THE SCOTS COLLEGE

YEAR 12

2/3 UNIT MATHEMATICS

HIGHER SCHOOL CERTIFICATE TRIAL

AUGUST 2000

TIME ALLOWED: 3 HOURS
[PLUS 5 MINUTES READING TIME]

DIRECTIONS TO STUDENTS:

- ALL QUESTIONS ARE OF EQUAL VALUE.
- START EACH QUESTION IN A NEW BOOKLET.
- MARKS MAY BE DEDUCTED FOR CARELESS OR UNTIDY WORK.
- ONLY BOARD APPROVED CALCULATORS MAY BE USED.
- TABLE OF STANDARD INTEGRALS IS PROVIDED.

QUESTION 1

a. Solve for x; $(x+2)^2 = 9$

[2 MARKS]

b. Express $\dfrac{3}{\sqrt{5}+2}$ in the form of $a\sqrt{5}+b$

[3 MARKS]

c. Differentiate the following:

[4 MARKS]

(i) $y = x^2 e^{2x}$

(ii) $y = \dfrac{\sin 3x}{x}$

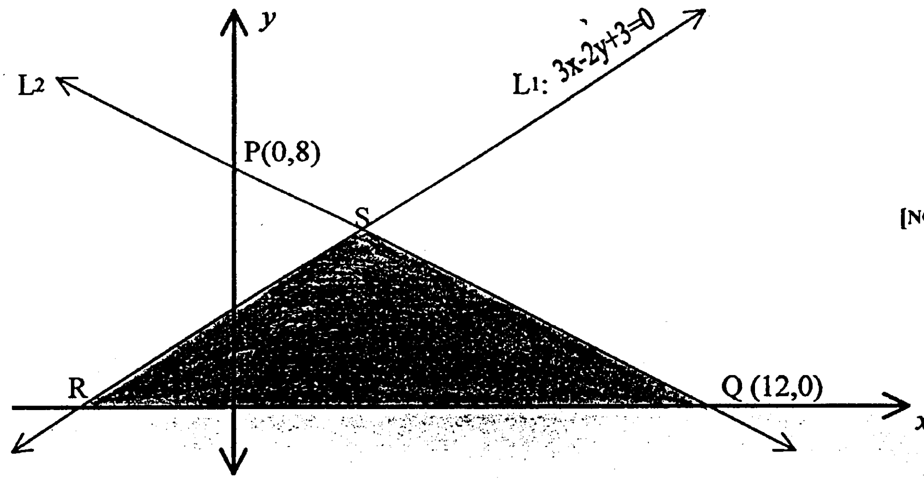
d. Find:

[3 MARKS]

(i) $\int \dfrac{dx}{x+5}$

(ii) $\int \sec^2 3x \; dx$

QUESTION 2 [START A NEW BOOKLET]



[NOT TO SCALE]

In the diagram, L_1 has equation $3x - 2y + 3 = 0$.

a. Find the coordinates of R. [1 MARK]

b. Find the gradient of line L_2. [1 MARK]

c. Show that line L_1 is perpendicular to line L_2. [2 MARKS]

d. Find the equation of line L_2, in general form. [2 MARKS]

e. Find the coordinates of S. [2 MARKS]

f. Hence calculate the area of ΔRSQ [2 MARKS]

g. Give the inequalities which represent this shaded area. [2 MARKS]

QUESTION 3 [START A NEW BOOKLET]

a. Differentiate:

[4 MARKS]

(i) $y = \ln(6x^2 - 3)$

(ii) $y = (\cos x + \sin 2x)^3$

b. Find the equation of the normal to the curve $y = x^3 - x + 5$ at $x = 2$. **[4 MARKS]**

c. In Australia, Roulette is played on a wheel with 37 equal slots. The slots are numbered 0, 1, 2, ... 36 and are randomly distributed around the wheel. The slots are coloured alternately red and black except the 0 slot which is usually green.

[4 MARKS]

What is the probability that:

(i) Zero shows in one spin?

(ii) A red shows in one spin?

(iii) The number 13 does not show in 5 spins?

QUESTION 4 [START A NEW BOOKLET]

a. Evaluate: [3 MARKS]

$$\int_{-1}^{1} (e^{x} - e^{-x})^2 \, dx$$

b. Use Simpson's Rule, with five function values to approximate $\int_1^5 x \ln x \, dx$, correct to 2 decimal places.

