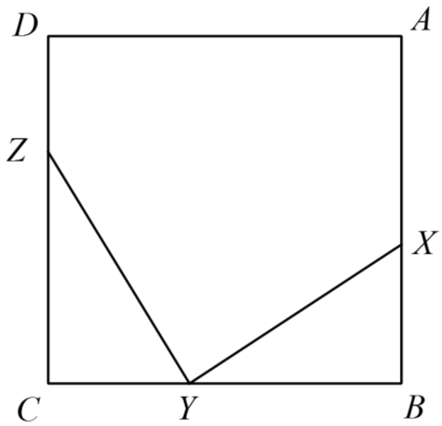
Which test proves $\Delta BXY\equiv\Delta CYZ$?

(A) AAA

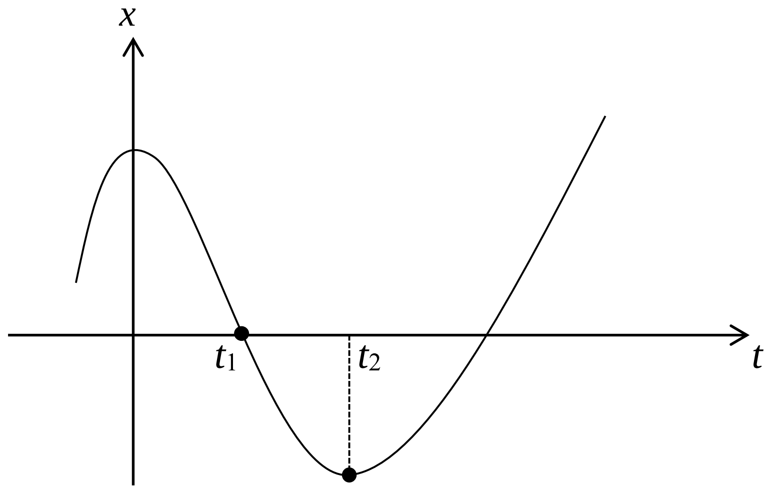
(B) AAS

(C) SAS

(D) RHS



8 The graph shows the displacement x of a particle moving along a straight line as a function of time t.



Which statement best describes the motion of the particle moving between $t = t_1$ and $t = t_2$?

(A) The particle is moving towards the origin and is stationary at $t = t_2$.

(B) The particle is moving away from the origin and is stationary at $t = t_2$.

(C) The particle is moving towards the origin and reaches minimum velocity at $t = t_2$.

(D) The particle is moving away from the origin and reaches minimum velocity at $t = t_2$.

9 Let $f(x) = x^4 - 1$. Which of the following statements is NOT true for the curve $y = f(x)$?

(A) $f(x)$ is an even function.

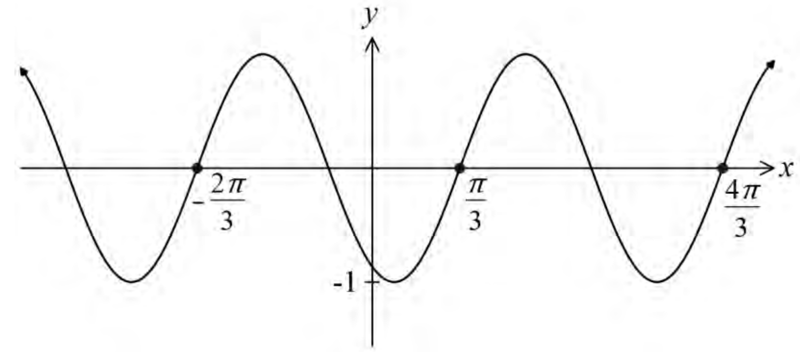
(B) $f(x)$ has a minimum turning point when $x = 0$.

(C) $f(x)$ has a horizontal point of inflexion when $x = 0$.

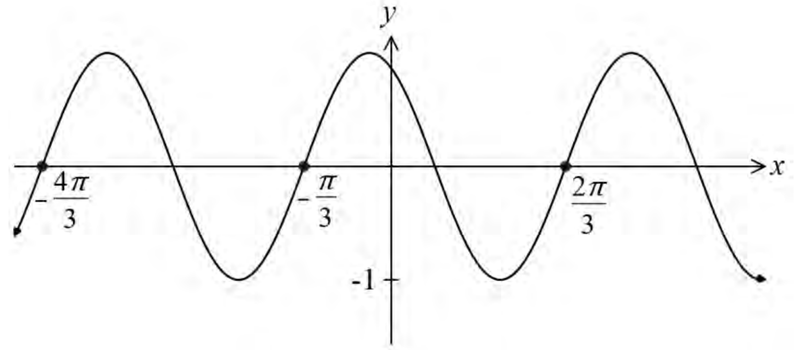
(D) $\lim\limits_{x \to \infty} \dfrac{1}{f(x)} = 0$.

10 Which diagram shows the graph of $y = \sin\left(2x - \dfrac{\pi}{3}\right)$?

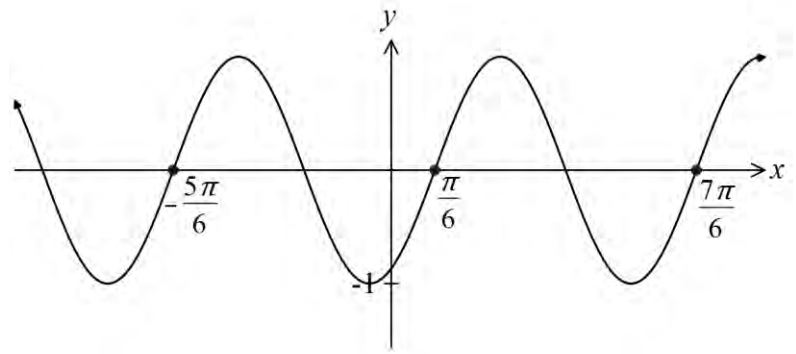
(A)



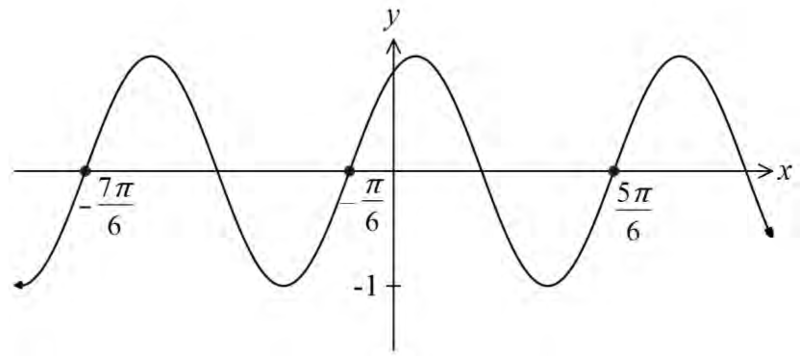
(B)



(C)



(D)



Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hours 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\dfrac{2\cdot 7-4\times 1\cdot 5^2}{\sqrt[3]{e}}$, correct to two significant figures. **1**

