NAME: _____

STUDENT No: _____

YEAR 12 TRIAL HSC EXAMINATION

2004

MATHEMATICS

Time Allowed: 3 hours
(plus 5 minutes reading time)

INSTRUCTIONS FOR CANDIDATES

- Start each question in a new booklet.
- All questions are to be answered.
- All questions are of equal value.
- The marking scheme has been given on the right hand side of the page.
- Standard integrals are printed on the back page.
- Approved calculators may be used.

DHL.LJF 19.7.04

Question 1.
(Start a new booklet)

Marks

(a) $|3-x|=7$ **2**

(b) Differentiate $y=2\sqrt{x}+\cos x$ with respect to x. **2**

(c) An arc of length 8 cm subtends an angle of θ at the centre of a circle with radius 3 cm. Find θ to the nearest degree. **2**

(d) Differentiate $f(x)=\ln\left(\frac{x+1}{x-1}\right)$. Answer in simplest form. **2**

(e) Evaluate $e^{-2.6}$ correct to 3 significant figures. **2**

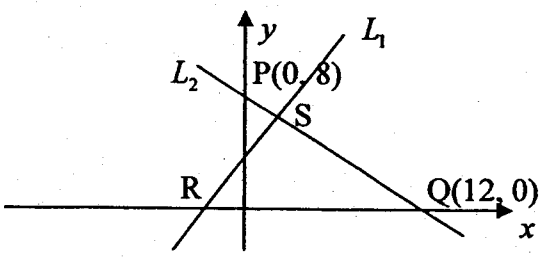
(f) At a sale, 30% discount is given off the marked price. If Anna pays \$175 for a coat at this sale, what was the original marked price? **2**

Question 2.
(Start a new booklet) ***Marks***

(a) Find the equation of the normal to $y = 2e^{2x}$ at the point $x = 0$ **3**

(b) Line L_1 has the equation $3x - 2y + 3 = 0$

Line L_2 passes through P(0, 8) and Q(12, 0)



(i) L_1 cuts the x axis at R. Find the coordinates of R. **1**

(ii) Find the gradient of line L_2 **1**

(iii) Show L_1 and L_2 are perpendicular **1**

(iv) Show the equation of L_2 is $2x + 3y - 24 = 0$ **1**

(v) Find S, the point of intersection of L_1 and L_2 **2**

(vi) Find the area of ΔRSQ **1**

(vii) Copy the diagram and shade the region where $2x + 3y - 24 \leq 0$, $3x - 2y + 3 \geq 0$ and $y \geq 0$ hold simultaneously. **2**

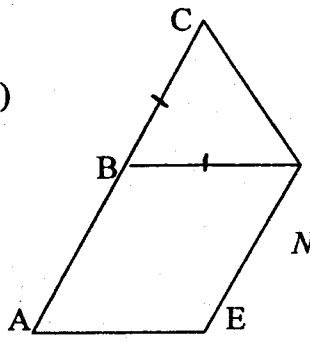
Question 3.
(Start a new booklet)

Marks

(a) Find $\int \frac{2x}{x^2-4}dx$ **1**

(b) Evaluate $\int_0^{\frac{\pi}{2}} 1+\cos x dx$ **2**

(c)



Not to scale

A, B and C are collinear

ABDE is a parallelogram

ΔBCD is isosceles with BC = BD

∠BCD = 58°

(i) Copy the diagram

(ii) Find the size of ∠AED giving reasons **3**

(d) Differentiate with respect to x **2**

(i) $(3x^2-5)^7$

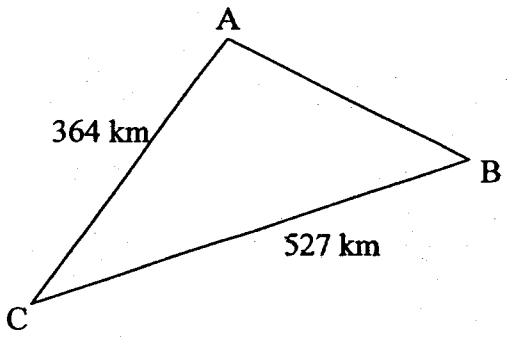
(ii) $x^3 \tan x$

(e) $\int_0^1 (3x+1)^3 dx$ **2**

Question 4.
(Start a new booklet) ***Marks***

(a) A plane is flying from town A on a bearing of 237°T to town C which is 364 km away.