[3 MARKS]

c. After t years, the value V, in dollars, of a car worth $30000 is given by the formula $V = Ae^{-0.2t}$

[6 MARKS]

(i) Find the value of A.

(ii) Find the value of the car after 6 years.

(iii) Find the rate, in dollars per year, at which the car is depreciating after 6 years.

(iv) How long will it take before the car is worth $12000?

<u>QUESTION 5</u> [START A NEW BOOKLET]

a. Solve for $0 \le x \le 2\pi$, $\tan x = -\sqrt{3}$. **[2 MARKS]**

b. Find the value(s) of m for which the equation $x^2 + 4mx + 8 - 4m = 0$ has equal roots. **[2 MARKS]**

c. The graph of $f(x) = 7 + 5x - x^2 - x^3$ is defined in the domain $-3 \le x \le 3$. **[8 MARKS]**

(i) Find the coordinates of the stationary points and determine their nature.

(ii) Find the coordinates of any points of inflexion.

(iii) Hence sketch $y = f(x)$ for the given domain.

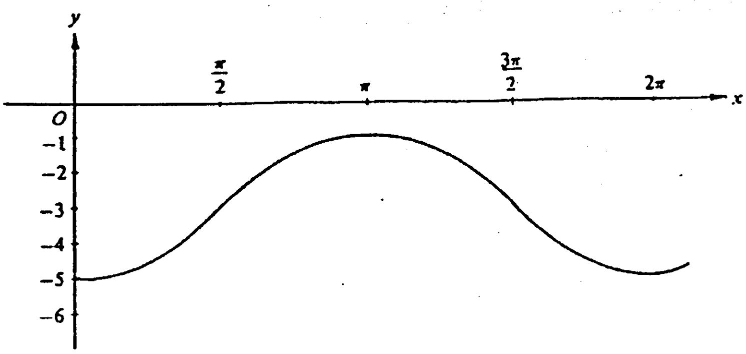
QUESTION 6 **[START A NEW BOOKLET]**

a. A graduate earns $28000 per annum in her first year, then in each successive year her salary rises by $1600. **[3 MARKS]**

(i) What is her salary in her 10th year?

(ii) What are her total earnings over ten years?

b. The graph of $y = A + B\cos Cx$ is given below. **[3 MARKS]**



(i) State the amplitude and period of the curve.

(ii) Hence or otherwise, determine the value of the constants A, B and C.

c. The velocity v m/s of a particle moving in a straight line is given by: $v = 3t^2 - 5t - 2$. The particle is initially 10m to the left of the origin. **[6 MARKS]**

Find:

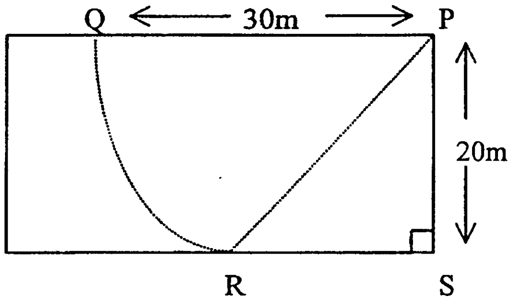
(i) The displacement and acceleration at any time t.

(ii) The direction in which the particle will initially move.

(iii) When and where the particle comes to rest.

QUESTION 7 [START A NEW BOOKLET]

a. **[5 MARKS]**



[DIAGRAM NOT TO SCALE]

The diagram represents a rectangular paddock 45m by 20m. It is fenced in completely and a horse is tied to one of the corner posts, P, with a rope 30m long.

(i) Show that $\angle QPR$ is $41^0 49'$, to the nearest minute.

(ii) The total area over which the horse can graze within the paddock is bounded by the arc QR and fences SR, SP and PQ. Calculate this area, to the nearest square metre.

b. For the function $f(x) = 2\cos x + 1$: **[7 MARKS]**

(i) State the range of $f(x)$.

