Candidate Number ______________________

ROSEVILLE COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2003

MATHEMATICS

Time Allowed - 3 hours
(plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for carelessness or badly arranged work.
- Standard integrals are provided
- Board-approved calculators may be used
- Start each question on a new page where directed to do so.

QUESTION 1

(a) Simplify $|4| - |-6|$ (1)

(b) Expand and simplify $3(2x-4)-(6-x)$ (2)

(c) Solve $\dfrac{x+1}{2} - \dfrac{3x}{5} = 8$ (2)

(d) Evaluate, correct to three significant figures,
$\log_e 1{\cdot}1$ (2)

(e) Convert 225° to radians. Answer in exact form. (1)

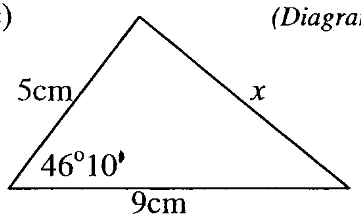
(f) Find a primitive of $4 - \dfrac{1}{x}$ (2)

(g) What is the exact value of $\cos \dfrac{5\pi}{6}$ (2)

QUESTION 2 (Start a new page)

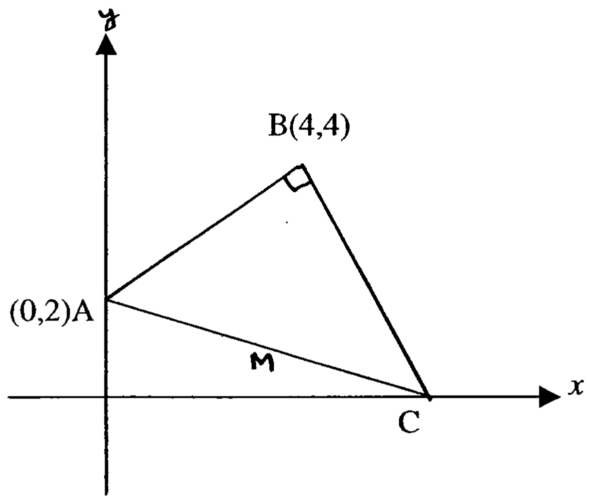
(a) Given $(2\sqrt{3} - 1)^2 = x + y\sqrt{3}$, find x and y (2)

(b) Solve $\tan\vartheta = -\sqrt{3}$ for $0 \le \vartheta \le 2\pi$ (2)

(c) 

(Diagram not to scale)

Find x to 2 decimal places (2)

(d) 

A is the point (0,2)
B is the point (4,4)
M is the midpoint of AC
$\angle ABC = 90^\circ$

(i) Find the gradient of the line AB (1)

(ii) Show that the equation of the line BC is $2x + y - 12 = 0$ (2)

(iii) Show that the coordinates of C are (6, 0) (1)

(iv) A circle with centre M passes through points A, B and C. Find the radius of this circle. Answer in surd form. (2)

QUESTION 3 (Start a new page)

(a) Differentiate with respect to x:

(i) $(2x-1)^3$ (2)

(ii) $\dfrac{e^{2x}}{x}$ (2)

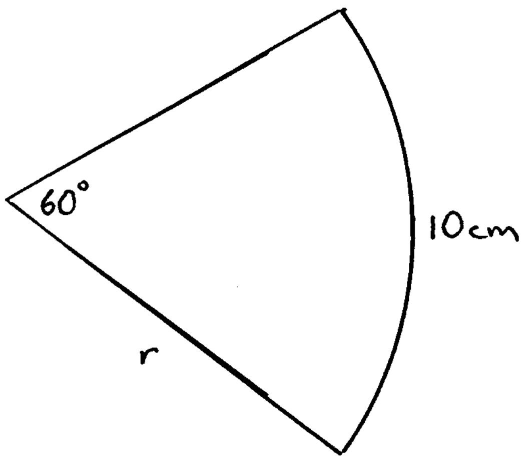
(b) Find the values of p for which $x^2 - 4x + p = 0$ is positive definite (2)

(c) Find:

(i) $\int \sqrt{2x+5}\, dx$ (2)

(ii) $\int_0^{\frac{\pi}{8}} \sin 4x \, dx$ (2)

(d) Find the length of the radius of the sector of the circle shown in this diagram. Give your answer to the nearest millimetre. (2)

