NORTH SYDNEY GIRLS HIGH SCHOOL

2016 TRIAL HSC EXAMINATION

Mathematics

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black pen.
- Board-approved calculators may be used.
- A reference sheet is provided.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.

Total Marks – 100

Section I Pages 3 - 6

10 marks

- Attempt Questions 1 -10
- Allow about 15 minutes for this section

Section II Pages 8 - 19

90 Marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:_______________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

QUESTION	MARKS
1-10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1- 10.

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1 What is 1.04976238 correct to four significant figures?

(A) 1.049
(B) 1.0497
(C) 1.0498
(D) 1.050

2 The quadratic equation $3x^2 + 5x - 2 = 0$ has roots α and β.

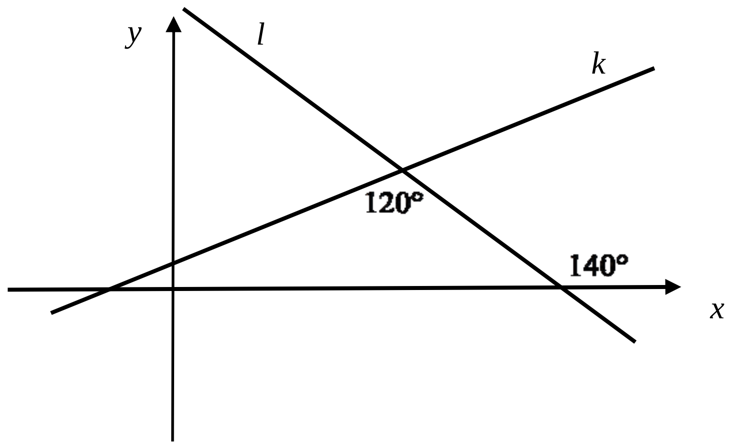
What is the value of $\alpha^2\beta + \alpha\beta^2$?

(A) $-\dfrac{10}{9}$
(B) $-\dfrac{9}{10}$
(C) $\dfrac{9}{10}$
(D) $\dfrac{10}{9}$

3 The first three terms of an arithmetic series are 12, 19 and 26.
What is the value of the 51st term?

(A) 350
(B) 357
(C) 362
(D) 369

4 The diagram shows two straight lines, l and k. Line l is inclined at an angle of $140°$ with the x-axis and the obtuse angle between the two lines is $120°$.



NOT TO SCALE

What is the gradient of line k correct to 2 decimal places?

(A) 0.16

(B) 0.36

(C) 0.84

(D) 1.73

5 What is the solution to the equation $\log_5 x^3 = 9$?

(A) $x = \dfrac{5}{3}$

(B) $x = 125$

(C) $x = \dfrac{5}{\sqrt[5]{3}}$

(D) $x = \sqrt[3]{5^9}$

6 How many solutions does the equation $\sin 2x = 0$ have in the domain $0 \le x \le 2\pi$?

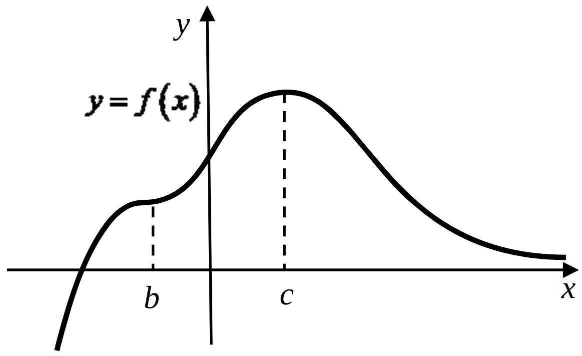
(A) 2

(B) 4

(C) 5

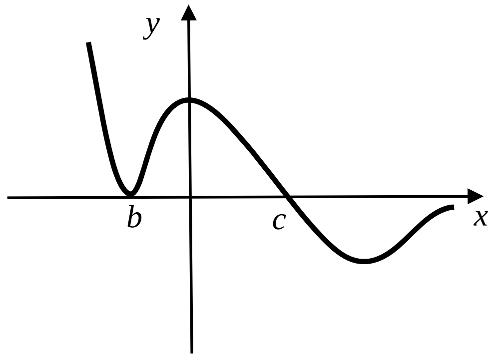
(D) 6

7 Given below is the graph of $y = f(x)$.

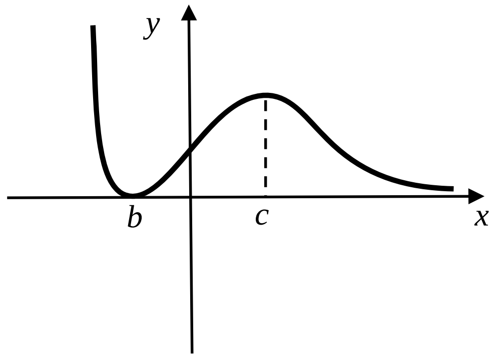


Which of the following could be the graph of $y = f'(x)$?

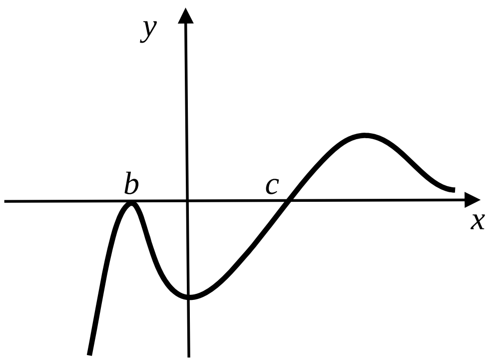
(A)



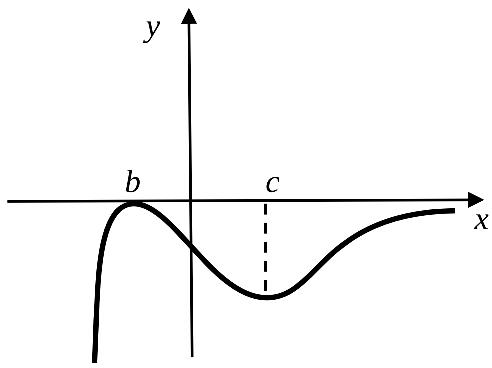
(B)



(C)



(D)



8 The first three terms of a geometric series are

$$3+\frac{3x}{2}+\frac{3x^2}{4}+\ldots$$

Which of the following represents all the possible values for x such that the series has a limiting sum?

(A) $|x| < 2$

(B) $|x| < \frac{1}{2}$

(C) $|x| \leq 2$

(D) $|x| \leq \frac{1}{2}$

9 How many solutions are there to the equation $3x^2 - |x-1| = 0$?

(A) 0

(B) 1

(C) 2

(D) 3

10 Let f and g be functions where $f'(2) = 2$, $g(2) = 1$, $f'(1) = 3$ and $g'(2) = -2$.

What is the gradient of the tangent to the curve $y = f\left[g(x)\right]$ at the point where $x = 2$?