(b) Find the values of x for which $|2x-3|>5$. **2**

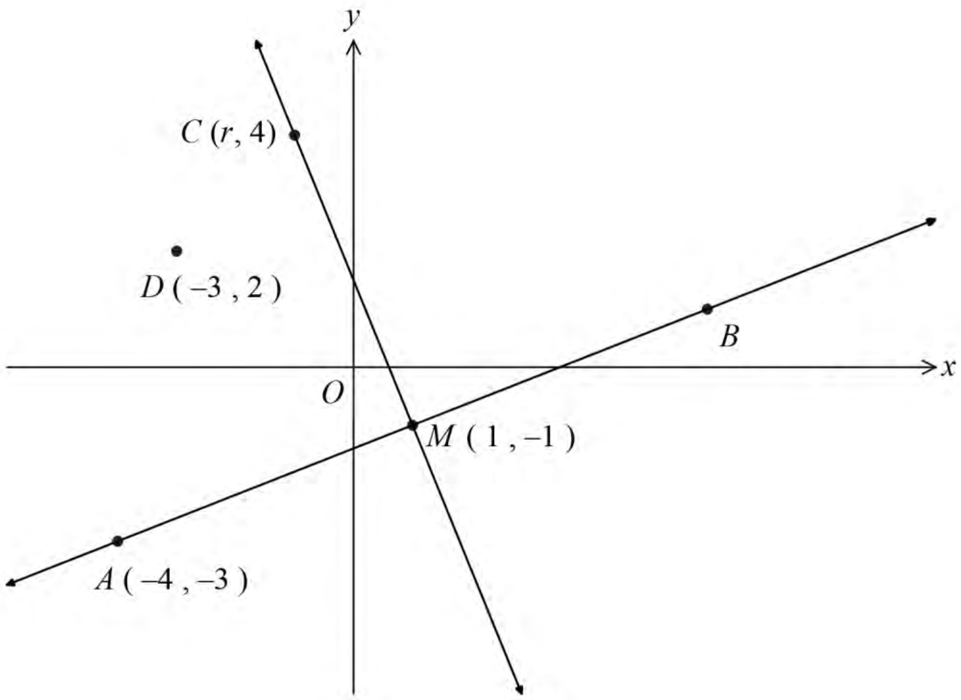
(c) Express $\sqrt{3}-\dfrac{4}{\sqrt{3}+1}$ in the form $a+b\sqrt{3}$ where a and b are integers. **2**

(d) Prove the identity: **2**

$$\frac{1-\tan^2\theta}{1+\tan^2\theta}=2\cos^2\theta-1$$

Question 11 continues on page 8

(e) The diagram shows points $A(-4,-3)$, $B(p, q)$, $C(r, 4)$, and $D(-3, 2)$ in the Cartesian plane.



(i) $M(1, -1)$ is the midpoint of AB. Find the coordinates of B. **2**

(ii) Show that the equation of the line perpendicular to AB and passing through the point M is $5x+2y-3=0$. **2**

(iii) If $C(r, 4)$ lies on the line $5x+2y-3=0$, find the value of r. **1**

(iv) Find the length of CM. **1**

(v) Hence find the area of ΔCDM. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Differentiate $y = x\log_e x$, with respect to x. **1**

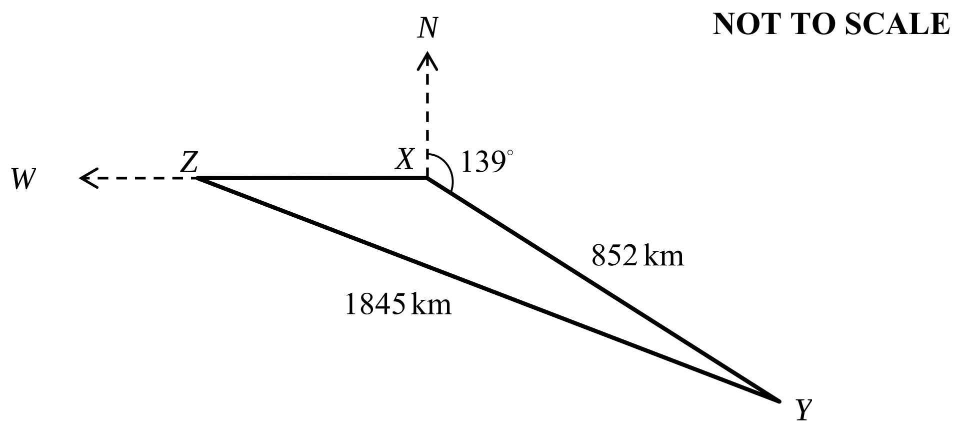
(b) Evaluate $\int_0^{\frac{\pi}{3}} \left(1 - \sec^2 \frac{x}{2}\right) dx$. **2**

(c) A plane flies from Town X on a bearing of 139°T for 852 km to Town Y. **3**

The plane then turns and flies on a bearing of α °T for 1845 km to Town Z, which is due west of Town X.

Copy the diagram into your writing booklet.

Find the value of α, to the nearest degree.

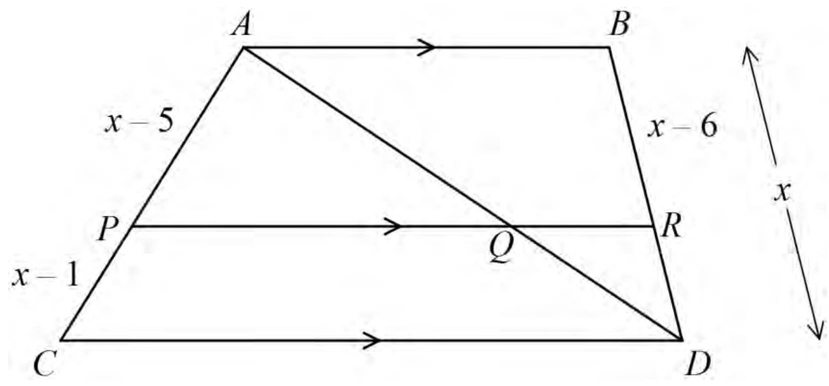


Question 12 continues on page 10

Question 12 (continued)

(d) In the diagram below, $AP = (x-5)$, $PC = (x-1)$, $BR = (x-6)$, and $BD = x\,\text{cm}$. **3**

Find the value of x in the diagram, giving reasons.



(e) Rapunzel has discovered something interesting about the way her hair grows.

Initially her hair is 18 metres long.

She measures her hair every year and records how much her hair grows during each year, as below.

10 m **9 m** **8.1 m** **7.29 m ….**

(i) Find how much Rapunzel's hair will grow during 10^{th} year, correct to 2 decimal places. **2**

(ii) Find the length of Rapunzel's hair after ten years, correct to 2 decimal places. **2**

(iii) If the tower is 110 metres high, assuming that her hair continues to grow in a similar pattern, will Rapunzel's hair ever be long enough to reach the ground? Justify your answer. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) A parabola has the equation $2y = x^2 - 6x + 7$.

(i) Find the coordinates of the vertex. **2**

(ii) State the coordinates of the focus and the equation of the directrix. **2**

(b) The amount, Q (in mg), of caffeine left in the blood t hours after consuming coffee is calculated using the formula $Q = Q_0 e^{-kt}$ where Q_0 is the original quantity of caffeine in the blood, k is a positive constant and t is the time in hours after consuming coffee.

Yuna likes her coffee strong. At 8 a.m. she drank 2 cups of coffee and had a total of 25 mg of caffeine in her blood.
At 12 p.m., the amount of caffeine left in her blood is 13 mg.