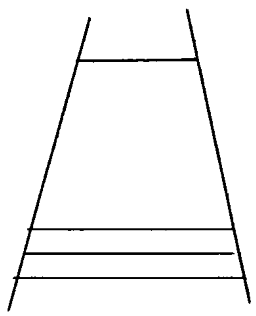


QUESTION 4 (Start a new page)

(a) Evaluate exactly $\int_0^1 \frac{dx}{x+1}$ (2)

(b) Find the values of k for which $x^2 - 2kx + 1 = 0$ has real roots. (2)

(c) A ladder tapers in from bottom to top as shown in the diagram. The ladder has 20 steps. The bottom step is 1250mm long. Each subsequent step is 27mm shorter.



(i) Calculate the length of the top step (2)

(ii) Calculate the total length of all 20 steps (2)

(d) A parabola has equation $(x-1)^2 = -8y$

Find the:

(i) coordinates of the focus (1)

(ii) the equation of the directrix (1)

(iii) the gradient of the tangent to the parabola at the point (-3, -2) (2)

QUESTION 5 (Start a new page)

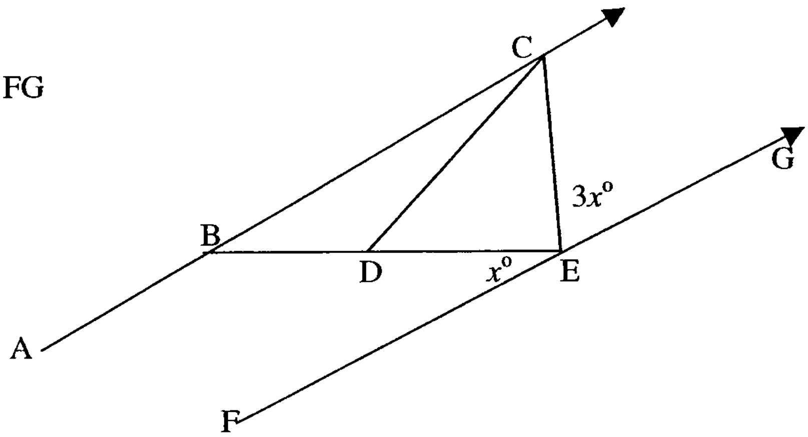
(a) Find $\sum_{n=1}^{\infty} \frac{2}{3^n}$ (3)

(b) In a quality control check of appliances, it was found that 90% were perfect whilst 10% were defective. If 2 appliances were selected at random find the probability that:

(i) both were perfect (1)

(ii) only one was perfect (2)

(c) AC is parallel to FG
CD = CE
$\angle BCD = 20^\circ$
$\angle DEF = x^\circ$
$\angle CEG = 3x^\circ$

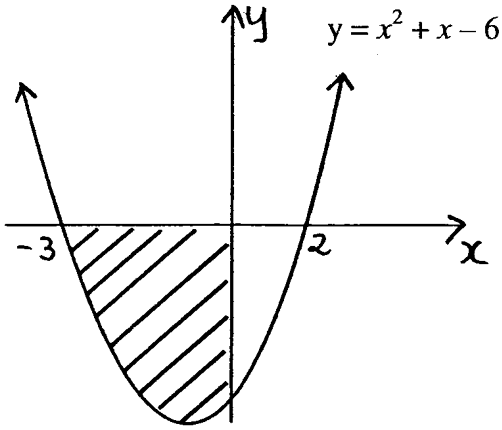


(i) Give a reason why $\angle CBE$ equals x° (1)
(ii) Find the value of x (2)

(d) By using the substitution $u = 3^x$, or otherwise, solve $3 \times 3^{2x} - 7 \times 3^x + 2 = 0$
Give your answer to 3 significant figures where appropriate. (3)

QUESTION 6 (Start a new page)

(a) The diagram shows the graph of the function $y = x^2 + x - 6$



(Diagram not to scale)

(i) State the range of the function $y = x^2 + x - 6$ (1)

(ii) Find the area of the shaded region (2)

(b) The gradient function for a curve is given by $3e^x - 2$.
If the curve passes through (0,5), find the equation of the curve. (2)

(c) The number, N, of bacteria in a culture is growing exponentially according to the formula

$$N = 150e^{kt}$$

where t is the time in hours.