(A) 6

(B) 3

(C) 2

(D) –6

Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hours 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Rationalise the denominator of $\dfrac{1}{2\sqrt{2}+3}$. **2**

(b) Fully factorise $x^4 - 8x$. **2**

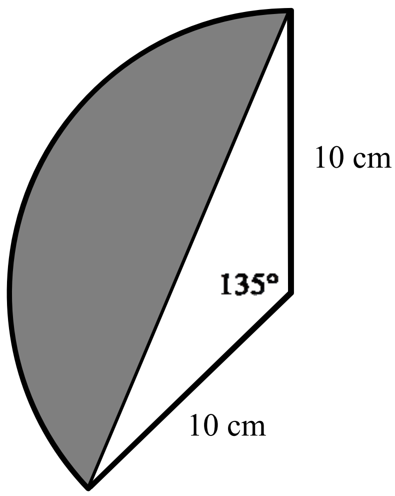
(c) Solve $|3x+1| \geq 4$. **2**

(d) Differentiate $\left(x+5e^{2x}\right)^4$. **2**

Question 11 continues on page 9

Question 11 (Continued)

(e) The diagram below shows a sector of a circle of radius 10 cm.
Find the area of the shaded segment. **2**



(f) Find the equation of the tangent to the curve $y = x^3 - 5x$ at the point $(1, -4)$. **2**

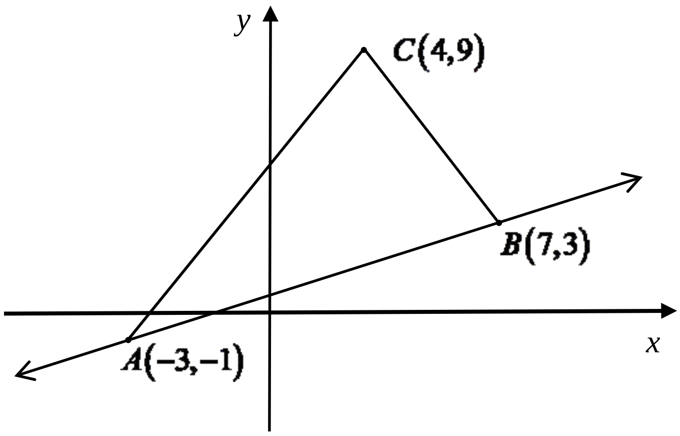
(g) Evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{3}\,dx$. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) In the diagram below $A(-3,-1)$, $B(7,3)$ and $C(4,9)$ are the vertices of a triangle.

NOT TO SCALE



(i) Find the distance *AB*. **1**

(ii) Show that the equation of line *AB* is $2x-5y+1=0$. **2**

(iii) Find the perpendicular distance from *C* to the line *AB*. **1**

(iv) Find the area of ΔABC. **1**

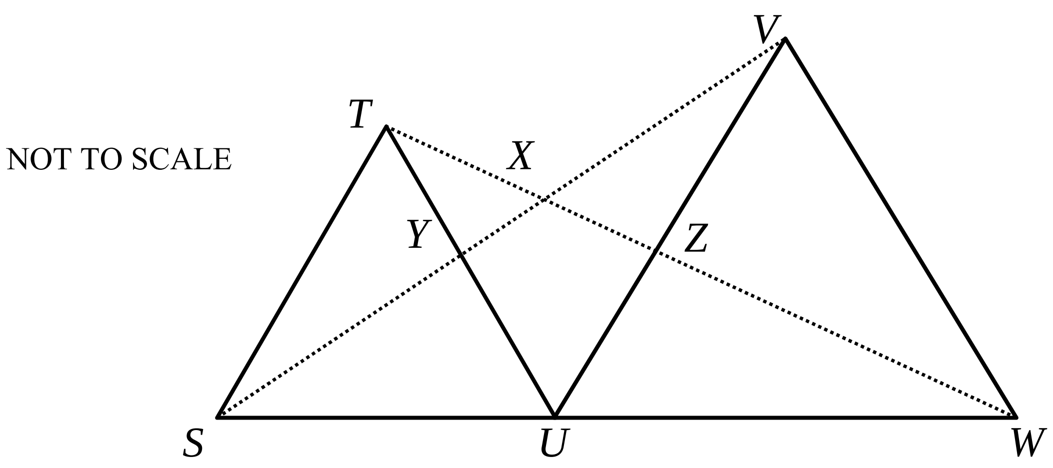
(b) T_n is the n^{th} term of a geometric series where $T_5=3$ and $T_8=24$. **2**

Find the first term and common ratio of this series.

Question 12 continues on page 11

(c) Solve the equation $\sin^2 x + 2\cos x = 1$ for $0 \leq x \leq 2\pi$. **3**

(d) In the diagram below, *STU* and *UVW* are both equilateral triangles.
SW is a straight line. *SV* intersects *TW* at *X*.



(i) Prove that $\Delta SVU \equiv \Delta TWU$. **3**

(ii) Hence or otherwise show that $\angle SXT = \angle SUT$. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$.

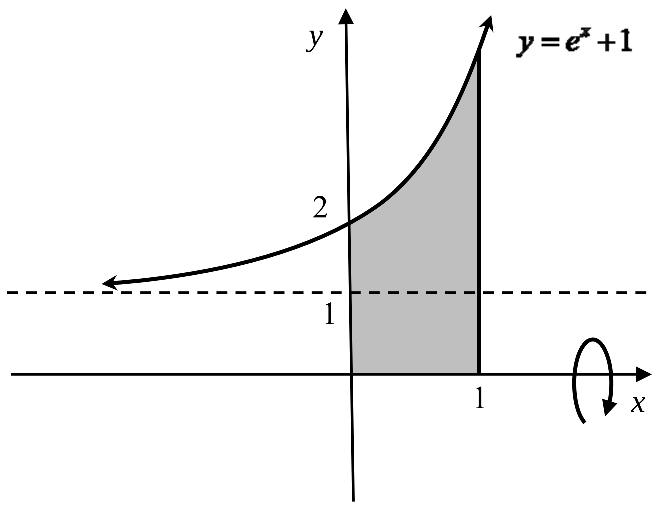
(i) Find the coordinates of any stationary points and determine their nature. **3**

(ii) Find the coordinates of any points of inflexion. **2**

(iii) Sketch the graph of $y = f(x)$. **2**

Do not attempt to find the x-intercepts

(b) The diagram below shows the function $f(x) = e^x + 1$. The region between the function, the x-axis, y-axis and the line $x = 1$ has been shaded.



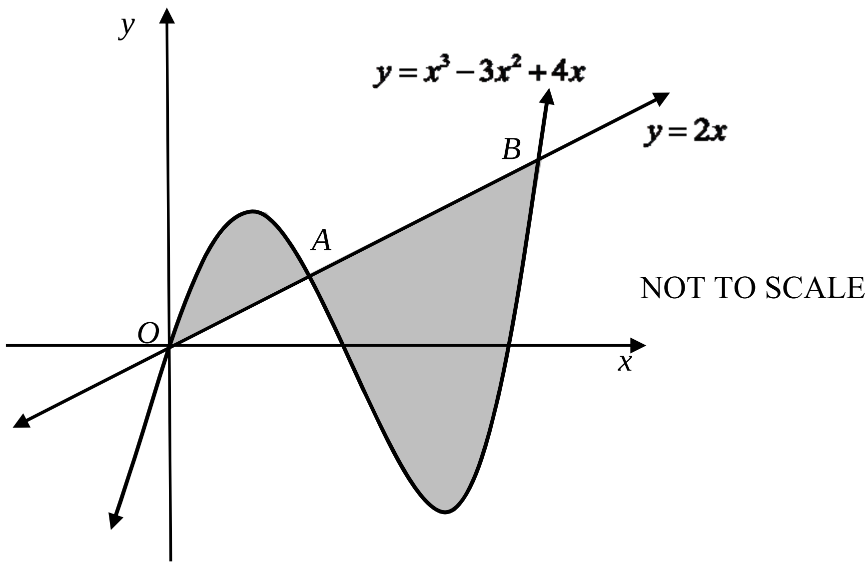
Find the volume of the solid generated when the shaded region is rotated around the x-axis. **3**

Question 13 continues on page 13

(c) The diagram shows the graphs of $y = 2x$ and $y = x^3 - 3x^2 + 4x$.

The functions intersect at the origin O, the point A and the point B.

The region between the two functions is shaded.