At town C, the plane changes its course and travels 527 km on a bearing of 077°T to town B.



(i) Copy the diagram and show $\angle ACB=20°$ **2**

(ii) Find the distance of B from A. Answer to the nearest km. **2**

(b) A parabola has the point (2, 1) as its focus and its directrix is $y = -3$. Write down: **5**

(i) the vertex

(ii) the equation of its axis

(iii) the focal length

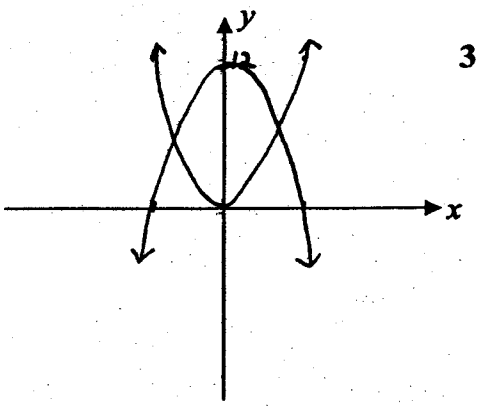
(iv) the equation of the parabola; and

(v) draw its sketch.

(c) The curves $y = 2x^2$ and $y = 12 - x^2$ are drawn **3**

(i) Show the curves intersect at (-2, 8) and (2, 8)

(ii) Hence, find the area between the curves



Question 5.
(Start a new booklet)

Marks

(a) The first 3 terms of a geometric series are $x-2, x+1, 3x-3$.

(i) Find the 2 values of x — **2**

(ii) Write down the 2 series and find their ratios — **1**

(iii) Explain why these series don't have a limiting sum — **1**

(b) The gradient function of a curve is given by $3x^2-11$. If the curve passes through (3, 4) find its equation. — **3**

(c) Find k if $4x^2-kx+1=0$ has real roots. — **3**

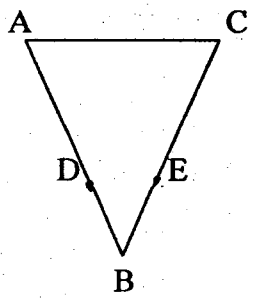
(d) Solve $2\sin x = -\sqrt{3}$ for $0 \le x \le 2\pi$ — **2**

Question 6.
(Start a new booklet) ***Marks***

(a) Solve $\log_3(2x-4)=4$ **2**

(b) ABC is a triangle with AB = BC. **1**

D and E are points of AB and CB such that AD = CE. **1**



Copy the diagram.

(i) Why does DB = EB? **1**

(ii) Prove ΔABE is congruent to ΔCBD. **2**

(iii) Why does ∠BAE = ∠BCD? **1**

(iv) Prove ∠CAE = ∠ACD. **1**

(c) A point P (x, y) moves so that its distance from A(3, 0) is twice its distance from B(0, 3).

(i) Show the equation of the locus of P is $x^2+2x+y^2-8y+9=0$ **2**

(ii) Show this locus is a circle and write down its centre and radius. **2**

(d) Change $\frac{3\pi}{5}$ radians to degrees. **1**

Question 7.
(Start a new booklet)

Marks

(a) For the curve $y = \frac{1}{3}x^3 - x^2 - 8x + 12$,

(i) Find any turning points. **2**

(ii) Determine their nature **1**

(iii) Find any points of inflexion **2**

(iv) Sketch the curve showing all relevant features **1**

(v) For what values of x is the curve concave up? **1**

(vi) When is the curve decreasing? **1**

(b) An arithmetic progression has a first term of 1 and a last term of 14. If the sum of the series is 90,

(i) Find the number of terms in the series **2**

(ii) Show the difference is $\frac{13}{11}$ **1**

(iii) Find an expression for the nth term as a single fraction. **1**

Question 8.
(Start a new booklet) ***Marks***

(a) Find A, B and C if $3x^2 + 4x + 2 \equiv Ax(x-1) + B(x+1) + C$ **3**

(b) Estimate $\int_0^1 \sin(1+x^2)\,dx$ using Simpson's Rule with 3 function values.