(i) What were the initial number of bacteria? (1)

(ii) After 8 hours the number of bacteria has tripled. Calculate the value of k to 3 decimal places (2)

(iii) How many bacteria will there be after 16 hours? (2)

(iv) At what rate will the bacteria be increasing after 24 hours? (2)

QUESTION 7 (Start a new page)

(a) (i) On the same set of axes, sketch the graphs of

$y = 2\sin x$ and $y = -\cos x$ for $0 \leq x \leq 2\pi$

Clearly label each graph. (2)

(ii) Find the area of the region bounded by the curves $y = 2\sin x$ and $y = -\cos x$ over the interval $0 \leq x \leq \frac{\pi}{2}$ (3)

(b) (i) Sketch $y = \ln x$ (1)

(ii) Use the trapezoidal rule with 3 function values to estimate

$$\int_1^2 \ln x \, dx$$

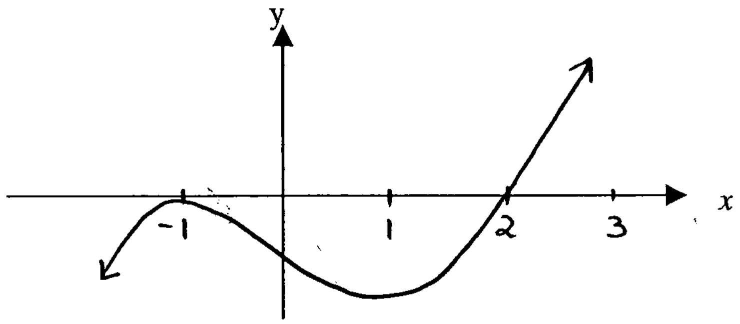
Answer to 2 decimal places (3)

(iii) Find the volume of the solid formed when the curve $y = \ln x$ is rotated about the y-axis between $y = 0$ and $y = 4$ (3)

QUESTION 8 (Start a new page)

(a) Show that $\cot\alpha\ (1-\cos^2\alpha) = \cos\alpha \sin\ \alpha$ (2)

(b) The graph shows the curve $y = f'(x)$



Write down the x values of any stationary points on the function $y = f(x)$and determine their nature (2)

(c) The quadratic equation $x^2 + mx + n = 0$ has one root that is twice the other.

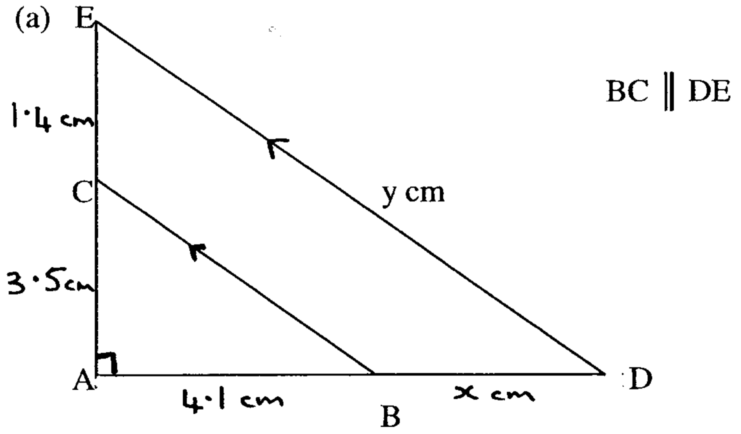
Find the value of $\frac{m^2}{n}$. (3)

(d) (i) Find $\frac{dy}{dx}$, given $y = x \ln x$ (2)

(ii) Find the coordinates of the stationary point and determine its nature (3)

QUESTION 9 (Start a new page)

(a)



(Diagram not to scale)

(i) Prove Δ ABC is similar to Δ ADE (2)

(ii) Find the values of x and y to 1 decimal place (2)

(b) Helen borrows \$100 000 at 12% p.a. reducible interest over 20 years, repaid in monthly instalments, \$M.

(i) Show that the amount owing, A_2, at the end of the second month is given by

$$A_2 = 100\,000 \times 1.01^2 - M\,(1 + 1.01)$$ (2)

(ii) Show that the amount owing, A_n, at the end of the n^{th} month is given by:

$$A_n = 100\,000 \times 1.01^n - M\left[\frac{1\cdot01^n - 1}{0\cdot01}\right]$$

and hence show that \$M = \$1101.09. (3)

At the end of the tenth year, the interest rate is reduced to 6% p.a. **Using your results from part ii),** answer the following questions.