(i) By solving simultaneously, show that the x-coordinates of A and B are $x = 1$ and $x = 2$ respectively. **2**

(ii) Find the area of the shaded region. **3**

End of Question 13

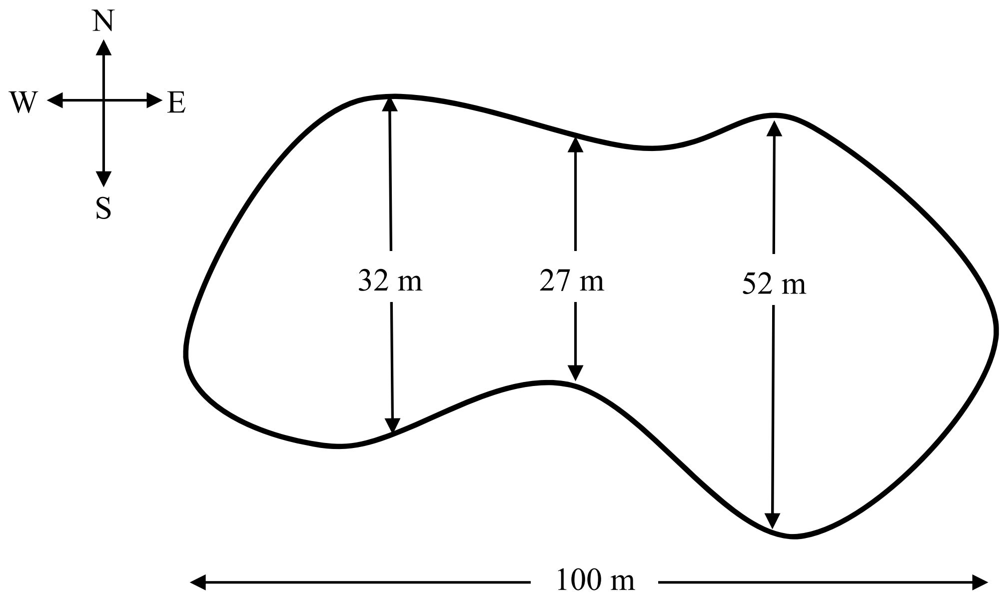
Question 14 (15 marks) Use a SEPARATE writing booklet

(a) (i) Differentiate $x^2 \ln x$. **2**

(ii) Hence, or otherwise, find $\int 5x\left(1+\ln x^2\right)dx$. **2**

(b) Find the coordinates of the focus of the parabola $y = -\frac{1}{8}x^2 + x - 1$. **3**

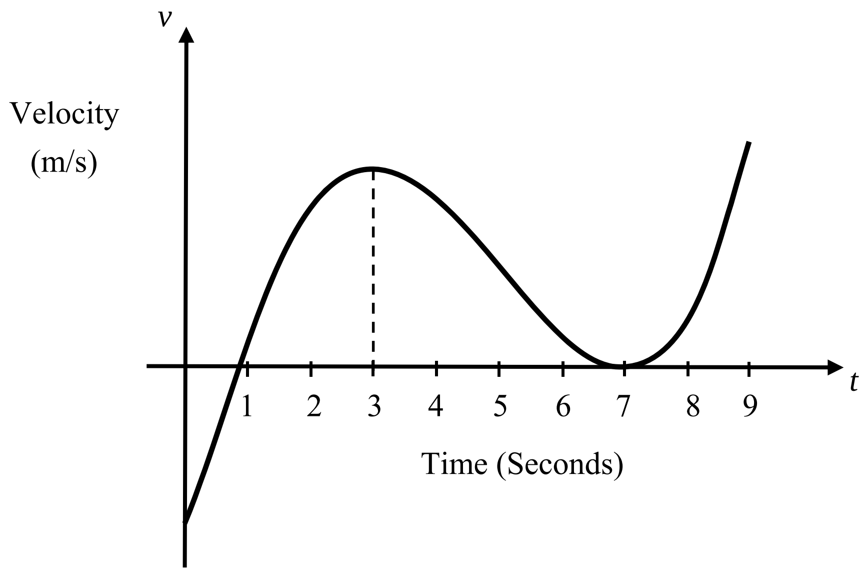
(c) The diagram below shows a lake that is 100 metres across from its most easterly point to its most westerly point. Three equally spaced measurements have been taken in a North – South direction across the lake as shown. **3**



Use Simpson's Rule with five function values to determine an approximate value for the area of the lake.

Question 14 continues on page 15

(d) The graph below is the velocity-time graph for an object moving in a straight line for $0 \le t \le 9$.



(i) At what time(s) is the object stationary? **1**

(ii) At what time(s) does the object change direction? **1**

(iii) At what time(s) is the acceleration of the object zero? **1**

(iv) Draw a possible displacement-time graph given that the object was initially at $x = 0$. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) Prove that $\dfrac{\cos\alpha}{1-\tan\alpha}+\dfrac{\sin\alpha}{1-\cot\alpha}=\sin\alpha+\cos\alpha$. **3**

(b) The mass M (in grams) of a radioactive carbon isotope called Carbon 14 $\left(C_{14}\right)$ that is found in a fossil is given by the equation

$$M = Ae^{-kt}$$

where A and k are positive constants and t is the number of years since the fossil began forming.

It is known that at the start a particular fossil contained 200 grams of Carbon 14 and it then decayed to 150 grams 2500 years later.

(i) Show that the rate of decay of C_{14} is proportional to the mass present in the fossil at any particular time. **1**

(ii) Find the exact value of k. **2**

(iii) Find the amount of time, to the nearest year, that it will take for the C_{14} to decay to 10 grams. **2**

Question 15 continues on page 17

(c) Alice decides to start saving for a deposit for a new apartment by investing \$1000 into a bank account at the start of every month.
Interest is paid at a rate of 3% per annum compounded monthly.

(i) Show that the amount A_n in the account after n months is: **2**

$$A_n = 401000\left(1.0025^n - 1\right)$$

(ii) Calculate the amount in the account at the end of 12 months to the nearest dollar. **1**

(iii) Determine how many deposits in total Alice will have made when her account first exceeds \$60 000. **2**

(iv) After twelve months Alice changes her contributions to \$1500 per month. Determine if Alice will have saved \$60 000 by the end of the third year.
Justify your answer. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) Consider the relation $\frac{y}{x}+\frac{x+2}{y+1}=2$ where $x \neq 0$ and $y \neq -1$.

(i) Show that $y^2+y(1-2x)+x^2=0$. **2**

(ii) Find the greatest integer value for x so that y is real and rational. **2**

(b) The level of a local dam is being monitored. After t days the rate at which the volume of water in the dam is changing is given by **4**

$$R=\frac{1000}{1+t}-500 ,$$

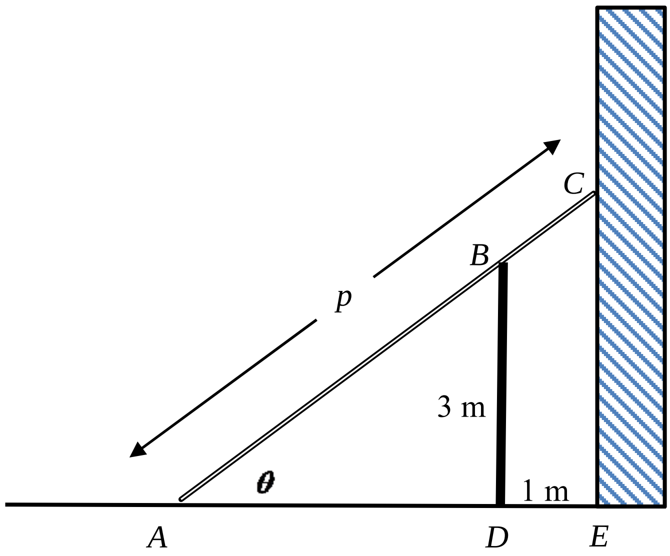
where R is the rate of change of the volume of water in the dam measured in megalitres/day.