Answer to three decimal places. **3**

(c) Solve $2\sin^2 x - 3\sin x + 1 = 0$ for $0 \le x \le 2\pi$ **2**

(d) When Tom started university, his parents borrowed \$30000 to pay for his education. They repaid the loan by making 48 equal monthly repayments. Interest was charged at the rate of 1.1% per month on balance owing.

(i) Write an expression for the amount owing after 1 monthly repayment of \$$m$. **1**

(ii) Show that after 2 monthly repayments, the amount owing is \$30663.63 – 2.011m. **1**

(iii) Calculate the value of each monthly repayment. Answer to the nearest cent. **2**

Question 9.
(Start a new booklet)

Marks

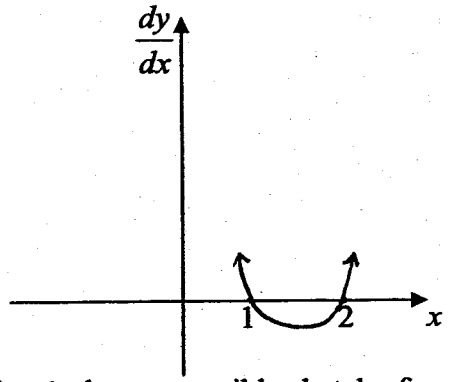
(a) If α and β are roots of $3x^2 + 5x - 2 = 0$, find:

(i) $\alpha + \beta$ **1**

(ii) $\alpha\beta$ **1**

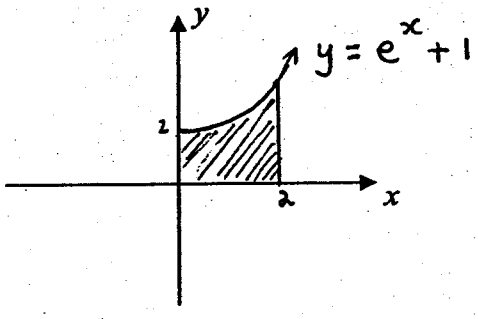
(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **2**

(b) The sketch shows the graph of $\frac{dy}{dx}$ which is the gradient function of $y = f(x)$.



If $f(0) = 1$, draw a possible sketch of $y = f(x)$. **2**

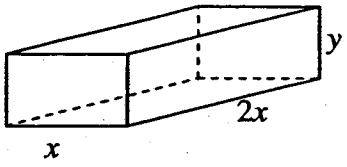
(c) What is the volume of the solid formed when the shaded area is rotated about the x axis? **3**



(d) Solve $\frac{1}{2}(5x-1) = 4 - \frac{1}{3}(x-2)$ **3**

Question 10.
(Start a new booklet) ***Marks***

(a) Boxes in the shape of rectangular prisms are to be made so that the width (x metres) of the base is half the length.



Each box holds a volume of 4 cubic metres. The wood used to make the top and bottom of the box costs \$15 per m^2 while the wood for the other 4 sides costs \$10 per m^2.