(i) Show that the amount of caffeine left in the blood decreases at a rate proportional to the amount of caffeine remaining in the blood. **1**

(ii) Find the exact value of k. **2**

(iii) What is the rate of change of the amount of caffeine in her blood, in mg per hour, when $t = 2$. **1**

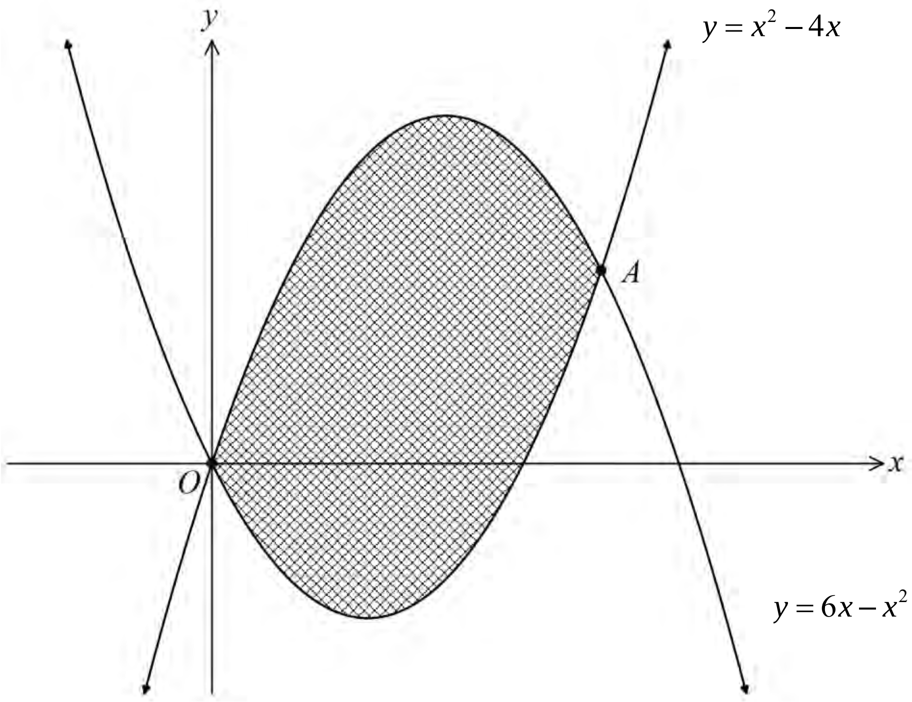
(iv) At what time has the caffeine in her blood reduced by 80%? Answer to the nearest minute. **3**

Question 13 continues on page 12

(c) The diagram shows the parabolas $y = 6x - x^2$ and $y = x^2 - 4x$.

The parabolas intersect at the origin O and the point A.

The region between the two parabolas is shaded.



(i) Find the x-coordinate of the point A. **1**

(ii) Find the area of the shaded region. **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Show that the function $f(x) = 2e^{-x} - 3e^{-2x}$ has a point of inflexion at $x = \log_e 6$. **3**

(b) A particle is moving in a straight line. Its velocity, v metres per second, at time t seconds is given by $v = 1 - \dfrac{2t}{t^2 + 1}$.

(i) Find the acceleration of the particle when $t = 2$. **3**

(ii) If the particle is initially at the origin, find the position of the particle when $t = 2$. **3**

(iii) Describe fully the motion of the particle when $t = 2$. **1**

Question 14 continues on page 14

(c) David invests \$3000 into his account at the beginning of every month to save up for a deposit on his dream house in 5 years' time.
The contributions are compounded monthly at an interest rate of 9% per annum.

(i) Let A_n be the amount in the account at the end of n months. **2**
Show that the expression for the amount in his account at the end of the second year is

$$A_{24} = \frac{\$3000 \times 1 \cdot 0075\left(1 \cdot 0075^{24} - 1\right)}{0.0075}$$

(ii) David requires \$300 000 as a deposit in order to buy his dream house. **3**
Two years after David started investing, he realised that he would not be able to save \$300 000 by the end of the 5^{th} year.
For the remaining 3 years, David decides to invest \$$K$ at the beginning of each month at the same interest rate of 9% per annum, compounded monthly.

Find the value of K, to the nearest dollar, so that David will have enough for the deposit by the end of the 5^{th} year.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) If α and β are the roots of $2x^2 + 4x - 1 = 0$, evaluate $\left(\alpha - \beta\right)^2$. **2**

(b) A filter pumps water into or out of a pond.
The flow rate of water, in litres per minute, is given by

$$\frac{dV}{dt} = 1 - \sqrt{2}\sin\frac{\pi t}{60}$$

(i) If the pump started at 11 a.m., what is the first time after 11 a.m. at which the flow rate is zero? **2**

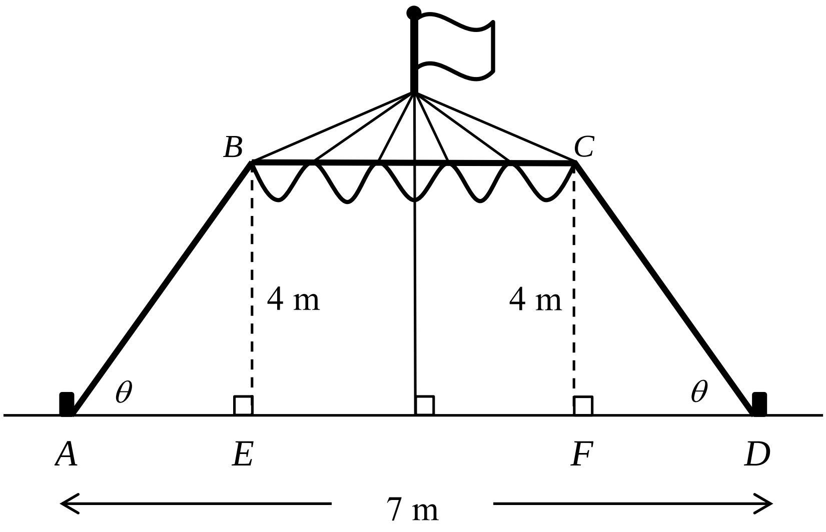
(ii) Initially there was $\dfrac{60\sqrt{2}}{\pi}$ litres of water in the pond. **2**

Find an expression for the volume, *V*, of water in the pond after *t* minutes.

(iii) By sketching the graph of $\dfrac{dV}{dt}$ for $0 \le t \le 120$, or otherwise, find when the pond has the least amount of water in first 120 minutes. **2**

Question 15 continues on page 16

(c) A circus marquee is supported by a centre pole, and secured by the rope $ABCD$.
George, a curious monkey, wants to know how quickly he can climb up and down the marquee along the rope $ABCD$.
Along AB and CD George moves at 2 m/min but along BC, he doubles his speed.