The volume of water in the dam at the end of the fifth day of monitoring was half the volume at the end of the fourth day.

Find an expression for the volume of water W in the dam t days after the monitoring began.

Question 16 continues on page 19

(c) The diagram below shows an extension ladder AC which has a variable length p metres. The ladder leans against a wall EC whilst also touching a 3 metre high fence DB that is 1 metre from the wall.
Let θ be the angle of inclination of the ladder with the horizontal ground.



(i) Show that $p = \dfrac{3}{\sin\theta} + \dfrac{1}{\cos\theta}$. **2**

(ii) Show that p has a stationary point when $\tan\theta = \sqrt[3]{3}$. **3**

(iii) For safety reasons θ must remain between 60° and 75°. **2**
Find the minimum value of p to the nearest centimetre so that the ladder can be placed safely.

End of paper

NORTH SYDNEY GIRLS HIGH SCHOOL

2016 TRIAL HSC EXAMINATION

Mathematics

SOLUTIONS

Multiple Choice

1.	A	B	C	●
2.	A	B	C	●
3.	A	B	●	D
4.	A	●	C	D
5.	A	●	C	D
6.	A	B	●	D
7.	●	B	C	D
8.	●	B	C	D
9.	A	B	●	D
10.	A	B	C	●

Multiple Choice

1	What is 1.04976238 correct to four significant figures?

Answer is (D) 1.050

2	The quadratic equation $3x^2+5x-2=0$ has roots α and β. What is the value of $\alpha^2\beta+\alpha\beta^2$?

Sum of roots $\alpha+\beta=\frac{-b}{a}=\frac{-5}{3}$

Product of roots $\alpha\beta=\frac{c}{a}=\frac{-2}{3}$

$$\begin{aligned}\alpha^2\beta+\alpha\beta^2&=\alpha\beta(\alpha+\beta)\\&=\frac{-2}{3}\times\frac{-5}{3}\\&=\frac{10}{9}\end{aligned}$$

Amswer is (D) $\frac{10}{9}$

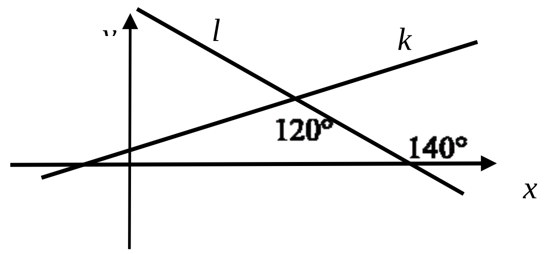
3	The first three terms of an arithmetic series are 12, 19 and 26. What is the value of the 51^{st} term?

$T_n=a+(n-1)d$

$$\begin{aligned}T_{51}&=(12)+((51)-1)(7)\\&=12+350\\&=362\end{aligned}$$

Answer is (c) 362

4	What is the gradient of line *k* correct to 2 decimal places?



Using exterior angle of a triangle the angle of inclination of line k is: $140°-120°=20°$

So $m_k=\tan 20°=0.3639702...$

Answer is (B) 0.36

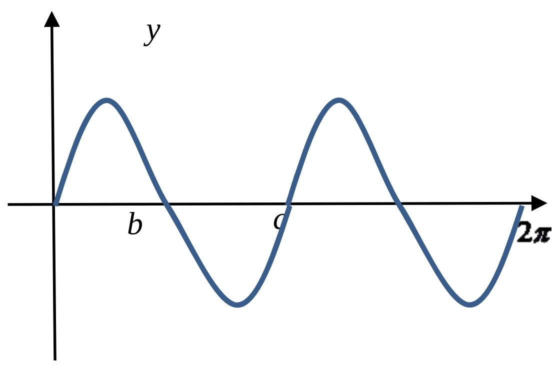
5.	What is the solution to the equation $\log_5 x^3=9$?

$$\begin{aligned}\log_5 x^3&=9\\3\log_5 x&=9\\\log_5 x&=3\\x&=5^3\\x&=125\end{aligned}$$

Answer is (B) $x=125$

6.	How many solutions does the equation $\sin 2x=0$ have in the domain $0\le x\le 2\pi$?

Sketch the graph of $y=\sin 2x$



Answer is (C) 5

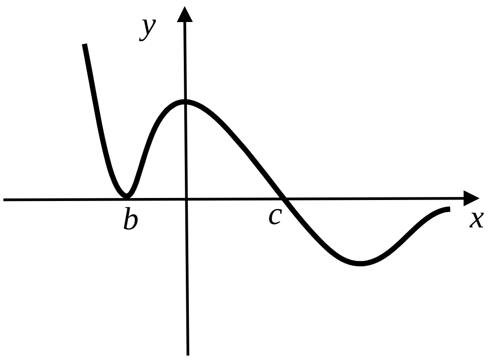
7. Given below is the graph of $y = f(x)$.

Which of the following could be the graph of $y = f'(x)$?

At $x = b$ the gradient is zero and to the left and right of $x = b$ the gradient is positive. So the answer can only be either A or B.

At $x = c$ the gradient is also zero.

So the answer must be A.



8. The first three terms of a geometric series are

$$3 + \frac{3x}{2} + \frac{3x^2}{4} + ...$$

Which of the following represents all the possible values for x such that
the series has a limiting sum?

The common ratio is $\frac{x}{2}$. If a geometric series has a limiting sum then $|r| < 1$

$\left|\frac{x}{2}\right| < 1$

$|x| < 2$

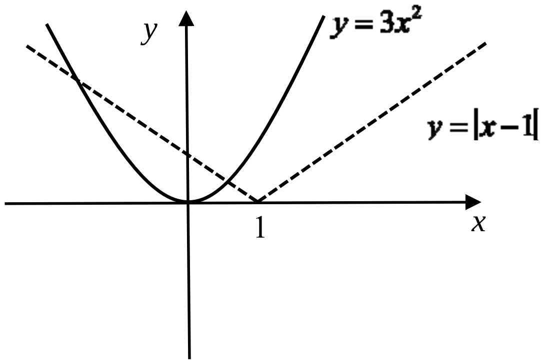
Answer is (A) $|x| < 2$

9. How many solutions are there to the equation $3x^2 - |x-1| = 0$

$3x^2 - |x-1| = 0$

$3x^2 = |x-1|$

Number of solutions are the number of intersections on the graphs of $y = 3x^2$ and $y = |x-1|$



Two intersections

Answer is (C) 2

10 Let f and g be functions where $f'(2) = 2$, $g(2) = 1$, $f'(1) = 3$ and $g'(2) = -2$.

What is the gradient of the tangent to the curve $y = f[g(x)]$ at the point where $x = 2$?