(i) Find an expression for y in terms of x. **1**

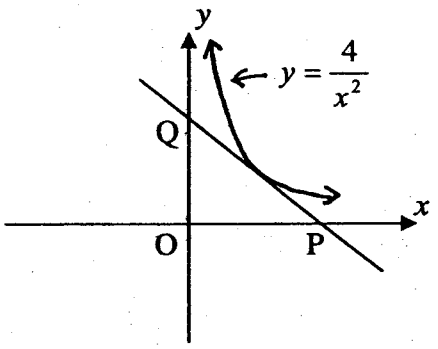
(ii) Show the cost \$$C$ of building each box is given by $C = 60x^2 + \frac{120}{x}$. **2**

(iii) What are the measurements of the cheapest boxes that can be constructed? **3**

(b) PQ is a tangent to $y = \frac{4}{x^2}$.

It cuts the x axis at P and the y axis at Q.

$\angle OPQ = \angle OQP$



(i) Explain why the gradient of PQ is -1. **1**

(ii) Show PQ is a tangent touching the curve at the point (2, 1). **2**

(iii) Find the equation of this tangent. **1**

(c) Differentiate $y = \cot x$. Answer in the simplest form. **2**

Solutions to Year 12 Trial Mathematics 2004

Question 1

$|3-x| = 7$

$3-x = 7$ or $3-x = -7$

$x = -4$ or $x = 10$

$y = 2\sqrt{x} + \cos x = 2x^{\frac{1}{2}} + \cos x$

$\frac{dy}{dx} = x^{-\frac{1}{2}} - \sin x$

$= \frac{1}{\sqrt{x}} - \sin x$

$l = 8 \qquad r = 3$

$l = r\theta \qquad \therefore 8 = 3\theta$

$\theta = \frac{8}{3}$ radians

$= \frac{8}{3} \times \frac{180}{\pi}$

$= 152^\circ 47'$

$= 153^\circ$ (to nearest deg.)

$f(x) = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$

$f'(x) = \frac{1}{x+1} - \frac{1}{x-1} = \frac{x-1-(x+1)}{(x+1)(x-1)}$

$= \frac{x-1-x-1}{x^2-1} = \frac{-2}{x^2-1}$

-2.6

$= 0.074273578$

$= 0.0743$ to 3 Sig. figs.

$70\% = \$175$

$1\% = \frac{175}{70} = 2.5$

$100\% = 250$

$\therefore$ Original price $= \$250$

Question 2

$y = 2e^{2x}$

$\frac{dy}{dx} = 2e^{2x} \times 2 = 4e^{2x}$

When $x = 0$, $\frac{dy}{dx} = 4 \qquad y = 2$

$\therefore m = 4 \qquad Pt = (0,2) \quad m_{normal} = -\frac{1}{4}$

Eqtn of normal is

$y - 2 = -\frac{1}{4}(x-0)$

$4y - 8 = -x$

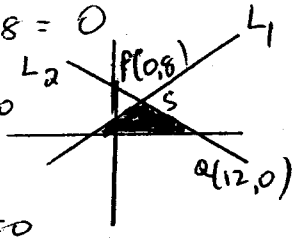
$x + 4y - 8 = 0$

b) i) L_1 is $3x - 2y + 3 = 0$

Cuts x axis when $y = 0$. ie $3x + 3 = 0$

$x = -1$

$\therefore R = (-1,0)$



ii) L_2 has $m = \frac{8-0}{0-12} = -\frac{2}{3}$

iii) $3x - 2y + 3 = 0$

$2y = 3x + 3$

$y = \frac{3}{2}x + \frac{3}{2}$

$m = \frac{3}{2}$

$m_{L_1} \times m_{L_2} = \frac{3}{2} \times -\frac{2}{3} = -1$

$\therefore L_1$ and L_2 are $\perp$

iv) Equation of L_2 is

$y - 8 = -\frac{2}{3}(x-0)$

$3y - 24 = -2x$

$2x + 3y - 24 = 0$

v) $3x - 2y = -3$ ①

$2x + 3y = 24$ ②

×① by 3. $9x - 6y = -9$ ③

×② by 2 $4x + 6y = 48$ ④

③ + ④ $13x = 39$

$x = 3, \quad y = 6$

$\therefore S = (3,6)$

vi) $R = (-1,0) \quad Q = (12,0)$

$\therefore RQ = 13$ Height is 6

Area $= \frac{1}{2} \times 13 \times 6$

$= 39$ u^2

vii) See diagram.