(ii) Sketch $y = f(x)$ for $0 \le x \le 2\pi$

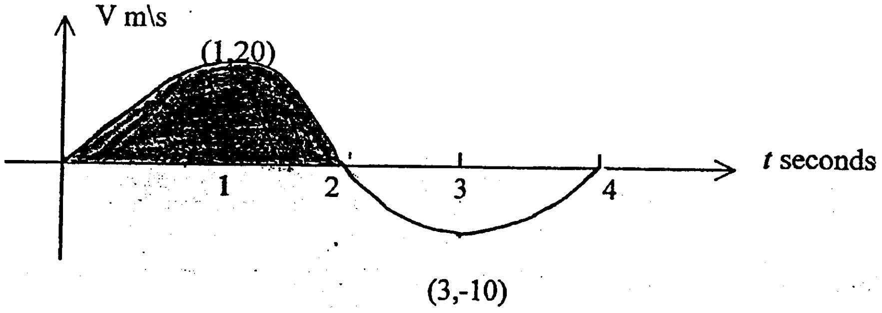
(iii) Calculate the exact area of the region bounded by the curve $y = 2\cos x + 1$, the y axis and the line $y = 1$.

QUESTION 8 **[START A NEW BOOKLET]**

a. Find the sum of 10 terms of the series $\log_m 3 + \log_m 6 + \log_m 12 + ... + ...$ given that $\log_m 3 = 0.48$ and. $\log_m 2 = 0.30$.

[3 MARKS]

b. **[4 MARKS]**



(i) What is the velocity of the particle after 1 second?

(ii) What is the acceleration of the particle after 3 seconds?

(iii) When does the particle change direction?

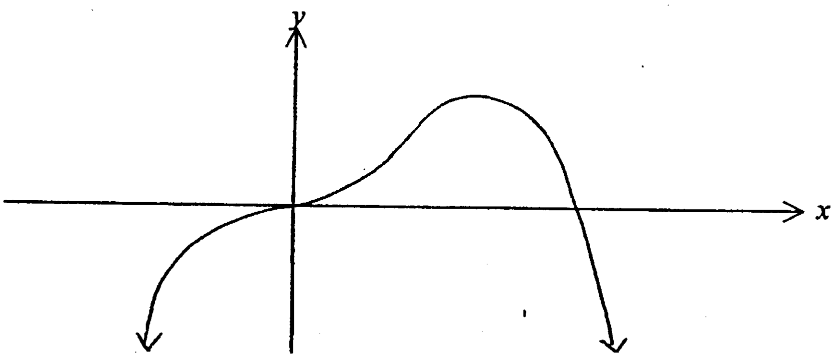
(iv) Explain what is represented by the shaded area in the diagram.

c. (i) On the same diagram, sketch $y = \log_e x$ and $y = \log_e 2$. **[5 MARKS]**

(ii) The area bounded by $y = \log_e x$, $y = \log_e 2$ and the x and y axes is rotated about the y axis. Calculate the volume of the solid generated.

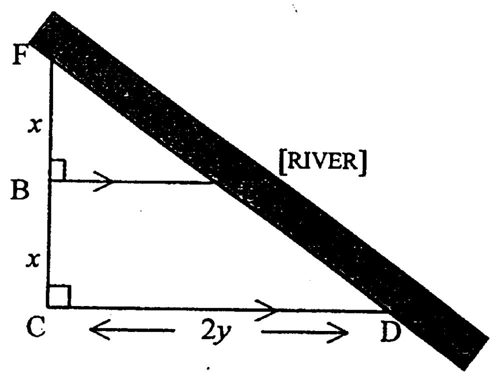
QUESTION 9 **[START A NEW BOOKLET]**

a. (i) Copy the graph of $y = f(x)$ below into your answer booklet. **[2 MARKS]**



(ii) On the same axes, sketch the possible graph for $y = f'(x)$

b. A farmer, who wishes to keep his animals separate, sets up his field so that fences exist at FC, CD and BE, as shown in the diagram below. The side FD is beside a river and no fence is needed there. **[8 MARKS]**



In the diagram, FB = BC = x metres and CD = $2y$ metres.

(i) If the area of the field FCD is 1200m^2, show that L, the amount of fencing needed, is given by $L = 2x + \dfrac{1800}{x}$.

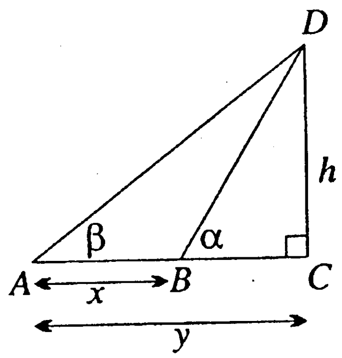
(ii) Hence, find the values of x and y so that the farmer uses the minimum amount of fencing.