$\frac{d}{dx} f[g(x)] = f'[g(x)] \times g'(x)$ Chain Rule

Gradient at $x = 2$ is :

$$\begin{aligned} m &= f'[g(2)] \times g'(2) \\ &= f'(1) \times (-2) \\ &= (3) \times (-2) \\ &= -6 \end{aligned}$$

Answer is (D) -6

Question 11

(a)	Rationalise the denominator of $\dfrac{1}{2\sqrt{2}+3}$.	**2**

SOLUTION:

$$\begin{aligned}\frac{1}{2\sqrt{2}+3} &= \frac{1}{2\sqrt{2}+3}\times\frac{2\sqrt{2}-3}{2\sqrt{2}-3}\\ &= \frac{2\sqrt{2}-3}{\left(2\sqrt{2}\right)^2-3^2}\\ &= \frac{2\sqrt{2}-3}{8-9}\\ &= 3-2\sqrt{2}\end{aligned}$$

(b)	Fully factorise x^4-8x .	**2**

SOLUTION:

$$\begin{aligned}x^4-8x &= x\left(x^3-2^3\right)\\ &= x(x-2)\left(x^2+2x+4\right)\end{aligned}$$

(c)	Solve $\left|3x+1\right|\geq 4$.	**2**

SOLUTION:

$\left|3x+1\right|\geq 4$

$3x+1\geq 4$ *OR* $3x+1\leq -4$

$3x\geq 3$ *OR* $3x\leq -5$

$x\geq 1$ *OR* $x\leq \dfrac{-5}{3}$

(d)	Differentiate $\left(x+5e^{2x}\right)^4$.	**2**

SOLUTION:

$$\frac{d}{dx}\left(x+5e^{2x}\right)^4 = 4\left(x+5e^{2x}\right)^3\left(1+10e^{2x}\right)$$

(e)	Find the area of the shaded segment.	**2**

SO
LU

TION:

Area of a segment is equal to:

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

The angle must be in radians

$$\theta = \frac{135\pi}{180} = \frac{3\pi}{4}$$

$$= \frac{1}{2}10^2\left(\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right)$$

$$= 50\left(\frac{3\pi}{4} - \frac{1}{\sqrt{2}}\right)\text{cm}^2$$

(f)	Find the equation of the tangent to the curve $y = x^3 - 5x$ at the point $(1,-4)$.	**2**

SOLUTION:

$$y = x^3 - 5x$$

$$\frac{dy}{dx} = 3x^2 - 5$$

Gradient at $(1,-4)$

$$m = 3(1)^2 - 5 = -2$$

Equation of tangent at $(1,-4)$

$$y - 4 = -2(x-1)$$

$$y = -2x + 2 - 4$$

$$y = -2x - 2$$

(g)	Evaluate $\int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{3}\,dx$.	**3**

SOLUTION:

$$\int_{\frac{\pi}{2}}^{\pi} \cos\frac{x}{3}\,dx = \left[3\sin\frac{x}{3}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \left(3\sin\frac{\pi}{3}\right) - \left(3\sin\frac{\pi}{6}\right)$$

$$= 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \frac{3\sqrt{3}-3}{2}$$

Question 12 (15 marks) Use a SEPARATE writing booklet

(a)

(i)	Find the distance *AB*.	**1**

SOLUTION:

$$d=\sqrt{(-3-7)^2+(-1-3)^2}$$
$$d=\sqrt{(-10)^2+(-4)^2}$$
$$d=\sqrt{100+16}$$
$$d=2\sqrt{29}$$

(ii)	Show that the equation of line *AB* is $2x-5y+1=0$.	**2**

SOLUTION:

Gradient of line

$$m=\frac{-1-3}{-3-7}$$
$$=\frac{-4}{-10}=\frac{2}{5}$$

Equation of line

$$y-3=\frac{2}{5}(x-7)$$
$$5y-15=2x-14$$
$$0=2x-5y-14+15$$
$$0=2x-5y+1$$

(iii)	Find the perpendicular distance from *C* to the line *AB*.	**1**

SOLUTION:

$$d=\left|\frac{2(4)-5(9)+1}{\sqrt{2^2+5^2}}\right|$$
$$d=\left|\frac{8-45+1}{\sqrt{4+25}}\right|$$
$$d=\left|\frac{-36}{\sqrt{29}}\right|$$
$$d=\frac{36}{\sqrt{29}}$$

(iv) Find the area of ΔABC.	1

$$Area_{ABC} = \frac{1}{2}(AB)\times(\text{Distance from } C \text{ to } AB)$$
$$= \frac{1}{2}\left(2\sqrt{29}\right)\times\left(\frac{36}{\sqrt{29}}\right)$$
$$= 36$$

(b) T_n is the n^{th} term of a geometric series where $T_5 = 3$ and $T_8 = 24$. Find the first term and common ratio of this series.	2

$T_n = ar^{n-1}$

$T_5 = 3 = ar^4$ $\quad I$

and

$T_8 = 24 = ar^7$ $\quad II$

$II \div I$

$$\frac{T_8}{T_5} = \frac{ar^7}{ar^4}$$
$$\frac{24}{3} = r^3$$
$$8 = r^3$$
$$r = 2$$

Sub in to I

$$3 = a(2)^4$$
$$3 = 16a$$
$$a = \frac{3}{16}$$

First term is $\frac{3}{16}$ and the common ratio is 2.

(c) Solve the equation $\sin^2 x + 2\cos x = 1$ for .	3

SOLUTION:

$$\sin^2 x + 2\cos x = 1$$
$$\left(1-\cos^2 x\right) + 2\cos x = 1$$

Let $u = \cos x$

$$\left(1-u^2\right) + 2u = 1$$
$$-u^2 + 2u = 0$$
$$u(2-u) = 0$$

$u = 0$ or $u = 2$

$\cos x = 0$ or $\cos x = 2$

Solve by graphing — NO Solution

$$x = \frac{\pi}{2} \text{ or } x = \frac{3\pi}{2}$$

(d)

(i) Prove that $\Delta SVU \equiv \Delta TWU$.	3

$\angle SUT = 60$ angle in a equilateral triangle
$\angle WUV = 60$ angle in a equilateral triangle

$\angle SUT + \angle TUV + \angle WUV = 180$ (Straight line)
$60 + \angle TUV + 60 = 180$
$\angle TUV = 60$

$\therefore \angle SUV = \angle TUW = 120°$

In ΔSVU and ΔTWU

$SU = TU$	Side of equilateral triangle STU
$WU = VU$	Side of equilateral triangle WVU
$\angle SUV = \angle TUW = 120°$	See above
$\therefore \Delta SVU \equiv \Delta TWU$	(SAS)

(ii) Hence or otherwise show that $\angle SXT = \angle SUT$.	2

In ΔTYX and ΔSYU

$\angle YSU = \angle YTX$	Matching angles in congruent triangles
$\angle UYS = \angle XYT$	Vertically Opposite angles
$\therefore \Delta TYX \parallel\!\mid \Delta SYU$	Equiangular
$\therefore \angle SXT = \angle SUT$	Matching angles in similar triangle

Question 13

(a)

(i)	Find the coordinates of any stationary points and determine their nature.	**3**

Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$.

$$f(x) = x^3 - 3x^2 - 9x + 5$$
$$f'(x) = 3x^2 - 6x - 9$$
$$f''(x) = 6x - 6$$

Stationary points where $f'(x) = 0$

$$3(x^2 - 2x - 3) = 0$$
$$3(x-3)(x+1) = 0$$
$$x = 3 \quad or \quad x = -1$$

Sub values into original function to find y-values and test *x*-values in the second derivative to determine their nature