(iii) Show that the amount still owing at the end of the 10^{th} year is \$76 745.39. (1)

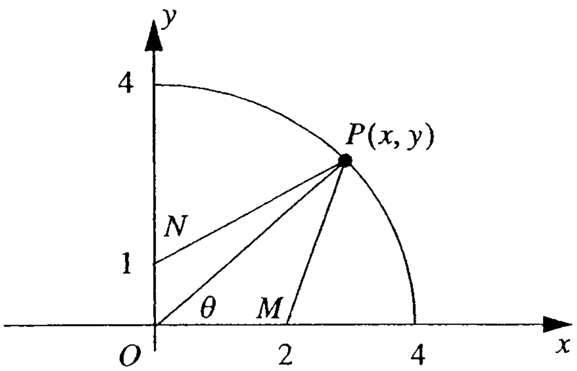
(iv) Hence, calculate the amount of each new monthly instalment for the remaining 10 years of the loan at the new rate of 6% p.a. (2)

QUESTION 10 (Start a new page)

(a) Given $y = \cos^2(3x)$ find $\dfrac{dy}{dx}$ (2)

(b) Solve $2\ln x = \ln(x + 6)$ (3)

(c)



The diagram shows the part of the circle $x^2 + y^2 = 16$ that lies in the first quadrant. The point $P(x,y)$ is on the circle, 0 is the origin, M is on the x-axis at $x = 2$ and N is on the y-axis at $y = 1$. The size of angle MOP is θ radians.

(i) Show that the area, A, of the quadrilateral OMPN is given by

$$A = 4\sin\theta + 2\cos\theta$$ (2)

(ii) Find the value of $\tan\theta$ for which A is a maximum. (3)

(iii) Hence determine in surd form the coordinates of P for which A is a maximum. (2)

Roseville College

Question 1

a) $|4|-|-6| = -2$ [2]

b) $3(2x-4)-6(6-x)$
$= 6x-12-6+x$ 1
$= 7x-18$ 1 [2]

c) $\frac{5}{10} \times \frac{(x+1)}{2} - \frac{3x}{5} \times 10^2 = 8\times 10$
$5x+5-6x = 80$
$5-x = 80$
$x = -75$ [2]

d) $\ln 1.1 = 0.095310179\ldots$ 1
$= 0.0953$ to 3 s.f. 1 [2]

e) $225° = \frac{225}{180}\times \pi^c$
$= \frac{5\pi}{4}$ [1]

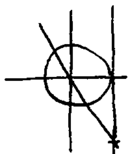
f) $\int 4-\frac{1}{x}\,dx = 4x - \ln x + c$ [2]

g) $\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}$
$= -\frac{\sqrt{3}}{2}$ [2]

Question 2

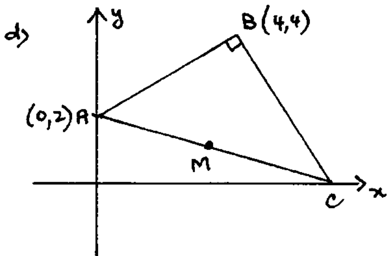
a) $(2\sqrt{3}-1)^2 = 12-4\sqrt{3}+1$
$x+y\sqrt{3} = 13-4\sqrt{3}$ 1
$\therefore x = 13 \quad y = -4$ 1 [3]

b) $\tan\theta = -\sqrt{3} \quad 0 \leq \theta \leq 2\pi$
acute $\theta = \frac{\pi}{3}$
$\therefore \theta = \frac{2\pi}{3}, \frac{5\pi}{3}$ [2]



c) $x^2 = 5^2+9^2-2\times5\times9\times\cos 46°10'$ 1
$x^2 = 43.669\ldots$
$x = 6.608277\ldots$ 1
$\therefore x = 6.61$ to 2 dec. pl. [2]

d)



i) gradient AB $= \frac{4-2}{4}$
$= \frac{1}{2}$ [1]

ii) gradient of BC $= -2 \quad (m_1m_2=-1)$ 1
through $(4,4)$
$y-4 = -2(x-4)$ 1
$y-4 = -2x+8$
$\therefore 2x+y-12 = 0$. [2]

iii) C: $y=0 \quad \therefore 2x+0-12 = 0$
$2x = 12$
$x = 6$.
$\therefore C(6,0)$ [1]

iv) radius $= \frac{1}{2}\times AC$
$= \frac{1}{2}\sqrt{2^2+6^2}$
$= \frac{1}{2}\sqrt{40}$
$= \frac{1}{2}\times 2\sqrt{10}$
$= \sqrt{10}$ units. [2]