(i) If $BE = CF = 4$ metres and $AD = 7$ metres, show that **2**

$$BC = 7 - \frac{8}{\tan\theta}$$

(ii) Hence show that the total time (t) needed for George to climb up and down the marquee, in minutes, is given by: **2**

$$t = \frac{7}{4} + \frac{4 - 2\cos\theta}{\sin\theta}$$

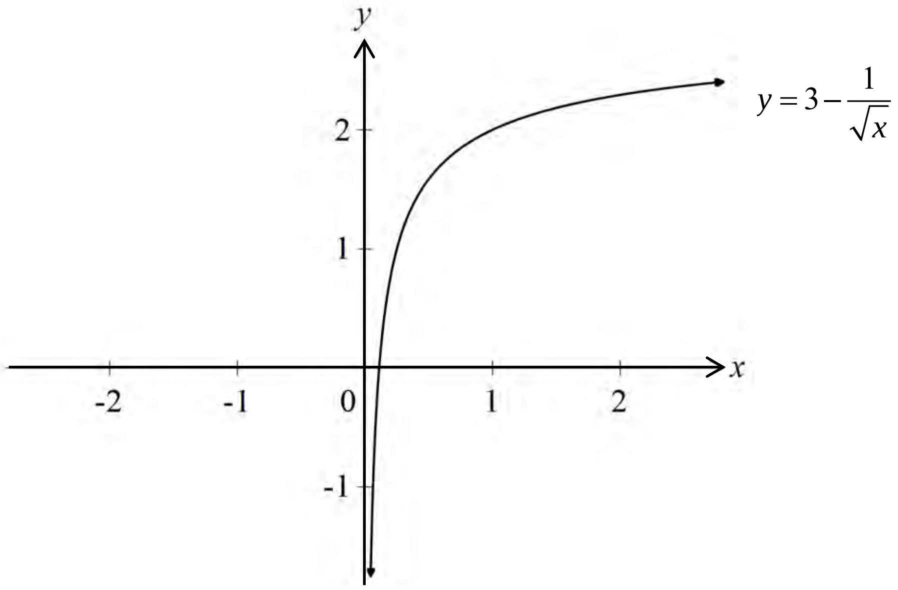
(iii) Given that $0 < \theta < \frac{\pi}{2}$, find value of θ for which the time taken by George to climb up and down the marquee is a minimum. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) The curved sides of a vase are formed by rotating the part of the curve $y = 3 - \dfrac{1}{\sqrt{x}}$ between $y = -1$ and $y = 2$ about the y-axis. **4**

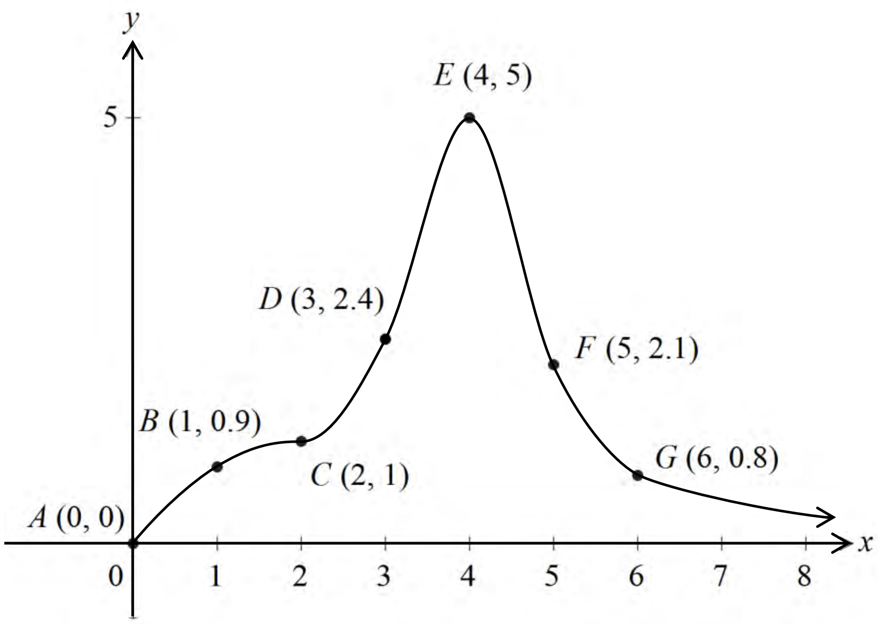
Find the exact volume of the vase.



Question 16 continues on page 18

Question 16 (continued)

(b) The graph shows the function $y = f(x)$.

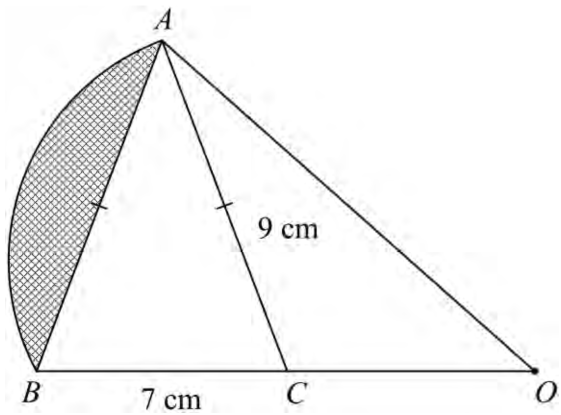


(i) Using Simpson's rule, with 6 sub-intervals, estimate the area under the graph of $y = f(x)$ between $x = 0$ and $x = 6$. **2**

(ii) Sketch a possible graph of $y = f'(x)$, given that $\lim_{x \to \infty} f(x) = \frac{1}{2}$, showing all important features. **2**

Question 16 continues on page 19

(c) In the diagram, ΔABC is an isosceles triangle with $AB = AC = 9$ cm and $BC = 7$ cm.



OA and OB are the radii of a circle with the centre O.

(i) Prove that $\Delta ABC \parallel\!| \Delta OAB$. **3**

(ii) Hence, find the area of the shaded segment, correct to 2 decimal places. **4**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \qquad x > 0$

2015 Mathematics Trial Solution

Section I

1 **(D)**

$x^2+(2-k)x+k(2-k)$

Positive definite → $\Delta<0$

$$\begin{aligned}\Delta&=(2-k)^2-4k(2-k)\\&=k^2-4k+4-8k+4k^2\\&=5k^2-12k+4\\&=(5k-2)(k-2)\end{aligned}$$

$\therefore (5k-2)(k-2)<0$

$\therefore \frac{2}{5}<k<2$

2 **(D)**

$$\begin{aligned}m&=\tan\theta\\&=\tan 150^\circ\\&=-\frac{1}{\sqrt{3}}\end{aligned}$$

3 **(B)**

4 **(D)**

$\log_a 54+\log_a 18+\log_a 6+.....$

Coomon difference is

$$\begin{aligned}\log_a 18-\log_a 54&=\log_a\frac{18}{54}\\&=\log_a\frac{1}{3}\\&=-\log_a 3\end{aligned}$$

5 **(B)**

$(\sin x+1)(\tan^2 x-3)=0$

$\therefore \sin x=-1,\quad \tan x=\pm\sqrt{3}$

But $0<x<\pi$

$\therefore$ No solution for $\sin x=-1$

as $\sin x>0$ for all $0<x<\pi$

Hence $\tan x=\pm\sqrt{3}$

Therefore 2 solutions

6 **(C)**

$$\begin{aligned}f(x)&=x^4-2x^3\\f'(x)&=4x^3-6x^2\\f''(x)&=12x^2-12x\end{aligned}$$

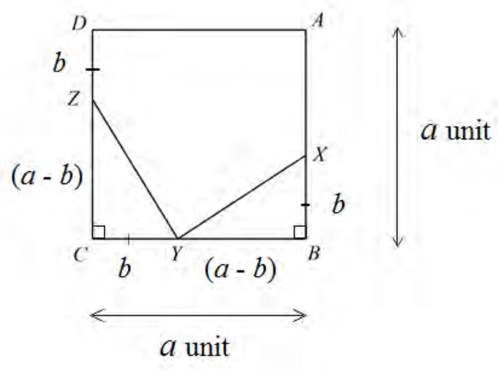
For concave down:

$f''(x)<0$

$\therefore 12x(x-1)<0$

$\therefore \quad 0<x<1$

7 **(C)**



$\therefore$ **SAS**

8 **(B)**

9 **(C)** There is no HPOI.