At $x = 3$

$$\begin{aligned} f(3) &= (3)^3 - 3(3)^2 - 9(3) + 5 \\ &= 27 - 27 - 27 + 5 \\ &= -22 \end{aligned}$$

$$\begin{aligned} f''(3) &= 6(3) - 6 \\ &= 12 \end{aligned}$$

$\therefore$ Minimum turning point at $(3, -22)$

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 - 9(-1) + 5 \\ &= -1 - 3 + 9 + 5 \\ &= 10 \end{aligned}$$

$$\begin{aligned} f''(-1) &= 6(-1) - 6 \\ &= -12 \end{aligned}$$

$\therefore$ Maximum turning point at $(10, -12)$

(ii)	Find the coordinates of any points of inflexion.	**2**

$\therefore$ Maximum turning point at $(-1,10)$

Possible point of inflexion where $f'''(x)=0$

$$\begin{aligned} 6x-6&=0 \\ x-1&=0 \\ x&=1 \end{aligned}$$

Test for change in concavity at $x=1$

x	0.9	1	1.1
$f'''(x)$	–0.6	0	0.6

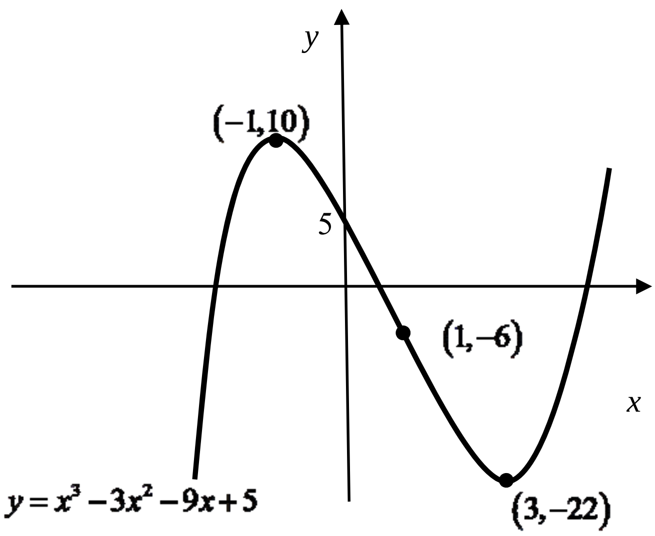
$\therefore$ Change in concavity at $x=1$

y- coordinate:

$$\begin{aligned} f(1)&=(1)^3-3(1)^2-9(1)+5 \\ &=1-3-9+5 \\ &=-6 \end{aligned}$$

(iii)	Sketch the graph of $y=f(x)$. Do not attempt to find the x-intercepts	**2**

Point of inflexion is at $(1,–6)$



(b)

Find the volume of the solid generated when the shaded region is rotated around the x-axis.	**3**

$$V = \pi\int_0^1 y^2 dx$$

$$= \pi\int_0^1 \left(e^x + 1\right)^2 dx$$

$$= \pi\int_0^1 \left(e^{2x} + 2e^x + 1\right) dx$$

$$= \pi\left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^1$$

$$= \pi\left[\left(\frac{e^{2(1)}}{2} + 2e^{(1)} + (1)\right) - \left(\frac{e^{2(0)}}{2} + 2e^{(0)} + (0)\right)\right]$$

$$= \pi\left[\left(\frac{e^2}{2} + 2e + 1\right) - \left(\frac{1}{2} + 2\right)\right]$$

$$= \pi\left(\frac{e^2}{2} + 2e - 1\frac{1}{2}\right)\text{units}^3$$

(c)

(i)	By solving simultaneously, show that the *x*-coordinates of *A* and *B* are $x = 1$ and $x = 2$ respectively.	**2**

$y = x^3 - 3x^2 + 4x$ and $y = 2x$

$2x = x^3 - 3x^2 + 4x$

$0 = x^3 - 3x^2 + 2x$

$0 = x\left(x^2 - 3x + 2\right)$

$0 = x(x-2)(x-1)$

$x = 0 \quad x = 1 \quad x = 2$

Since O is the origin then A and B have x coordinates of $x = 1$ and $x = 2$ respectively.

(ii)	Find the area of the shaded region.	**3**

$$A = \int_0^1 \left(\left(x^3 - 3x^2 + 4x\right) - (2x)\right)dx + \int_1^2 \left((2x) - \left(x^3 - 3x^2 + 4x\right)\right)$$

$$= \int_0^1 \left(x^3 - 3x^2 + 2x\right)dx + \int_1^2 \left(-x^3 + 3x^2 - 2x\right)dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 + \left[-\frac{x^4}{4} + x^3 - x^2\right]_1^2$$

$$= \left(\left(\frac{1}{4} - 1 + 1\right) - (0)\right) + \left(\left(-\frac{2^4}{4} + 2^3 - 2^2\right) - \left(-\frac{1}{4} + 1 - 1\right)\right)$$

$$= \frac{1}{4} + \left((0) - \left(-\frac{1}{4}\right)\right)$$

$$= \frac{1}{2} units^2$$

Question 14

(a)

(i)	Differentiate $x^2 \ln x$.	**2**

$$\frac{d}{dx} x^2 \ln x = 2x \ln x + x^2 \frac{1}{x} \qquad \text{Product Rule}$$
$$= 2x \ln x + x$$

(ii)	Hence, or otherwise, find $\int 5x\left(1+\ln x^2\right)dx$.	**2**

$$\int 5x\left(1+\ln x^2\right)dx = 5\int x\left(1+2\ln x\right)dx$$
$$= 5\int \left(x+2x\ln x\right)dx$$
$$= 5x^2 \ln x + C \qquad \text{From part i)}$$

(b)	Find the coordinates of the focus of the parabola $y = -\frac{1}{8}x^2 + x - 1$.	**3**

$$y = -\frac{1}{8}x^2 + x - 1$$
$$-8y = x^2 - 8x + 8$$
$$-8y = x^2 - 8x + 16 - 8 \qquad \text{Completing the square}$$
$$-8y = \left(x-4\right)^2 - 8$$
$$-8y + 8 = \left(x-4\right)^2$$
$$-8\left(y-1\right) = \left(x-4\right)^2$$

Using the standard form $\left(x-h\right)^2 = -4a\left(y-k\right)$ the vertex is $\left(4,1\right)$ focal length is 2 and parabola is concave down.

Focus is 2 units below the vertex.

Therefor the focus is $\left(4,-1\right)$

(c)

Use Simpson's Rule with five function values to determine an approximate value for the area of the lake.

y	0	32	27	52	0
k	1	4	2	4	1
ky	0	125	54	208	0

$$A \doteqdot \frac{h}{3}\sum ky$$
$$\doteqdot \frac{25}{3}(128+54+208)$$
$$\doteqdot \frac{25}{3}(390)$$
$$\doteqdot 3250\text{m}^2$$

(d)

(i)	At what time(s) is the object stationary?	**1**

$t=1$ and $t=7$

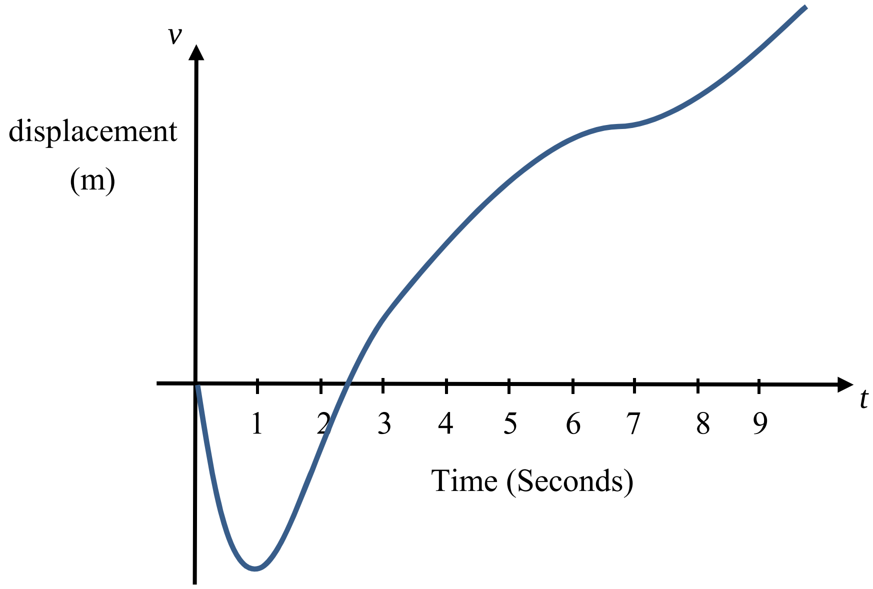
(ii)	At what time(s) does the object change direction?	**1**