Question 3

a) i) $y = (2x-1)^3$
$y' = 3(2x-1)^2\times 2$
$= 6(2x-1)^2$ [2]

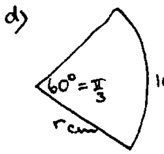
ii) $y = \frac{e^{2x}}{x}$
$y' = \frac{x\times 2e^{2x} - e^{2x}\times 1}{x^2}$
$= \frac{2xe^{2x}-e^{2x}}{x^2}$ [2]

b) $x^2-4x+p = 0$ positive definite
$a>0 \quad \Delta<0$ [2]
$a=1 \therefore a>0$
$\Delta = b^2-4ac$
$= 16-4\times1\times p$
$= 16-4p$ 1
$\therefore 16-4p<0$
$4p>16$
$p>4$. 1

c) i) $\int(2x+5)^{1/2}\,dx$
$= \frac{2(2x+5)^{3/2}}{3\times 2} + c$
$= \frac{(2x+5)^{3/2}}{3} + c$ [2]

ii) $\int_0^{\frac{\pi}{8}} \sin 4x\,dx$
$= \left[\frac{-1}{4}\cos 4x\right]_0^{\frac{\pi}{8}}$ 1 [2]
$= \left[-\frac{1}{4}\cos\frac{\pi}{2} + \frac{1}{4}\cos 0\right]$
$= \frac{1}{4}$ 1

d)

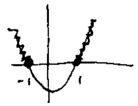


$l = r\theta$
$10 = r\times\frac{\pi}{3}$
$r = \frac{30}{\pi}$ cm
$r = 9.549\ldots$ cm
$r = 9.5$ cm nearest mm [2]

Question 4

a) $\int_0^1 \frac{1}{x+1}\,dx = \left[\ln(x+1)\right]_0^1$ 1
$= \ln 2 - \ln 1$
$= \ln 2$. 1 [2]

b) $x^2-2kx+1 = 0$
real roots: $\Delta \geq 0$
$\Delta = b^2-4ac$
$= 4k^2-4\times1\times1$
$= 4k^2-4$
$\therefore 4k^2-4 \geq 0$ 1 [2]
$4(k+1)(k-1) \geq 0$

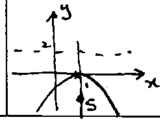


$\therefore k \leq -1$ or $k \geq 1$ 1

c) i) Arithmetic sequence 1
$a = 1250,\ d = -27$
$\therefore T_n = 1250-27\times(n-1)$
$= 1277-27n$ [2]
$\therefore T_{20} = 1277-27\times20$
$= 737$ 1
$\therefore$ top step is 737 mm long

ii) Total length $S_n = \frac{n}{2}(a+l)$
$S_{20} = \frac{20}{2}(1250+737)$ 1 [2]
$= 19870$ mm 1

d) $(x-1)^2 = -8y$ of the form $(x-h)^2 = -4a(y-k)$
$\therefore$ vertex $(1,0) \quad a=2$ [1]



i) focus $(1,-2)$ [1]
ii) $y = 2$.
iii) $y = -\frac{1}{8}(x-1)^2$ [2]
$y' = -\frac{1}{4}(x-1)$ 1
$\therefore x = -3,\ m = 1$ 1

Question 5. (Marks)

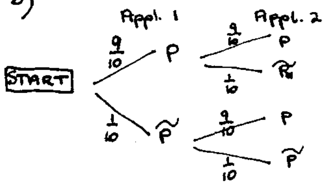
a) $\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \dots$ 1

geometric series $a = \frac{2}{3}$ $r = \frac{1}{3}$ $S = \frac{a}{1-r}$ [3]

$\therefore S = \dfrac{\frac{2}{3}}{1 - \frac{1}{3}}$ 1

$= 1$ 1

b)



i) $P(PP) = \frac{9}{10} \times \frac{9}{10}$ [1]