10 **(C)**

Section II

Question 11

(a) $\dfrac{2\cdot 7-4\times 1\cdot 5^2}{\sqrt[3]{e}}=-4.514147...$

$=-4.5\ (\text{to 2 sig.fig.})$

(b) $|2x-3|>5$

$$\begin{aligned} 2x-3&>5 &\quad \text{or} \quad 2x-3&<-5\\ 2x&>8 & 2x&<-2\\ x&>4 & x&<-1 \end{aligned}$$

$\therefore x>4,\ x<-1$

(c) $\sqrt{3}-\dfrac{4}{\sqrt{3}+1}=\sqrt{3}-\dfrac{4\left(\sqrt{3}-1\right)}{3-1}$

$$\begin{aligned} &=\sqrt{3}-2\left(\sqrt{3}-1\right)\\ &=\sqrt{3}-2\sqrt{3}+2\\ &=2-\sqrt{3}\\ &\therefore a=2,\ b=-1 \end{aligned}$$

(d) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=2\cos^2\theta-1$

$$\begin{aligned} \text{LHS}&=\frac{1-\tan^2\theta}{1+\tan^2\theta}\\ &=\frac{1-\tan^2\theta}{\sec^2\theta}\\ &=\frac{1}{\sec^2\theta}-\frac{\tan^2\theta}{\sec^2\theta}\\ &=\cos^2\theta-\frac{\dfrac{\sin^2\theta}{\cancel{\cos^2\theta}}}{\dfrac{1}{\cancel{\cos^2\theta}}}\\ &=\cos^2\theta-\sin^2\theta\\ &=\cos^2\theta-\left(1-\cos^2\theta\right)\\ &=\cos^2\theta-1+\cos^2\theta\\ &=2\cos^2\theta-1\\ &=\text{RHS} \end{aligned}$$

Marker's Comments

Question 11(a)

Successful students correctly read the question and rounded their answers to two significant figures. Others misread the question and rounded to three significant figures. No part marks are given in the Higher School Certificate examination, hence no part marks were awarded.

Question 11(b)

A mark was awarded to students who correctly identified that $x>4$ and the second mark was awarded to students who also identified that $x<-1$

Question 11(c)

Successful students rationalised the denominator, expanded correctly and completely simplified. Some students tried to combine the two terms but only wrote down the numerator. Errors made along the way, such as a negative sign turning into a positive sign, were penalised. No marks were deducted for not explicitly stating the values of a and b.

Question 11(d)

Extension 1 students are encouraged to show how $\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$.

They were not penalised for doing this.

(e)

(i) $M(1,-1)$ is a midpoint of $A(-4,-3)$ and $B(p,q)$

$$\frac{-4+p}{2}=1 \qquad \text{and} \qquad \frac{-3+q}{2}=-1$$

$$\therefore\ p=6 \qquad\qquad \therefore\ q=1$$

$$\therefore B(6,1)$$

(ii) If two lines are perpendicular,

then $m_1\times m_2=-1$

$$m_{AB}=\frac{1+3}{6+4} \qquad \therefore m_{AB}=\frac{2}{5}$$

$\therefore$ line perpendicular to AB has

gradient of $-\frac{5}{2}$.

$\therefore$ the equation of the line is:

$$y+1=-\frac{5}{2}(x-1)$$

$$2y+2=-5x+5$$

$$\therefore\ 5x+2y-3=0$$

(iii) $C(r,4)$ lies on $5x+2y-3=0$

$$\therefore 5r+8-3=0 \qquad \therefore r=-1$$

(iv) $C(-1,4)$ and $M(1,-1)$

$$d_{CM}=\sqrt{(1+1)^2+(-1-4)^2}$$

$$=\sqrt{4+25}$$

$$\therefore\ d_{CM}=\sqrt{29}$$

(v) Perpendicular distance from $D(-3,2)$ to $5x+2y-3=0$.

$$=\frac{|(-3)\times5+2\times2-3|}{\sqrt{5^2+2^2}}$$

$$=\frac{14}{\sqrt{29}}$$

$$\therefore\ \text{Area}=\frac{1}{2}\times\sqrt{29}\times\frac{14}{\sqrt{29}}$$

$$=7\,\text{u}^2$$

Question 11(e)

(i) Most students answered this question correctly.

(ii) Successful students show the gradient of AB, and then substituted the gradient and $(-1,1)$ into the point-gradient formula.

(iii) Most students answered this question correctly. Students are asked to answer the question as some students wrote $C(-1,4)$ rather than $r=-1$. They were not penalised for this.

(iv) Most students answered this question correctly.

(v) Students are encouraged not to be misled by the diagram. Students who assumed that $CD\perp DM$ made the question too easy, hence were not awarded any marks. Successful students used the perpendicular distance to find the appropriate height and then used the area of a triangle.

Question 12

(a) $y = x\log_e x$

$$\frac{dy}{dx} = (\log_e x) + \left(x \times \frac{1}{x}\right)$$

$$= \log_e x + 1$$

(b) $\int_0^{\frac{\pi}{3}} \left(1 - \sec^2 \frac{x}{2}\right) dx$

$$= \left[x - 2\tan\frac{x}{2}\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi}{3} - 2\tan\frac{\pi}{6}\right) - (0 - 0)$$

$$= \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$

(c) $\angle YXZ = 360^\circ - 90^\circ - 139^\circ$ (angles at a point)

$\therefore \angle YXZ = 131^\circ$

In ΔXYZ, let $\angle XZY = \theta$

Using sine rule

$$\frac{\sin\theta}{852} = \frac{\sin 131^\circ}{1845}$$

$\therefore \theta \approx 20^\circ$ (to the nearest degree)

$\therefore \alpha^\circ = 270^\circ + 20^\circ$

$\therefore \alpha = 290$

(d) $\dfrac{x-5}{x-1} = \dfrac{x-6}{x-(x-6)}$

(parallel line preserve ratios of intercepts)

$$6(x-5) = (x-1)(x-6)$$

$$6x - 30 = x^2 - 7x + 6$$

$$x^2 - 13x + 36 = 0$$

$$(x-4)(x-9) = 0$$

$$x = 4, 9$$

But

$x > 6$ as $x - 6 > 0$

$\therefore x = 9$

Marker's Comments

Question 12 (a)
Extremely well done

Question 12 (b)
Very well done. Only a few students had trouble integrating $\sec^2 \dfrac{x}{2}$.