Question 3

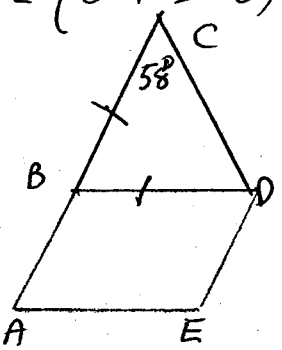
$\int \frac{2x}{x^2-4}\,dx = \ln(x^2-4) + C$

) $\int_0^{\frac{\pi}{2}} 1 + \cos x\,dx = \left[x + \sin x\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} + \sin\frac{\pi}{2} - (0 + \sin 0)$

$= \frac{\pi}{2} + 1$

) i)



ii) $\angle BDC = \angle BCD = 58^\circ$

Base L's of isos. Δ) $\angle ABD = \angle BDC + \angle BCD = 58^\circ + 58^\circ = 116^\circ$

$\angle AED = 116^\circ$ (opp. L's of parm. =)

1) i) $\frac{d}{dx}(3x^2-5)^7 = 7(3x^2-5)^6 \times 6x$

$= 42x(3x^2-5)^6$

ii) $\frac{d}{dx} x^3\tan x = x^3 \times \sec^2 x + \tan x \times 3x^2$

or $= x^2(x\sec^2 x + 3\tan x)$

) $\int_0^1 (3x+1)^3\,dx = \left[\frac{(3x+1)^4}{4\times 3}\right]_0^1$

$= \frac{256}{12} - \frac{1}{12}$

$= \frac{255}{12}$ or $21\frac{1}{4}$

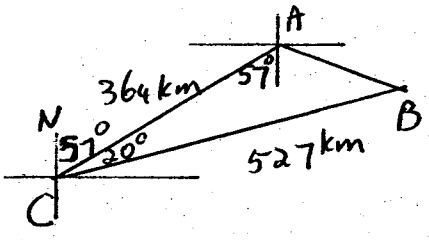
Question 4

i) $\angle NCA = 57^\circ$ (alt L's =)

$\angle NCB = 77$

$= 57 + 20$

$\therefore \angle ACB = 20^\circ$



ii) $AB^2 = 364^2 + 527^2 - 2\times 364\times 527\times \cos 20^\circ$

$= 49706.28788$

$AB = 222.949$

$= 223$ km to nearest km.

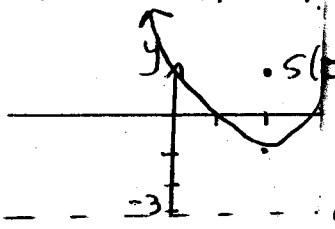
b) i) $V = (2, -1)$

ii) axes is $x = 2$

iii) $a = 2$

iv) $(x-2)^2 = 8(y+1)$

v) See sketch



c) i) $y = 2x^2$ and $y = 12 - x^2$

intersect when $2x^2 = 12 - x^2$

$3x^2 = 12$

$x^2 = 4$

$x = 2$ or $x = -2$

$y = 8$ $y = 8$.

ii) Area $= 2\int_0^2 12 - x^2 - 2x^2\,dx$

$= 2\int_0^2 12 - 3x^2\,dx$

$= 2\left[12x - x^3\right]_0^2$

$= 2(24 - 8)$

$= 32u^2$

Question 5

a) $x-2$, $x+1$, $3x-3$

i) $\frac{x+1}{x-2} = \frac{3x-3}{x+1}$

$x^2 + 2x + 1 = 3x^2 - 9x + 6$

$2x^2 - 11x + 5 = 0$

$(2x-1)(x-5) = 0$

$x = \frac{1}{2}$ or 5

When $x = \frac{1}{2}$, series is $-\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}$

ratio is -1

When $x = 5$, series is $3, 6, 12$

ratio is 2

For a limiting sum, $|r| < 1$.