$= \frac{81}{100}$

ii) $P(\text{only 1 Perfect}) = P\tilde{P} + \tilde{P}P$ [2]

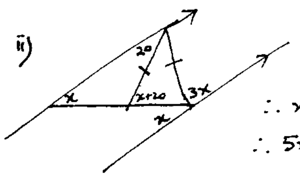
$= \frac{9}{10} \times \frac{1}{10} + \frac{9}{10} \times \frac{1}{10}$

$= \frac{18}{100}$

c) i) $C\hat{B}E = x^\circ$ as

$\angle C\hat{B}E = \angle BEF$ [alternate angles, given $AC \parallel FG$] [1]

ii)



$\therefore x^\circ + x + 20^\circ + 3x^\circ = 180^\circ$ [2]

$\therefore 5x = 160$

$x = 32$

d) $u = 3^x$

$u^2 = (3^x)^2 = 3^{2x}$

$\therefore 3 \times 3^{2x} - 7 \times 3^x + 2 = 0$ [3]

$3u^2 - 7u + 2 = 0$

$(3u - 1)(u - 2) = 0$

$\therefore u = \frac{1}{3}$ or 2

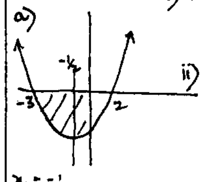
$\therefore 3^x = \frac{1}{3}$ $\therefore x = -1$ or

$3^x = 2$

$x = \frac{\ln 2}{\ln 3}$ 0.631 to 3 s.f.

Question 6 (Marks)

a) i) range : $y \geq -6\frac{1}{4}$. [1]



$x = -\frac{1}{2}$

$y = \frac{1}{4} - \frac{1}{2} - 6$

$= -6\frac{1}{4}$

ii) Area $= \left| \int_{-3}^{0} x^2 + x - 6 \, dx \right|$ 1 [2]

$= \left[\frac{x^3}{3} + \frac{x^2}{2} - 6x \right]_0^{-3}$

$= \frac{-27}{3} + \frac{9}{2} + 18$ u^2

$= 13\frac{1}{2}$ units^2 1

b) $\frac{dy}{dx} = 3e^x - 2$ [2]

$y = 3e^x - 2x + c$ 1

$x = 0, y = 5$ $5 = 3e^0 - 0 + c$

$5 = 3 + c$

$\therefore c = 2$. 1

$\therefore y = 3e^x - 2x + 2$.

c) $N = 150e^{kt}$

i) $t = 0$, $N = 150e^0$ [1]

$\therefore 150$ bacteria.

ii) $t = 8$ $N = 450$

$\therefore 450 = 150e^{8k}$ 1 [2]

$e^{8k} = 3$

$8k = \ln 3$

$k = \frac{1}{8} \ln 3$

$= 0.1373265\dots$

$k = 0.137$ to 3 dec. pl. 1

iii) $t = 16$, $N = ?$

$N = 150e^{0.137\dots \times 16}$ 1 [2]

$= 1350$. 1

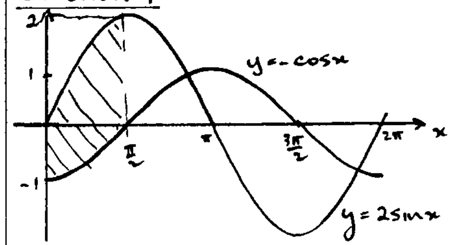
iv) $\frac{dN}{dt} = 150e^{kt} \times k$ [2]

$t = 24$, $\frac{dN}{dt} = 150 \times e^{0.137\dots \times 24} \times 0.137\dots$ 1

$= 556.17\dots$ bacteria/hour. 1

Year 12 Mathematics Trial 2003

Question 7 (Marks)



Area $= \int_0^{\frac{\pi}{2}} 2\sin x - (-\cos x) \, dx$ [3]

$= \int_0^{\frac{\pi}{2}} 2\sin x + \cos x \, dx$

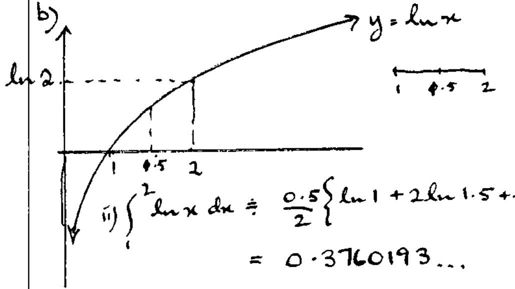
$= \left[-2\cos x + \sin x \right]_0^{\frac{\pi}{2}}$