Question 12 (c)
The main problem in this question was having α in the wrong place – some students thought $\angle XYZ$ was α. Many students who did not get it fully correct still got 2 marks. A few students incorrectly used the cosine rule.

Question 12 (d)
Many students did not us "parallel lines preserves ratio" and spent a lot of time proving triangles similar. Some students incorrectly thought that equiangular trapeziums would be similar. Most students who started correctly were able to explain why x = 4 was no a valid solution.

(e) (i) $r = \frac{9}{10} = \frac{8.1}{9} = 0.9$

The sequence is GP

with $a = 10$ and $r = 0.9$

$T_{10} = ar^9$

$= 10 \times (0.9)^9$

$= 3.87$

$\therefore$ 3.87 metres

(ii) $S_{10} = \frac{a(1-r^{10})}{1-r}$

$= \frac{10(1-0.9^{10})}{1-0.9}$

$= 65.13$

Total length of her hair is 83.13 m

(iii) $r = 0.9 < 1 \quad \therefore S_\infty$ exists

$S_\infty = \frac{a}{1-r}$

$= \frac{10}{1-0.9}$

$= 100$

The max length of her hair is 118 metres.
Yes it will be long enough to reach the ground.

Question 12 (e)

(i) A surprising number of students used $a = 18$, instead of . Quite a few students had the formula $T_n = ar^{n-1}$ incorrect.

(ii) The most common error was not adding 18 m to their answer of 65.13 metres, only getting the total of the additional growth. Again, far too many students did not have the S_n formula correct.

(iii) Many students correctly found $S_\infty = 100$ metres but again forgot to add on the 18 metres. If this was done in parts (ii) and (iii) it was only penalised once.
A good number of students were awarded full marks or close to it for this question.

Question 13

(a) $2y = x^2 - 6x + 7$

(i)

$$2y = (x-3)^2 - 2$$

$$(x-3)^2 = 2(y+1)$$

$\therefore$ Vertex $(3, -1)$

(ii) $4a = 2 \qquad \therefore a = \frac{1}{2}$

$\therefore$ focus $\left(3, -\frac{1}{2}\right)$

and directrix $y = -\frac{3}{2}$

(b) $Q = Q_0 e^{-kt}$

(i)

$$\frac{dQ}{dt} = Q_0 e^{-kt} \times (-k)$$

$$= -k\left(Q_0 e^{-kt}\right)$$

$$= -kQ$$

$$\therefore \frac{dQ}{dt} \propto Q$$

Therefore the amount of caffeine left in the blood decreases at a rate proportional to the amount of caffeine remaining in her blood.

(ii) Let $t = 0$ correspond to 8 a.m.

At $t = 0$, $Q = 25 \quad \therefore Q_0 = 25$

At $t = 4$, $Q = 13$

$\therefore 13 = 25e^{-4k}$

$$e^{-4k} = \frac{13}{25}$$

$$4k = \log_e\left(\frac{13}{25}\right)$$

$$\therefore k = -\frac{1}{4}\log_e\left(\frac{13}{25}\right)$$

Marker's comments

Question 13 (a)

(i) Generally well done.

Some students used $x = -\frac{b}{2a}$ to find the coordinates of the vertex.

(ii) Most students answered this question correctly.

Question 13 (b)

(i) Many students did not answer this question correctly.

Students differentiated Q, $\frac{dQ}{dt} = -kQ_o e^{-kt}$ correctly but did not conclude that $\frac{dQ}{dt} = -kQ$.

(ii) Most students correctly answered this question.

(iii) At $t = 2$

$$\frac{dQ}{dt} = -kQ$$

$$= \frac{1}{4}\log_e\left(\frac{13}{25}\right)\times e^{\frac{1}{2}\log_e\left(\frac{13}{25}\right)}$$

$$= \frac{1}{4}\log_e\left(\frac{13}{25}\right)\times\sqrt{\frac{13}{25}}$$

$$= -2.947...$$

Therefore it decreases at a rate of 3 mg/h.

(iv) $$0.2\cancel{Q_O} = \cancel{Q_O}e^{-kt}$$

$$-kt = \log_e 0.2$$

$$t = \frac{\log_e 0.2}{-k}$$

$$t = \frac{\log_e 0.2}{\frac{1}{4}\log_e\left(\frac{13}{25}\right)}$$

$$t = 9\text{hr } 51\text{min}$$

$\therefore$ 5:51pm

(c) $y = 6x - x^2 \qquad y = x^2 - 4x$

(i) For intersection, solve simultaneously:

$$6x - x^2 = x^2 - 4x$$

$$2x^2 - 10x = 0$$

$$x(x-5) = 0$$

$$\therefore x = 0,\ 5$$

$\therefore x-$coordinate of A is 5.

(ii) $$A = \int_0^5 \left[\left(6x - x^2\right) - \left(x^2 - 4x\right)\right] dx$$

$$= \int_0^5 \left(10x - 2x^2\right) dx$$

$$= \left[5x^2 - \frac{2}{3}x^3\right]_0^5$$

$$= 5\times 25 - \frac{2}{3}\times 125$$

$$= \frac{125}{3}\ \text{u}^2$$

(iii) Some students did not cancel negative sign for $(-k)$.

e.g.

$$\frac{dQ}{dt} = -kQ$$

$$= -\frac{1}{4}\log_e\left(\frac{13}{25}\right)\times e^{\frac{-1}{2}\log_e\left(\frac{13}{25}\right)}$$

$$= -\frac{1}{4}\log_e\left(\frac{13}{25}\right)\times\sqrt{\frac{25}{13}}$$

$$= 2.947...$$

(iv) Generally well done. Some students put 0.8 instead of 0.2.

Question 13 (c)

(i) Extremely well done.

(ii) Many students answered this question correctly.
Only a few student divided the area in to three sections and calculated the area separately but incorrectly.

Question 14

(a) $f(x) = 2e^{-x} - 3e^{-2x}$

$f'(x) = -2e^{-x} + 6e^{-2x}$

$f''(x) = 2e^{-x} - 12e^{-2x}$

To find the point of inflexion, $f''(x) = 0$

$2e^{-x} - 12e^{-2x} = 0$

$2e^{-2x}\left(e^{x} - 6\right) = 0$

$\therefore e^{x} = 6 \quad \left(\text{as } e^{-2x} \neq 0\right)$

$\therefore x = \log_e 6$

Concavity check

x	$\log_e 5$	$\log_e 6$	$\log_e 7$
$f''(x)$	$-\dfrac{2}{25}$	0	$\dfrac{2}{49}$

Concavity changes

$\therefore$ point of inflexion occurs at $x = \log_e 6$

(b) $v = 1 - \dfrac{2t}{t^2+1}$

(i) Acceleration $(a) = \dfrac{dv}{dt}$

$$a = -\frac{2\left(t^2+1\right) - 2t \times 2t}{\left(t^2+1\right)^2}$$

$$= \frac{2\left(t^2-1\right)}{\left(t^2+1\right)^2}$$

When $t = 2 \quad \therefore a = \dfrac{6}{25}$ m/s^2

(ii) $x = \int v \; dt$

$$x = \int \left(1 - \frac{2t}{t^2+1}\right) dt$$

$$= t - \log_e\left(t^2+1\right) + C$$

When $t = 0$, $x = 0$

$\therefore x = t - \log_e\left(t^2+1\right)$

When $t = 2$

$x = 2 - \log_e 5$

(iii) At $t = 2$, the particle is 0.39m to the right of the origin, travelling in the positive direction as $v > 0$ and the particle is speed up as $a > 0$.