$\frac{dy}{dx} = 3x^2 - 11$

$y = x^3 - 11x + C$

Curve passes thru $(3, 4)$

$\therefore 4 = 27 - 33 + C$

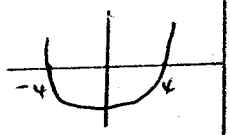
$C = 10$

Curve is $y = x^3 - 11x + 10$

$4x^2 - kx + 1 = 0$ has real roots $\therefore \Delta \geq 0$

$k^2 - 16 \geq 0$

$(k+4)(k-4) \geq 0$



$k \leq -4$ or $k \geq 4$

$2 \sin x = -\sqrt{3}$ for $0 \leq x \leq 2\pi$

$\sin x = -\frac{\sqrt{3}}{2}$

x is in quads 3 or 4

$x = \pi + \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3}$

$= \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

Question 6

$\log_3(2x - 4) = 4$

$2x - 4 = 3^4$

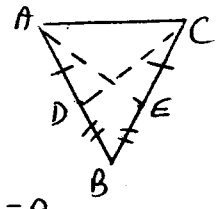
$2x - 4 = 81$

$2x = 85$

$x = 42.5$

b) i) $AB = AD + DB$

$CB = CE + EB$



But $AB = CB$

$\therefore AD + DB = CE + EB$

But $AD = CE$, $\therefore DB = EB$

ii) $AB = BC$ given.

$EB = DB$ (proven above)

LB is common.

$\therefore \triangle ABE \equiv \triangle CBD$ (SAS)

iii) $\angle BAE = \angle BCD$ (corresp. $\angle$'s of $\equiv \triangle$s)

iv) $\angle BAC = \angle BCA$ (base $\angle$'s of isos $\triangle$)

$\angle BAE + \angle CAE = \angle BCD + \angle ACD$

But $\angle BAE = \angle BCD$ (proven above)