$= \left[-2\cos\frac{\pi}{2} + \sin\frac{\pi}{2} \right] - \left[-2\cos 0 + \sin 0 \right]$

$= 1 + 2$

$= 3$ units^2

b)



[1]

ii) $\int_1^2 \ln x \, dx \doteq \frac{0.5}{2} \{ \ln 1 + 2\ln 1.5 + \ln 2 \}$ [3]

$= 0.3760193\dots$

$= 0.38$ to 2 dec. pl.

iii) $V = \int_0^4 \pi x^2 \, dy$ $x = e^y$, $x^2 = e^{2y}$ [3]

$= \pi \int_0^4 e^{2y} \, dy$ 1

$= \pi \left[\frac{e^{2y}}{2} \right]_0^4$ 1

$= \pi \left[\frac{e^8}{2} - \frac{e^0}{2} \right]$

$= \frac{\pi(e^8 - 1)}{2}$ units^3. 1

Question 8 (Marks)

a) $\cot\alpha(1 - \cos^2\alpha) = \cos\alpha \sin\alpha$ [2]

LHS $= \cot\alpha(1 - \cos^2\alpha)$

$= \frac{\cos\alpha}{\sin\alpha} \times \sin^2\alpha$

$= \cos\alpha \sin\alpha$

$=$ RHS.

b) stationary points : $f'(x) = 0$ [2]

ie $x = -1$ horizontal pt of inflexion

$x = 2$. minimum turning pt.

c) $x^2 + mx + n = 0$ [3]

roots are α & β where $\beta = 2\alpha$

$\alpha + \beta = \frac{-b}{a}$ $\alpha\beta = \frac{c}{a}$

$3\alpha = \frac{-m}{1}$ $2\alpha^2 = \frac{n}{1}$ 1

$\alpha = -\frac{m}{3}$ $\therefore 2 \times \left(-\frac{m}{3}\right)^2 = n$ 1

$\frac{2m^2}{9} = n$

$\therefore \frac{m^2}{n} = \frac{9}{2}$ 1

d) i) $y = x \ln x$ [2]

$y' = x \times \frac{1}{x} + \ln x \times 1$

$\therefore y' = 1 + \ln x$

ii) Stationary pts: $y' = 0$ [3]

$1 + \ln x = 0$

$\ln x = -1$

$\therefore x = e^{-1}$

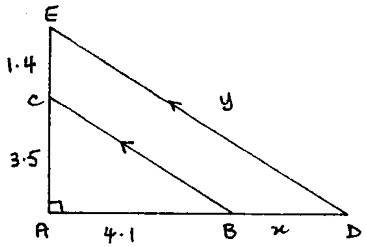
$\therefore y = -e^{-1}$

$y'' = \frac{1}{x}$

$x = e^{-1}$, $y'' > 0$

$\therefore (e^{-1}, -e^{-1})$ is a minimum turning pt.

Question 9



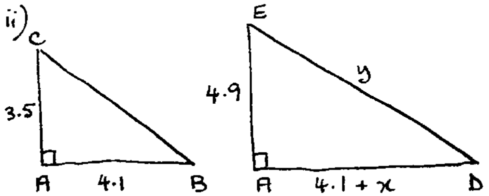
i) In ΔABC & ΔADE

$\angle CAB = \angle EAD$ [given]

$\angle ABC = \angle ADE$ [corresponding angles given $BC \parallel DE$]

$\therefore \Delta ABC \ ||| \ \Delta ADE$ [they are equiangular] [2]

ii)



$\frac{4.1+x}{4.1} = \frac{4.9}{3.5}$ $\qquad y^2 = 4.9^2 + (4.1+1.64)^2$

$4.1 + x = 5.74$ $\qquad y = 7.5470\ldots$

$x = 1.64$ $\qquad = 7.5$ to 1 dec. pl.

ie $x = 1.6$ to 1 dec pl. [2]

b) \$100 000 at 12% p.a = 1% p.m for 20 yrs ie 240 months.

i) After 1 month, balance owing

$A_1 = \$100000 \times 1.01 - \$M$

After 2 months, balance owing

$A_2 = (\$100000 \times 1.01 - \$M) \times 1.01 - \$M$

(A_2 in dollars)