Marker's comments

Question 14 (a)

Well done. Some students tried to substitute the values into second derivative and jumped steps. Some students did not show the change in concavity.

Question 14 (b)

(i) Lots of errors made differentiating using quotient rule to find acceleration expression.

(ii) Generally well done. Some students neglected to include constant when integrating.

(iii) Students need to describe the motion using position, direction of motion and whether speeding up or slowing down. Some answers merely repeated the numbers without interpretation.

(c) $r = 9\%$ p.a.

$= \dfrac{9}{12}\%$ per month $\quad \therefore r = 0.0075$

(i) For first 24 months

First amount: 3000×1.0075^{24}

Second amount: 3000×1.0075^{23}

Third amount: 3000×1.0075^{22}

The other amounts follow this pattern

Second last amount: 3000×1.0075^{2}

So last amount : 3000×1.0075

Total amount (A_{24})

$$A_{24} = 3000(1.0075)^{24} + 3000(1.0075)^{23} + ... + 3000(1.0075)^{2} + 3000(1.0075)$$
$$= 3000\left(1.0075^{24} + 1.0075^{23} + ... + 1.0075^{2} + 1.0075\right)$$
$$= 3000\left(1.0075 + 1.0075^{2} + ... + 1.0075^{23} + 1.0075^{24}\right)$$

Now $1.0075 + 1.0075^{2} + ... + 1.0075^{23} + 1.0075^{24}$ is a GP with $a = 1.0075,\ r = 1.0075,\ n = 24$

$$A_{24} = 3000 \times \frac{1.0075\left(1.0075^{24} - 1\right)}{1.0075 - 1}$$

$$\therefore A_{24} = \frac{\$3000 \times 1.0075\left(1.0075^{24} - 1\right)}{0.0075}$$

(ii) If this amount continues to earn interest at 9% p.a. it will grow to $\$A_{24} \times 1.0075^{36}$.

(iii) Let B_{36} be the amount in the account for last 3years, with $\$K$ invested each month.

$$B_{36} = \frac{K \times 1.0075\left(1.0075^{36} - 1\right)}{0.0075}$$

$$A_{24} \times 1.0075^{36} + B_{36} = 300\,000$$

$$B_{36} = 300\,000 - A_{24} \times 1.0075^{36}$$

$$B_{36} = 300\,000 - \frac{3000 \times 1.0075^{37}\left(1.0075^{24} - 1\right)}{0.0075}$$

$$\frac{K \times 1.0075\left(1.0075^{36} - 1\right)}{0.0075} = 196\ 414.63$$

$\therefore \$K = \$4737$

Question 14 (c)

(i) Generally well done. Standard question and most students showed good development of the series.

(ii) Most students were able to adapt their thinking to this non-standard question. However they did not realise the first two years' worth of savings would earn further interest over the following three years.

Question 15

(a) $2x^2 + 4x - 1 = 0$

$\alpha + \beta = -2$ and $\alpha \times \beta = -\frac{1}{2}$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$
$$= (-2)^2 - 4 \times \left(-\frac{1}{2}\right)$$
$$= 4 + 2$$
$$= 6$$
$$\therefore (\alpha - \beta)^2 = 6$$

(b) $\frac{dV}{dt} = 1 - \sqrt{2}\sin\frac{\pi t}{60}$

(i)

$$\frac{dV}{dt} = 0$$
$$1 - \sqrt{2}\sin\frac{\pi t}{60} = 0$$
$$\sin\frac{\pi t}{60} = \frac{1}{\sqrt{2}}$$
$$\frac{\pi t}{60} = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$$
$$t = 15, 45, \ldots$$
$$\therefore t = 15$$

Therefore, 11:15 a.m

(ii)

$$V = \int \frac{dV}{dt}\, dt$$
$$= \int \left(1 - 2\sin\frac{\pi t}{60}\right) dt$$
$$= t + 2\cos\left(\frac{\pi t}{60}\right) \times \frac{60}{\pi} + C$$

When $t = 0$, $V = \frac{60\sqrt{2}}{\pi}$

$$\therefore \frac{60\sqrt{2}}{\pi} = 0 + \frac{\sqrt{2} \times 60}{\pi} \times 1 + C$$
$$\therefore C = 0$$
$$\therefore V = t + \frac{60\sqrt{2}}{\pi}\cos\left(\frac{\pi t}{60}\right)$$

Marker's Comments

Question 15 (a)

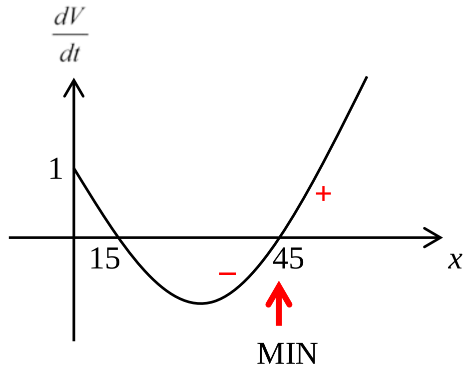
Generally well done.
Students need to take care when dealing with the negative signs.

Question 15 (b)

(i) Well done.

(ii) Generally well done. Some students had difficulty handing the coefficient of *t* when integrating a trig function.

(iii)



Alternatively,

$$V = t + \frac{60\sqrt{2}}{\pi}\cos\left(\frac{\pi t}{60}\right)$$

Min Volume is at t = 45

$\therefore$ 11.45 a.m.

(iii) Poorly done. Many students had difficulty correctly drawing the graph and if the graph was correct, then interpreted the graph incorrectly.

Question 15 (c)

(i) Well done. Some students used a congruence proof which was unnecessary.

(c)

(i) In ΔABE

$$\tan\theta = \frac{4}{AE}$$

$$\therefore AE = \frac{4}{\tan\theta}$$

Similarly $DF = \dfrac{4}{\tan\theta}$

$$\begin{aligned} BC &= EF \\ &= AD - (AE + DF) \\ &= 7 - 2\times\frac{4}{\tan\theta} \end{aligned}$$

$$\therefore\ BC = 7 - \frac{8}{\tan\theta}$$

(ii) In ΔABE

$\sin\theta = \dfrac{4}{AB}$ $\therefore AB = \dfrac{4}{\sin\theta}$

Similarly $CD = \dfrac{4}{\sin\theta}$

Speed =2 m/min

$$\begin{aligned} \therefore \text{time}_{AB} &= \frac{4}{\sin\theta}\times\frac{1}{2} \\ &= \frac{2}{\sin\theta} \end{aligned}$$

$$\therefore \text{time}_{CD} = \frac{2}{\sin\theta}$$

Speed 4 m/min along BC.

$$\therefore \text{time}_{BC} = \left(7 - \frac{8}{\tan\theta}\right) \times \frac{1}{4}$$

$$= \frac{7}{4} - \frac{2}{\tan\theta}$$

$$\therefore t = 2\left(\frac{2}{\sin\theta}\right) + \left(\frac{7}{4} - \frac{2}{\tan\theta}\right)$$

$$= \frac{7}{4} + \frac{4}{\sin\theta} - \frac{2\cos\theta}{\sin\theta}$$

$$\therefore t = \frac{7}{4} + \frac{4 - 2\cos\theta}{\sin\theta}$$

(iii) $t = \frac{7}{4} + \frac{4 - 2\cos\theta}{\sin\theta}$

$$\frac{dt}{d\theta} = \frac{2\sin^2\theta - \cos\theta(4 - 2\cos\theta)}{\sin^2\theta}$$

$$= \frac{2\sin^2\theta + 2\cos^2\theta - 4\cos\theta}{\sin^2\theta}$$

$$\therefore \frac{dt}{d\theta} = \frac{2 - 4\cos\theta}{\sin^2\theta}$$

$$\frac{dt}{d\theta} = 0 \quad \text{when } 2 - 4\cos\theta = 0$$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\therefore \quad \theta = \frac{\pi}{3} \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

At $\theta = \frac{\pi}{3}$, there is a stationary point.