$\therefore \angle CAE = \angle ACD$

c) $P(x, y)$ $A(3, 0)$ $B(0, 3)$

i) $PA = 2PB$

$\therefore (PA)^2 = 4(PB)^2$

$(x-3)^2 + y^2 = 4\left[x^2 + (y-3)^2\right]$

$x^2 - 6x + 9 + y^2 = 4x^2 + 4y^2 - 24y + 36$

$3x^2 + 3y^2 + 6x - 24y + 27 = 0$

$x^2 + y^2 + 2x - 8y + 9 = 0$

ii) $x^2 + 2x + 1 + y^2 - 8y + 16 = -9 + 1 + 16$

$(x+1)^2 + (y-4)^2 = 8$

This is a circle centre $(-1, 4)$

radius $= 2\sqrt{2}$

d) $\frac{3\pi}{5} = \frac{3 \times 180}{5} = 108^\circ$

Question 7

$y = \frac{1}{3}x^3 - x^2 - 8x + 12$

i) $\frac{dy}{dx} = x^2 - 2x - 8$

For turning pts, $\frac{dy}{dx} = 0$

ie $x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = 4$ or $x = -2$

$y = -14\frac{2}{3}$ $\quad$ $y = 21\frac{1}{3}$

ii) $\frac{d^2y}{dx^2} = 2x - 2$

When $x = 4$, $\frac{d^2y}{dx^2} > 0$

$\therefore$ min. at $(4, -14\frac{2}{3})$

When $x = -2$, $\frac{d^2y}{dx^2} < 0$

$\therefore$ max at $(-2, 21\frac{1}{3})$

iii) For a pt. of inflexion,

$\frac{d^2y}{dx^2} = 0$ ie $2x - 2 = 0$

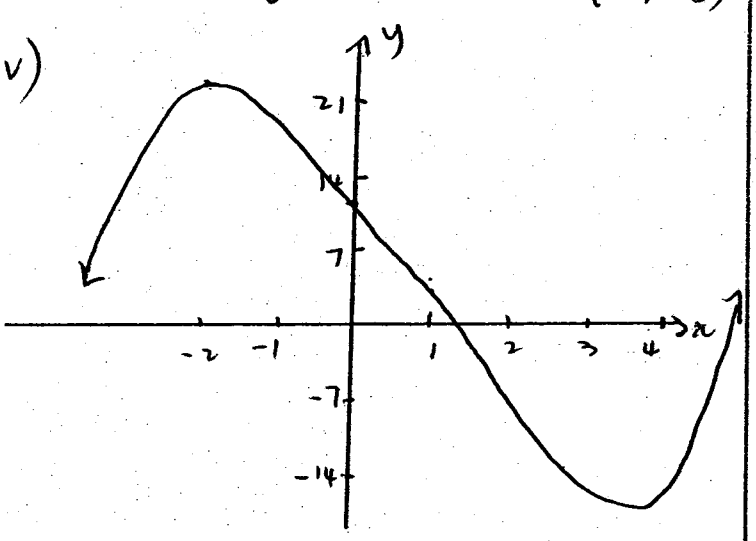
$x = 1$

$y = 3\frac{1}{3}$

When $x < 1$ $\frac{d^2y}{dx^2} < 0$, $x > 1$ $\frac{d^2y}{dx^2} > 0$ } Change in Concavity

$\therefore$ Pt of inflexion at $(1, 3\frac{1}{3})$

iv)

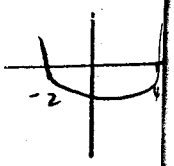


v) Curve is concave up for $\frac{d^2y}{dx^2} > 0$ ie $2x - 2 > 0$

$x > 1$

vi) curve is decreasing when $\frac{dy}{dx} < 0$ ie $(x-4)(x+2) < 0$

$-2 < x < 4$



b) 1, ... , 14

i) $90 = \frac{n}{2}(1 + 14)$

$180 = 15n$

$n = 12$ $\therefore$ 12 terms.

ii) $14 = 1 + 11d$

$13 = 11d$

$d = \frac{13}{11}$

iii) $t_n = a + (n-1)d$

$= 1 + (n-1)\frac{13}{11}$

$= \frac{11 + 13n - 13}{11}$

$= \frac{13n - 2}{11}$

Question 8

a) $3x^2 + 4x + 2 \equiv Ax(x-1) + B(x+1)$

$3x^2 + 4x + 2 \equiv Ax^2 - Ax + Bx + B + C$

$\therefore 3 = A$

$4 = -A + B$

$4 = -3 + B$

$B = 7$

$2 = B + C$

$2 = 7 + C$

$C = -5$

$\therefore A = 3, B = 7, C = -5$

$\int \sin(1+x^2)\,dx$

0 $\frac{1}{2}$ 1

$\frac{1-0}{2} = \frac{1}{2}$

$\int \sin(1+x^2) = \frac{1}{3} \times \frac{1}{2}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$

$= \frac{1}{6}\left[\sin 1 + 4\sin 1.25 + \sin 2\right]$

$= \frac{1}{6}\left[5.546\right]$

$\doteq 0.924$

$2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = 1$

$x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$ or $\frac{\pi}{2}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$.