$= 100\,000 \times 1.01^2 - M \times 1.01 - M$

$= 100\,000 \times 1.01^2 - M[1 + 1.01]$ [2]

$A_3 = 100\,000 \times 1.01^3 - M[1 + 1.01 + 1.01^2]$

$\therefore A_n = 100\,000 \times 1.01^n - M[1 + 1.01 + 1.01^2 + \ldots 1.01^{n-1}]$

↓ geometric series

$a = 1 \quad r = 1.01 \quad n = n$

$S_n = \frac{a(1-r^n)}{1-r} = \frac{1(1-1.01^n)}{1-1.01} = \frac{1.01^n - 1}{0.01}$

(A) $\therefore A_n = 100\,000 \times 1.01^n - M\left[\frac{1.01^n - 1}{0.01}\right]$

$A_{240} = 0$

$0 = 100\,000 \times 1.01^{240} - M\left[\frac{1.01^{240} - 1}{0.01}\right]$

$M = \frac{100\,000 \times 1.01^{240}}{\left[\frac{1.01^{240} - 1}{0.01}\right]}$

$= \$1101.09$ to nearest cent. [3]

iii) End of year 10, A_{120} is balance owing

$\therefore A_{120} = 100\,000 \times 1.01^{120} - \$1101.09 \times \left[\frac{1.01^{120} - 1}{0.01}\right]$

$= \$76\,745.39$ [1]

iv) new rate = 6% p.a = 0.5% p.m

adapting A_n from (A)

$A_n = \$76745.39 \times 1.005^n - M\left[\frac{1.005^n - 1}{0.005}\right]$

$\therefore n = 120 \quad A_n = 0$

$0 = \$76745.39 \times 1.005^{120} - M\left[\frac{1.005^{120} - 1}{0.005}\right]$

New \$M = \$852.03 [2]

Question 10

a) $y = (\cos 3x)^2$

$y' = 2\cos 3x \times -\sin 3x \times 3$

$= -6\cos 3x \sin 3x$ [2]

b) $2\ln x = \ln(x+6) \qquad x > 0$

$\therefore \ln x^2 = \ln(x+6)$

$\therefore x^2 = x + 6$

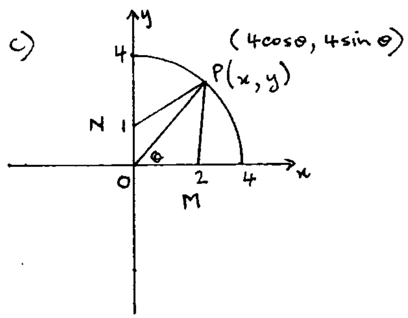
$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ or -2

but $x > 0 \quad \therefore x = 3$ [3]

c)



i) Area OMPN = $\Delta OMP + \Delta ONP$

$= \frac{1}{2} \times 2 \times 4\sin\theta + \frac{1}{2} \times 1 \times 4\cos\theta$

Area $= 4\sin\theta + 2\cos\theta$ [2]

ii) Maximum if $\frac{dA}{d\theta} = 0$

$A = 4\sin\theta + 2\cos\theta$

$\frac{dA}{d\theta} = 4\cos\theta - 2\sin\theta$

let $\frac{dA}{d\theta} = 0 \qquad 4\cos\theta - 2\sin\theta = 0$

$\therefore \frac{4\cos\theta}{\cos\theta} = \frac{2\sin\theta}{\cos\theta}$

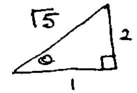
$\therefore \tan\theta = 2$

consider $\frac{d^2A}{d\theta^2} = -4\sin\theta - 2\cos\theta$

as θ is acute $\sin\theta$, $\cos\theta$ are both +ve $\therefore \frac{d^2A}{d\theta^2} < 0$ so $\tan\theta = 2$ gives a maximum value. [3]

iii) $\therefore P(4\cos\theta, 4\sin\theta)$

$\therefore \tan\theta = 2$



$\cos\theta = \frac{1}{\sqrt{5}}$

$\sin\theta = \frac{2}{\sqrt{5}}$

$\therefore P\left(\frac{4}{\sqrt{5}}, \frac{8}{\sqrt{5}}\right)$ [2]