QUESTION 10 **[START A NEW BOOKLET]**

a. (i) Given that $a^2 + b^2 = 23ab$, express $\left(\frac{a+b}{5}\right)^2$ in terms of a and b. **[4 MARKS]**

(ii) Hence, show that $\ln\left[\frac{1}{5}(a+b)\right] = \frac{1}{2}[\ln a + \ln b]$

b. Consider the diagram given: **[8 MARKS]**



(i) Show that $BD = \dfrac{x\sin\beta}{\sin(\alpha - \beta)}$.

(ii) Hence show that $h = \dfrac{x\sin\alpha\sin\beta}{\sin(\alpha - \beta)}$.

(iii) Prove that $h = \dfrac{x\tan\alpha\tan\beta}{\tan\alpha - \tan\beta}$.

(iv) By using parts **(ii)** and **(iii)**, show that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

TSC 2000 Solutions.
2/3 unit Trial.

Question 1

a) $(x+2)^2 = 9$

$x + 2 = \pm 3$

$x = 5$ or $x = -1$

b) $\dfrac{3}{\sqrt{5}+2} \times \dfrac{\sqrt{5}-2}{\sqrt{5}-2} = \dfrac{3(\sqrt{5}-2)}{5-4}$

$= 3(\sqrt{5}-2)$

c) (i) $y = x^2 e^{2x}$

$\dfrac{dy}{dx} = x^2 . 2e^{2x} + e^{2x} . 2x$

$= 2xe^{2x}(x+1)$

(ii) $y = \dfrac{\sin 3x}{x}$

$\dfrac{dy}{dx} = \dfrac{x . 3\cos 3x - \sin 3x . 1}{x}$

$= \dfrac{3x \cos 3x - \sin 3x}{x}$

d) (i) $\displaystyle\int \frac{dx}{x+5} = \ln(x+5) + c$

(ii) $\displaystyle\int \sec^2 3x \, dx = \frac{\tan 3x}{3} + c$

(15)

Question 2

a) $3x - 2y + 3 = 0$.

sub $y = 0$, $x = -1$ $\therefore R(-1, 0)$

b) $m_{L_2} = \frac{8-0}{0-12} = -\frac{2}{3}$

(c) $3x - 2y + 3 = 0$ $\therefore m_1 = \frac{3}{2}$ $m_2 = -\frac{2}{3}$

$2y = 3x + 3$ $\therefore$ Since $m_1 \times m_2 = -1$

$y = \frac{3}{2}x + \frac{3}{2}$ $\therefore L_1 \perp L_2$

d) $y - 8 = -\frac{2}{3}(x - 0)$

$3y - 24 = -2x$

$2x + 3y - 24 = 0$

e) $2x + 3y - 24 = 0$ —① sub $y = 6$

$3x - 2y + 3 = 0$ —② $2x + 18 - 24 = 0$

①×3 $6x + 9y - 72 = 0$ $2x = 6$

②×2 $6x - 4y + 6 = 0$ $x = 3$

$13y - 78 = 0$ $\therefore S(3, 6)$

$y = 6$

f) Area $\Delta RSQ = \frac{1}{2} \times 13 \times 6$

$= 39$ sq units.

g) $3x - 2y + 3 \geq 0$, $2x + 3y - 24 \leq 0$ & $y \geq 0$.

Question 3.

a) (i) $y = \ln(6x^2 - 3)$

$\frac{dy}{dx} = \frac{12x}{6x^2 - 3}$

(ii) $y = (\cos x + \sin 2x)^3$

$\frac{dy}{dx} = 3(\cos x + \sin 2x)^2 . (-\sin x + 2\cos 2x)$

$= 3(2\cos 2x - \sin x)(\cos x + \sin 2x)^2$

b) $y = x^3 - x + 5$

$\frac{dy}{dx} = 3x^2 - 1$

when $x = 2$, $\frac{dy}{dx} = 11$

when $x = 2$, $y = 11$

Let eqn of tangent be

$y - y_1 = m(x - x_1)$

$y - 11 = -\frac{1}{11}(x - 2)$

$11y - 121 = -x + 2$

$x + 11y - 123 = 0.$

c) i) $P(0) = \frac{1}{37}$

(ii) $P(\text{red}) = \frac{15}{37}$

(iii) $P(\text{no } 13) = \left(\frac{36}{37}\right)^5$

Question 4

a) $\int_{-1}^{1} (e^x - e^{-x})^2 \, dx$

$= \int_{-1}^{1} e^{2x} - 2 + e^{-2x} \, dx$

$= \left[\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2} \right]_{-1}^{1}$