$t=1$

(iii)	At what time(s) is the acceleration of the object zero?	**1**

$t=1$ and $t=7$

(iv)	Draw a possible displacement-time graph given that the object was initially at $x=0$.	**2**


v
displacement
(m)
t
1
2
3
4
5
6
7
8
9
Time (Seconds)

Question 15

(a)	Prove that $\dfrac{\cos\alpha}{1-\tan\alpha}+\dfrac{\sin\alpha}{1-\cot\alpha}=\sin\alpha+\cos\alpha$.	**3**

$$
\begin{aligned}
LHS &= \frac{\cos\alpha}{1-\tan\alpha}+\frac{\sin\alpha}{1-\cot\alpha}\\
&= \frac{\cos\alpha}{1-\dfrac{\sin\alpha}{\cos\alpha}}+\frac{\sin\alpha}{1-\dfrac{\cos\alpha}{\sin\alpha}}\\
&= \frac{\cos^2\alpha}{\cos\alpha-\sin\alpha}+\frac{\sin^2\alpha}{\sin\alpha-\cos\alpha}\\
&= \frac{\cos^2\alpha}{\cos\alpha-\sin\alpha}-\frac{\sin^2\alpha}{\cos\alpha-\sin\alpha}\\
&= \frac{\cos^2\alpha-\sin^2\alpha}{\cos\alpha-\sin\alpha}\\
&= \frac{(\cos\alpha-\sin\alpha)(\cos\alpha+\sin\alpha)}{\cos\alpha-\sin\alpha}\\
&= \sin\alpha+\cos\alpha\\
&= RHS
\end{aligned}
$$

	(i)	Show that the rate of decay of C_{14} is proportional to the mass present in the fossil at any particular time.	**1**

(b)

If $M = Ae^{-kt}$ then the rate of decay is $\dfrac{dM}{dt}$

$$
\begin{aligned}
\frac{dM}{dt} &= -kAe^{-kt}\\
&= -kM
\end{aligned}
$$

	(ii)	Find the exact value of k.	**2**

the rate of decay of C_{14} is proportional to the mass present in the fossil.

When $t=0$ $M=200$

Using $M = Ae^{-kt}$

$$
\begin{aligned}
200 &= Ae^{-k(0)}\\
200 &= A
\end{aligned}
$$

and when $t = 2500$ $M = 150$

$$150 = 200e^{-2500k}$$

$$\frac{150}{200} = e^{-2500k}$$

$$\frac{3}{4} = e^{-2500k}$$

$$-2500k = \ln\frac{3}{4}$$

$$k = \frac{\ln\frac{3}{4}}{-2500}$$

(iii)	Find the amount of time, to the nearest year, that it will take for the C_{14} to decay to 10 grams.	**2**

$$10 = 200e^{-kt}$$

$$\frac{10}{200} = e^{-kt}$$

$$\frac{1}{20} = e^{-kt}$$

$$-kt = \ln\frac{1}{20}$$

$$t = \frac{\ln\frac{1}{20}}{-k}$$

$$t = 2500\left(\frac{\ln\frac{1}{20}}{\ln\frac{3}{4}}\right)$$

$t = 26033$ years

(c)

(i)	Show that the amount A_n in the account after n months is: $$A_n = 401000\left(1.0025^n - 1\right)$$	**2**

Since A_n is the amount of money in the account after n months then

$$A_1 = 1000\left(1 + \frac{0.03}{12}\right)$$
$$= 1000(1.0025)$$

$$A_2 = (A_1 + 1000)(1.0025)$$
$$= \left(1000(1.0025) + 1000\right)(1.0025)$$
$$= \left(1000(1.0025) + 1000\right)(1.0025)$$
$$= 1000(1.0025)^2 + 1000(1.0025)$$

$$A_3 = (A_2 + 1000)(1.0025)$$
$$= \left(1000(1.0025)^2 + 1000(1.0025) + 1000\right)(1.0025)$$
$$= 1000(1.0025)^3 + 1000(1.0025)^2 + 1000(1.0025)$$

$$A_n = 1000(1.0025)^n + 1000(1.0025)^{n-1} + 1000(1.0025)^{n-1} + \ldots + 1000(1.0025)$$
$$A_n = 1000\left[(1.0025)^n + (1.0025)^{n-1} + (1.0025)^{n-2} + \ldots + (1.0025)\right]$$
$$A_n = 1000\left[\frac{a\left(r^n - 1\right)}{r - 1}\right] \qquad \begin{array}{l} r = 1.0025 \\ a = 1.0025 \end{array}$$
$$A_n = 1000\left[\frac{1.0025\left(1.0025^n - 1\right)}{1.0025 - 1}\right]$$
$$A_n = 1000\left[\frac{1.0025\left(1.0025^n - 1\right)}{0.0025}\right]$$
$$A_n = 1000\left[\frac{1.0025\left(1.0025^n - 1\right)}{0.0025}\right]$$
$$A_n = 1000\left[401\left(1.0025^n - 1\right)\right]$$
$$= 401000\left(1.0025^n - 1\right)$$

(ii)	Calculate the amount in the account at the end of 12 months to the nearest dollar.	**1**

$$
\begin{aligned}
A_{12} &= 401000\left(1.0025^{12}-1\right) \\
&= 12196.79872 \\
&= \$12197 \qquad \text{To the nearest dollar}
\end{aligned}
$$

(iii)	Determine how many deposits in total Alice will have made when her account first exceeds \$60 000.	**2**

$$
\begin{aligned}
60\,000 &= 401000\left(1.0025^{n}-1\right) \\
\frac{60}{401} &= 1.0025^{n}-1 \\
\frac{461}{401} &= 1.0025^{n} \\
n &= \log_{1.0025}\left(\frac{461}{401}\right) \\
n &= \frac{\ln\left(\frac{461}{401}\right)}{\ln 1.0025} \\
n &= 55.84433557....
\end{aligned}
$$

Alice will have made 56 deposits when her account first exceeds \$60000

(iv)	After twelve months (and after the twelfth deposit has been made), Alice changes her contributions to \$1500 per month. Determine if Alice will have saved \$60 000 by the end of the third year. Justify your answer.	**2**

$$A_{12} = 401000\left(1.0025^{12} - 1\right)$$

$$\begin{aligned} A_{13} &= \left(A_{12} + 1500\right)1.0025 \\ &= A_{12}\left(1.0025\right) + 1500\left(1.0025\right) \end{aligned}$$

$$\begin{aligned} A_{14} &= \left(A_{13} + 1500\right)1.0025 \\ &= \left(A_{12}\left(1.0025\right) + 1500\left(1.0025\right) + 1500\right)1.0025 \\ &= A_{12}\left(1.0025\right)^2 + 1500\left(1.0025\right)^2 1500\left(1.0025\right) \end{aligned}$$

$$\begin{aligned} A_{15} &= \left(A_{14} + 1500\right)1.0025 \\ &= \left(A_{12}\left(1.0025\right)^2 + 1500\left(1.0025\right)^2 1500\left(1.0025\right) + 1500\right)1.0025 \\ &= A_{12}\left(1.0025\right)^3 + 1500\left(1.0025\right)^3 1500\left(1.0025\right)^2 + 1500\left(1.0025\right) \end{aligned}$$

$$\begin{aligned} A_{36} &= A_{12}\left(1.0025\right)^{24} + 1500\left(1.0025\right)^{24} 1500\left(1.0025\right)^{23} + \ldots + 1500\left(1.0025\right) \\ &= A_{12}\left(1.0025\right)^{24} + 1500\left[\frac{1.0025\left(\left(1.0025\right)^{24} - 1\right)}{0.0025}\right] \\ &= 401000\left(1.0025^{12} - 1\right)\left(1.0025\right)^{24} + 1500\left[\frac{1.0025\left(\left(1.0025\right)^{24} - 1\right)}{0.0025}\right] \\ &= \$50096.89 \end{aligned}$$

Alice had not reach \$60000 by the end of the third year

Question 16

(a) Consider the relation $\dfrac{y}{x}+\dfrac{x+2}{y+1}=2$ where $x\neq 0$ and $y\neq -1$.