Test

θ	1	$\frac{\pi}{3}$	1.1
$\frac{dt}{d\theta}$	-0.2277	0	0.2337

Minimum time when $\theta = \frac{\pi}{3}$

Question 15 (c)

(ii) Some students used Pythagoras' theorem to determine AB = 4/(sin theta) rather than trigonometry, which was much more difficult. Those students who stated that the total time taken = AB/2 + CD/2 + BC/4 had the best responses.

(iii) Many students had trouble correctly differentiating, confusing the negative and positive signs. Many students had difficultly verifying their solution was a minimum. Many did not correctly evaluate the first derivative:- some chose theta=0, but the derivative does not exist here. The most successful responses tested values of 1 and 1.5.

Question 16

(a) $y = 3 - \dfrac{1}{\sqrt{x}}$

$$\frac{1}{\sqrt{x}} = 3 - y$$

$$\sqrt{x} = \frac{1}{3-y}$$

$$x = \frac{1}{(3-y)^2}$$

$$x^2 = \frac{1}{(3-y)^4}$$

$$V = \pi\int_{-1}^{2} x^2 \quad dy$$

$$\therefore V = \pi\int_{-1}^{2} \frac{1}{(3-y)^4}\, dy$$

$$= \pi\int_{-1}^{2} (3-y)^{-4} \quad dy$$

$$= \pi\left[\frac{(3-y)^{-3}}{3}\right]_{-1}^{2}$$

$$= \frac{\pi}{3}\left[\frac{1}{(3-y)^3}\right]_{-1}^{2}$$

$$= \frac{\pi}{3}\left(1 - \frac{1}{64}\right)$$

$$= \frac{21\pi}{64} \quad \text{u}^3$$

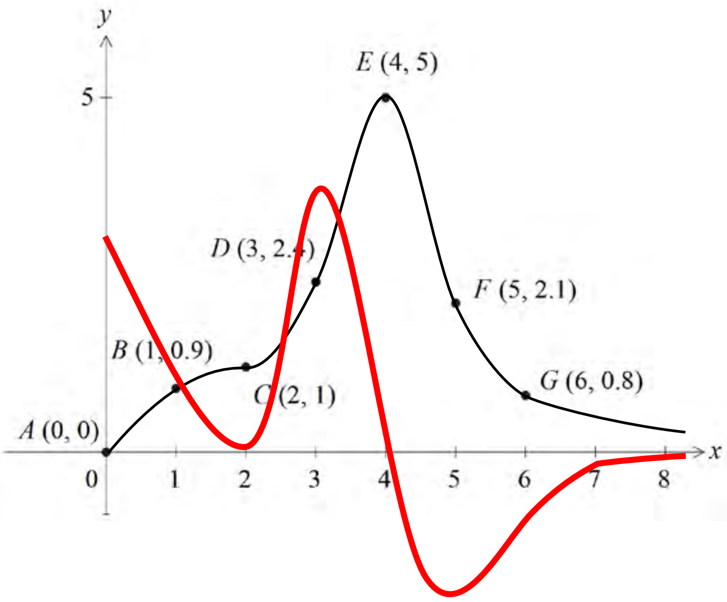
(b) (i) $h = 1$

x	0	1	2	3	4	5	6
$f(x)$	0	0.9	2.1	2.4	5	2.1	0.8

$$\therefore \text{ Area} \doteqdot \frac{1}{3}\left[0 + 4(0.9 + 2.4 + 2.1) + 2(2.1 + 5) + 0.8\right]$$

$$= \frac{61}{5} \quad \text{u}^2$$

(ii)



Marker's Comments

Question 16 (a)

The most costly errors occurred trying to integrate. Those that had expanded $(3-y)^2$ couldn't integrate correctly and failed to gain additional marks.
Volumes do not need to be split into parts above and below the axis.
If your volumes comes out to be negative, this means you have made a mistake. Don't just go back and add absolute value signs.
This is a simple definite integral. Integration by substitution is not required so don't waste time.

Question 16 (b)

(i) Generally well done. Some careless errors though. Some students simply just don't know Simpson's Rule.
Use of a table when calculating generally led to less errors.

(ii) Very poorly done.
This is an important skill which is often tested in the HSC.
Students should revise this.
x-values should be on the *x*-axis. Not points (*A*,*B*,*C*..)etc

(c) (i) Let $\angle ABC = x$ (i.e. $\angle ABO = x$)

$AB = AC$ (given)

$\therefore \angle ABC = \angle ACB$

(angles opposite equal sides in ΔABC)

$\therefore \angle ACB = x$ .... (1)

$OA = OB$ (radii of circle)

$\therefore \angle OAB = \angle OBA$

(angles opposite equal sides in ΔOAB)

$\therefore \angle OAB = x$ .... (2)

In ΔABC and ΔOAB

$\angle B$ is common

$\angle ACB = \angle OAB = x$ (from (1) and (2))

$\therefore \Delta ABC \,|||\, \Delta OAB$ (equiangular)

(ii) Let $\angle BAC = \theta$, then in ΔABC

$$\cos\theta = \frac{9^2 + 9^2 - 7^2}{2 \times 9 \times 9}$$

$$\theta = 0.7988...$$

$\therefore \theta = 0.799$ rad (to 3 decimal places)

In similar triangles ΔOAB and ΔABC

$\angle AOB = \angle BAC$

(matching angles in similar triangles)

$\therefore \angle AOB = 0.799$ rad

$\frac{AB}{OA} = \frac{BC}{AB}$ (matching sides in similar triangles)

$\therefore \frac{9}{OA} = \frac{7}{9}$

$\therefore OA = \frac{81}{7}$

Hence the area of segment

$$= \frac{1}{2} \times \left(\frac{81}{7}\right)^2 \times (0.799 - \sin 0.799) = 5.51...$$

$\therefore$ Area of shaded segment = 5.51 cm^2 (to 2 decimal places)

Question 16 (c)

(i) Please use 'angles opposite equal sides' rather than base angles of isosceles triangle. First need to establish ΔABO is isosceles by explaining why *AO* = *OB*. Many students did not provide sufficient reasoning.

(ii) This question is much simpler if students only work in radians rather than convert between radians and degrees. When using area of a sector or segment formula you must use radians. Many students rounded their answer too early and didn't maintain accuracy.