$= \frac{e^2}{2} - 2 + \frac{e^{-2}}{2} - \left(\frac{e^{-2}}{2} + 2 + \frac{e^2}{2} \right)$

$=$ 3.25

b) $\int_{1}^{5} x \ln x \, dx$

$x_1 = 1$	$y_1 = 0$
$x_2 = 2$	$y_2 = 2\ln 2$
$x_3 = 3$	$y_3 = 3\ln 3$
$x_4 = 4$	$y_4 = 4\ln 4$
$x_5 = 5$	$y_5 = 5\ln 5$

$\int_{1}^{5} x \ln x \, dx \doteq \frac{1}{3}(0 + 5\ln 5 + 4(2\ln 2 + 4\ln 4) + 2(3\ln 3))$

$= 14.12$ (2 d p)

c) $V = Ae^{-0.2t}$

i) $t = 0$, $V = 30000$ $\therefore A = 30000$

ii) $V = 30000e^{-0.2t}$

$t = 6$, $V = 30000e^{-0.2(6)}$

$V = \$9035.80$

(iii) $\frac{dV}{dt} = -0.2V$

$t = 6$, $\frac{dV}{dt} = -0.2(9035.80)$

$= -1807.15$ dollars/yr.

(iv)

$12000 = 30000e^{-0.2t}$

$0.4 = e^{-0.2t}$

$\ln(0.4) = -0.2t$

$t = \frac{\ln(0.4)}{-0.2}$

$t = 4.6$ yrs.

Question 5

a) $\tan x = -\sqrt{3}$

$x = \pi - \frac{\pi}{3}, \ 2\pi - \frac{\pi}{3}$

$x = \frac{2\pi}{3}, \ \frac{5\pi}{3}$

c) $f(x) = 7 + 5x - x^2 - x^3$

(i) $f'(x) = 5 - 2x - 3x^2$

$f''(x) = -2 - 6x$

for stationary pts, $f'(x) = 0$

$5 - 2x - 3x^2 = 0$

$3x^2 + 2x - 5 = 0$

$(3x+5)(x-1) = 0$

$\therefore x = -\frac{5}{3}$ or $x = 1$

when $x = -\frac{5}{3}, \ y = \frac{14}{27}, \ y'' > 0$ ∪

$\therefore \left(-\frac{5}{3}, \frac{14}{27}\right)$ min turning point

when $x = 1, \ y = 10, \ y'' < 0$ ∩

$\therefore (1, 10)$ max turning point.