(i)	Show that $y^2+y(1-2x)+x^2=0$.	**2**

$$\frac{y}{x}+\frac{x+2}{y+1}=2$$
$$\frac{y(y+1)+x(x+2)}{x(y+1)}=2$$
$$y(y+1)+x(x+2)=2x(y+1)$$
$$y^2+y+x^2+2x=2xy+2x$$
$$y^2+y-2xy+x^2=0$$
$$y^2+y(1-2x)+x^2=0$$

(ii)	Find the greatest integer value for x so that y is real and rational.	**2**

Using the quadratic formula for an equation in y

$$y=\frac{b\pm\sqrt{b^2 4ac}}{2a} \quad \text{Where } a=1\ ,\ b=1-2x \text{ and } c=x^2$$
$$y=\frac{-(1-2x)\pm\sqrt{(1-2x)^2\, 4(1)(x^2)}}{2}$$
$$y=\frac{2x-1\pm\sqrt{1-4x+4x^2-4x^2}}{2}$$
$$y=\frac{2x-1\pm\sqrt{1-4x}}{2}$$

If y is to be real and rational then the discriminant $1-4x$ must be positive and a perfect square

$$1-4x\geq 0$$
$$-4x\geq -1$$
$$x\leq\frac{1}{4}$$

Now test integer values for $x \le \frac{1}{4}$ to find the greatest integer that gives a perfect square

Let $x = 0$

$1-4(0)=1$ which is a perfect square but $x \neq 0$

Let $x = -1$

$1-4(-1)=5$ which is not a perfect square

Let $x = -2$

$1-4(-2)=9$ which is a perfect square

So the greatest integer value for x for which y is real and rational is $x = -2$

(b)	The level of a local dam is being monitored. After *t* days the rate at which the volume of water in the dam is changing is given by $$R = \frac{1000}{1+t} - 500 \; ,$$ where *R* is the rate of change of the volume of water in the dam measured in megalitres/day. The volume of water in the dam at the end of the fifth day of monitoring was half the volume at the end of the fourth day. Find an expression for the volume of water *W* in the dam *t* days after the monitoring began.	**4**

$W = \int R\,dt$

$W = \int \left(\frac{1000}{1+t} - 500 \right) dt$

$W = 1000 \ln(1+t) - 500t + C$

When $t = 4$

$W = 1000 \ln(5) - 2000 + C$

When $t = 5$

$W = 1000 \ln(6) - 2500 + C$

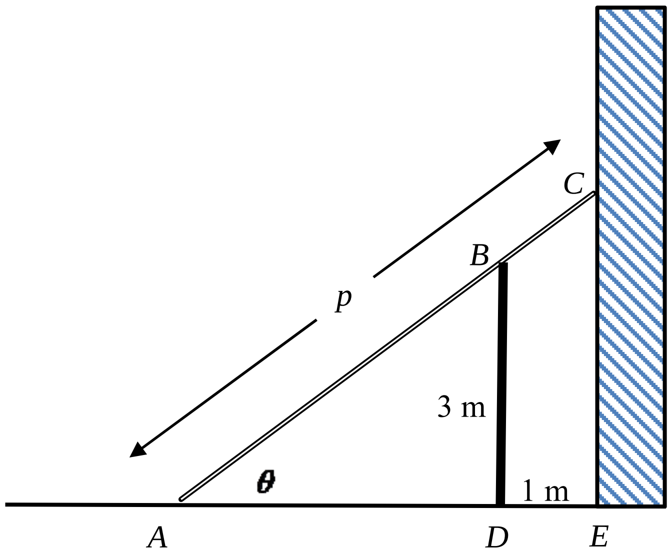
$$2\left(1000\ln(6)-2500+C\right)=1000\ln(5)-2000+C$$
$$2000\ln(6)-5000+2C=1000\ln(5)-2000+C$$
$$C=1000\ln(5)-2000\ln(6)+5000-2000$$
$$C=1000\left(\ln(5)-2\ln(6)\right)+3000$$
$$C=1000\left(\ln\left(\frac{5}{36}\right)\right)+3000$$

The expression for the volume of water W in the dam t days after the monitoring began is:

$$W=1000\ln(1+t)-500t+1000\left(\ln\left(\frac{5}{36}\right)\right)+3000$$

Question 16 (Continued)

(c) The diagram below shows an extension ladder *AC* which has a variable length *p* metres. The ladder leans against a wall *EC* whilst also touching a 3 metre high fence *DB* that is 1 metre from the wall.
Let θ be the angle of inclination of the ladder with the horizontal ground.



(i)	Show that $p = \dfrac{3}{\sin\theta} + \dfrac{1}{\cos\theta}$.	**2**

In ΔABD

$$\tan\theta = \frac{3}{AD}$$

$$AD = \frac{3}{\tan\theta} \qquad I$$

In ΔACE

$$\cos\theta = \frac{AD+1}{p}$$

$$p = \frac{AD+1}{\cos\theta} \qquad II$$

Sub I into II

$$p = \frac{\frac{3}{\tan\theta}+1}{\cos\theta}$$

$$= \frac{\frac{3}{\tan\theta}}{\cos\theta} + \frac{1}{\cos\theta}$$

$$= \frac{3}{\tan\theta\cos\theta} + \frac{1}{\cos\theta}$$

$$= \frac{3}{\sin\theta} + \frac{1}{\cos\theta}$$

(ii)	Show that p has a stationary point when $\tan\theta = \sqrt[3]{3}$.	**3**

$$p = 3\left(\sin\theta\right)^{-1} + \left(\cos\theta\right)^{-1}$$
$$\frac{dp}{d\theta} = -3\left(\sin\theta\right)^{-2} \times \cos\theta - \left(\cos\theta\right)^{-2} \times \left(-\sin\theta\right)$$
$$\frac{dp}{d\theta} = -\frac{3\times\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta}$$
$$\frac{dp}{d\theta} = \frac{-3\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta}$$

Stationary point where $\frac{dp}{d\theta} = 0$

$$\frac{-3\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta} = 0$$
$$-3\cos^3\theta + \sin^3\theta = 0$$
$$\sin^3\theta = 3\cos^3\theta$$
$$\frac{\sin^3\theta}{\cos^3\theta} = 3$$
$$\tan^3\theta = 3$$
$$\tan\theta = \sqrt[3]{3}$$

(iii)	For safety reasons θ must remain between $60°$ and $75°$. Find the minimum value of p to the nearest centimetre so that the ladder can be placed safely.	**2**

If $\tan\theta = \sqrt[3]{3}$ then $\theta = 55°15'51''$ which is not within the safe values for θ.

Test the two extreme values safe values for θ to see which one has the lowest value.

$$p = \frac{3}{\sin 60°} + \frac{1}{\cos 60°}$$
$$= 5.464101615...$$

$$p = \frac{3}{\sin 75°} + \frac{1}{\cos 75°}$$
$$= 6.446337718...$$

Thus the minimum value for p that is safe is 5.46m.