i) Amount owing after 1 mth

$30000\left(1 + \frac{1.1}{100}\right) - m$

$30000(1.011) - m$

Amt. owing after 2 months

$\left[30000(1.011) - m\right]1.011 - m$

$30000(1.011)^2 - m(1 + 1.011)$

$30663.63 - 2.011m$

Amt. owing after 48 payments

$30000(1.011)^{48} - m(1 + 1.011 + \ldots 1.011^{47})$

$30000(1.011)^{48} = \frac{m \times 1\left(1.011^{48} - 1\right)}{1.011 - 1}$

$719.97\ldots = 62.787m$

$m = 807.8060$

Repayments are $807.81

Question 9

a) $3x^2 + 5x - 2 = 0$

i) $\alpha + \beta = -\frac{5}{3}$

ii) $\alpha\beta = -\frac{2}{3}$

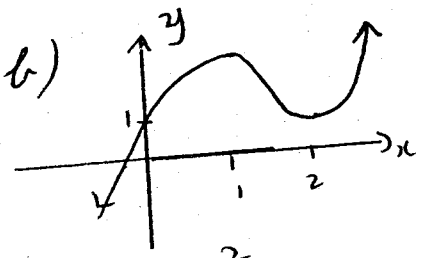
iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{(\alpha\beta)^2}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{\frac{25}{9} + \frac{4}{3}}{\frac{4}{9}} = \frac{37}{9} \times \frac{9}{4}$

$= \frac{37}{4}$

b)



c) $V = \pi \int_0^2 (e^x + 1)^2\,dx$

$= \pi \int_0^2 e^{2x} + 2e^x + 1\,dx$

$= \pi\left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^2$

$= \pi\left[\frac{e^4}{2} + 2e^2 + 2 - \left(\frac{1}{2} + 2\right)\right]$

$= \pi\left[\frac{e^4}{2} + 2e^2 - \frac{1}{2}\right]$ u^3

d) $\frac{1}{2}(5x - 1) = 4 - \frac{1}{3}(x - 2)$

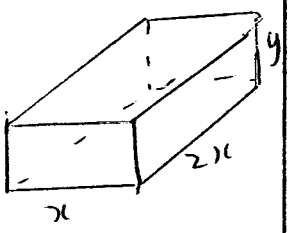
$\times 6 \quad 3(5x - 1) = 24 - 2(x - 2)$

$15x - 3 = 24 - 2x + 4$

$17x = 31$

$x = \frac{31}{17}$

Question 10



Volume = 4

$x \times 2x \times y = 4$

$2x^2 y = 4$

$y = \frac{4}{2x^2} = \frac{2}{x^2}$

i) $\text{Cost} = (2 \times 2x^2) \times 15 + (2x2x \times y + 2 \times x \times y) \times 10$

$C = 15 \times 4x^2 + 6xy \times 10$

$= 60x^2 + 6x \times \frac{2}{x^2} \times 10$

$= 60x^2 + \frac{120}{x}$

ii) For cheapest cost,

$\frac{dC}{dx} = 0$ ie $120x - \frac{120}{x^2} = 0$

$120x = \frac{120}{x^2}$

$x^3 = 1$

$x = 1$

$\frac{d^2C}{dx^2} = 120 + \frac{240}{x^3} > 0$

∴ Minimum cost when $x = 1$

∴ Measurements are

1m by 2m by 2m.

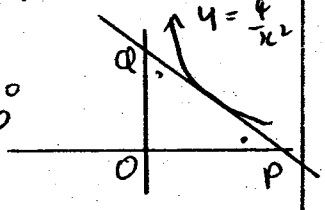
b) i) $\angle OPQ = \angle OQP$ (given)

$\angle OPQ + \angle OQP + \angle QOP = 180^\circ$ ($\angle$ sum of $\triangle$)

$\angle OPQ = \angle OQP = 45^\circ$ as $\angle QOP = 90^\circ$

∴ QP makes an $\angle$ of 135° with the +ve x axes so $m_{PQ} = \tan 135^\circ$

$= -1$



ii) m_{tangent} to $y = \frac{4}{x^2}$ is -1

∴ $\frac{dy}{dx} = -1$ ie $-\frac{8}{x^3} = -1$

$8 = x^3$

$x = 2$.

When $x = 2$, $y = 1$ so the tangent PQ touches the curve at $(2, 1)$

iii) Equation is $y - 1 = -1(x - 2)$ ie $x + y - 3 =$

c) $y = \cot x = \frac{\cos x}{\sin x}$.

$\frac{dy}{dx} = \frac{\sin x \times -\sin x - \cos x \times \cos}{(\sin x)^2}$

$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$

$= -\text{cosec}^2 x$