(ii) for points of inflexion, $y'' = 0$

$-2 - 6x = 0.$

$\therefore x = -\frac{1}{3}$

when $x = -\frac{1}{3}, \ y = 5\frac{7}{27}$

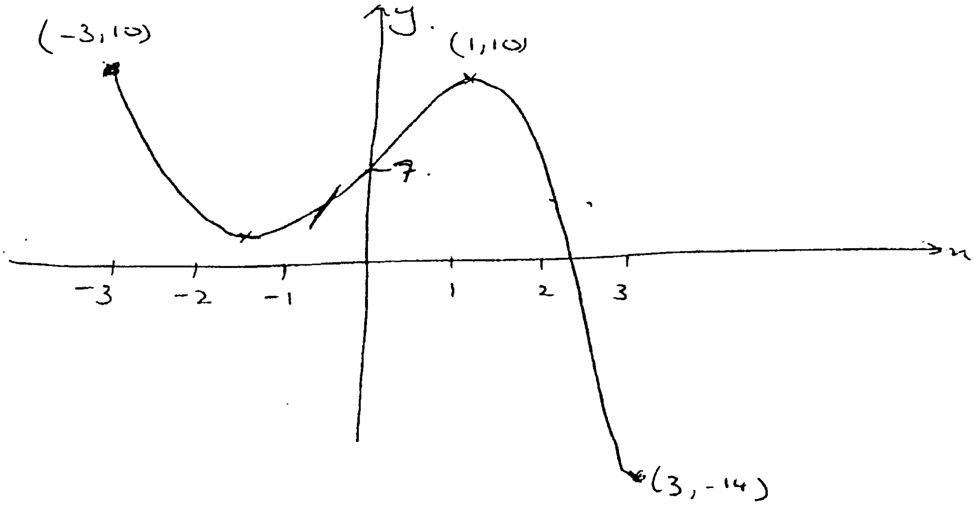
x	-1	$-\frac{1}{3}$	0
y''	$+$	0	$-$

$\therefore$ change in concavity

$\therefore \left(-\frac{1}{3}, 5\frac{7}{27}\right)$ is point of inflexion

(iii) $x = -3, \ y = 10$

$x = 3, \ y = -14.$



(iv) $f'(x) = 5 - 2x - 3x^2$

when $x = 0$, $f'(x) = 5$, $f(x) = 7$.

Let the eq'n of tangent be

$$y - y_1 = m(x - x_1)$$
$$y - 7 = 5(x - 0)$$
$$y = 5x + 7.$$

(b) $x^2 + 4mx + 8 - 4m = 0$

$$\Delta = (4m)^2 - 4(8 - 4m)$$
$$\Delta = 16m^2 + 16m - 32$$

For equal roots, $\Delta = 0$.

$$16m^2 + 16m - 32 = 0$$
$$m^2 + m - 2 = 0$$
$$\therefore (m+2)(m-1) = 0$$
$$m = -2 \quad \text{or} \quad m = 1$$

Question 6

a) $a = 28\,000 \quad d = 1600$

(i) $T_n = a + (n-1)d$

$T_{10} = 28\,000 + (10-1)\,1600$

$= 42\,400$

∴ She earns \$42 400 in the 10th year

(ii) $S_n = \frac{n}{2}(a + \ell)$

$S_{10} = \frac{10}{2}(28\,000 + 42\,400)$

$= 352\,000$

∴ total earnings for 10yrs are \$352 000.

(b) $v = 3t^2 - 5t - 2. \quad t = 0,\ x = -10.$

(i) $a = 6t - 5$

$x = \int 3t^2 - 5t - 2 \; dt$

$x = t^3 - \frac{5t^2}{2} + 2t + c$

when $t = 0$, $x = -10$, ∴ $c = -10$

∴ $x = t^3 - \frac{5t^2}{2} + 2t - 10.$

(ii) $t = 0, \quad v = -2$

∴ particle initially moves in ← direction.

(iii) for rest, $v = 0$.

$3t^2 - 5t - 2 = 0$

$(3t + 1)(t - 2) = 0$

∴ $t = -\frac{1}{3}$ or $t = 2$

but $t \geq 0$ ∴ $t = 2$

when $t = 2$

$x = 8 - \frac{5(4)}{2} + 2(2) - 10$

$x = -8$

∴ particle comes to rest after 2 seconds, 8m to left of origin

Q6

(b) (i) amplitude = 2

period = 2π

(ii) $y = A + B\cos Cx$.

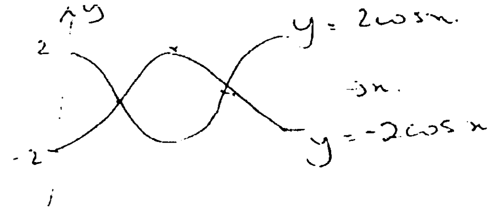
$y = A + 2\cos x$

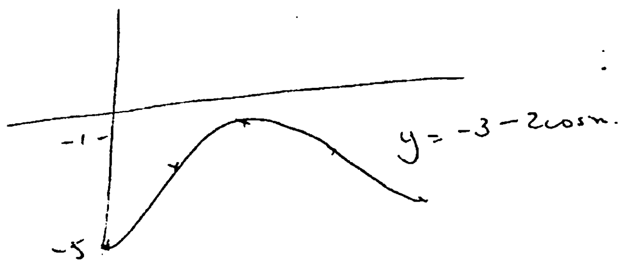
$y = -3 - 2\cos x$

$\therefore A = -3$

$B = -2$

$C = 1$.





Question 7

a) (i) $PR = 30.$

$\cos \angle RPS = \frac{20}{30}$

$\angle RPS = 48^\circ 11'$

$\therefore \angle QPR = 90 - 48^\circ 11'$

$= 41^\circ 49'$

(ii) Area $\triangle RPS$

$= \frac{1}{2} \times 30 \times 20 \times \sin 48^\circ 11'$

$= 223.58$ sq metres

$41^\circ 49' \times \frac{\pi}{180} = 0.73.$

Area $QPR = \frac{1}{2} \times 30^2 \times 0.73$

$= 328.43$ sq m

$\therefore$ Total Area

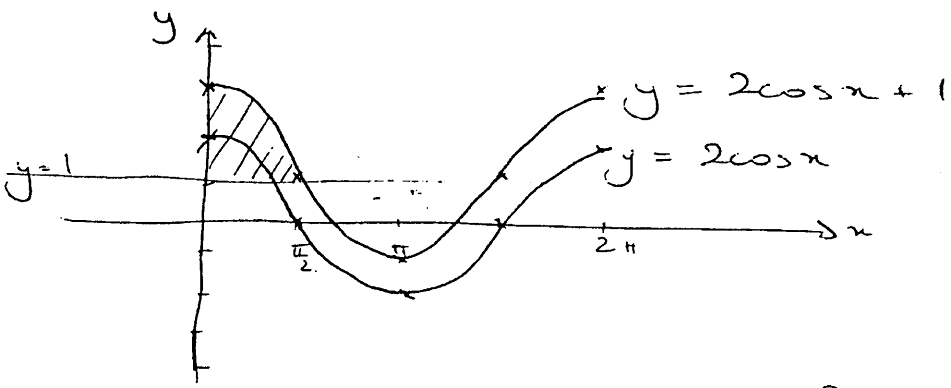
$= 223.58 + 328.43$

$= 552 \text{ m}^2$

b) $f(x) = 2\cos x + 1$

(i) $-1 \le f(x) \le 3$

(ii) $a = 2 \quad p = 2\pi$



(iii) $A = \int_0^{\frac{\pi}{2}} 2\cos x + 1 - 1 \, dx$

$= \int_0^{\frac{\pi}{2}} 2\cos x \, dx$

$= \left[2\sin x \right]_0^{\frac{\pi}{2}}$

$= 2\sin\frac{\pi}{2} - 2\sin 0$

$= 2$ sq units.

s(1
+ - 1

Question 8

a) $\log_m 3 + \log_m 6 + \log_m 12 + \dots$

AP $a = \log_m 3$ $d = \log_m 6 - \log_m 3$

$d = \log_m 2$

$\therefore S_n = \frac{n}{2}(2a + (n-1)d)$

$S_{10} = \frac{10}{2}(2\log_m 3 + (10-1)\log_m 2)$

$= 5(2(0.48) + 9(0.30))$

$= 18.3$

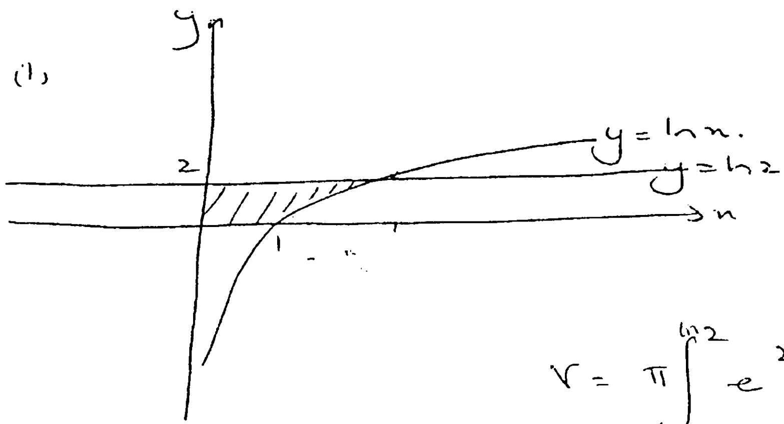
(b) (i) $t = 1$, $v = 20$ m/s → direction

(ii) $t = 3$, $a = 0$ m/s²

(iii) Changes direction, $v = 0$
ie $t = 2$ seconds

(iv) distance travelled in first 2 seconds.

c) (i)



$y = \ln x$
$x = e^y$
$x^2 = e^{2y}$

$V = \pi \int_0^{\ln 2} e^{2y}\, dy$

$= \frac{\pi}{2}\left[e^{2y}\right]_0^{\ln 2}$

~~$= \frac{\pi}{2}(e^4 - 1)$~~ cubic units.

$= \frac{\pi}{2}[4 - 1] = \frac{3\pi}{2}$ cubic units