THE SCOTS COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

YEAR 12 MATHEMATICS

AUGUST 2003

Time allowed – 3 hours

(plus 5 minutes reading time)

Instructions to Candidates:

- All questions are of equal value.
- All questions may be attempted.
- All necessary working should be shown in each question.
- Marks may not be awarded for careless of poorly arranged work.
- Approved non- graphical calculators may be used.
- The questions are not necessarily arranged in order of difficulty. Candidates are advised to read the whole paper carefully at the start of the examination.
- A table of standard integrals is provided at the end of the paper.
- Start each question in a **separate** booklet. Clearly indicate any non – attempted part of a question.

This is a trial paper only and does not necessarily reflect the content or format of the final H.S.C. examination for this subject.

Answer each question in SEPARATE writing booklets. Extra writing booklets are available.

Question 1 (12 Marks) – Use a SEPARATE writing booklet **Marks**

a. Calculate $\dfrac{97.3-6.42}{(0.25)^2}$ correct to one decimal place. **2**

b. Solve: $\dfrac{1}{2}(t-1)-\dfrac{1}{3}(t+2)=4$ **2**

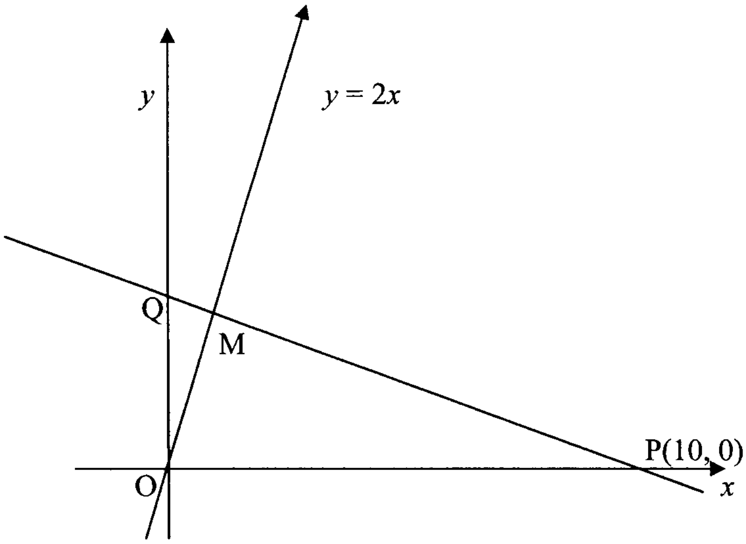
c. Rationalise the denominator of $\dfrac{\sqrt{2}}{\sqrt{2}-1}$. **2**

d. Solve: $x^2-3x-4<0$ **2**

e. Write down the primitive of $\dfrac{1}{x-2}$ **2**

f. Factorise fully: $x^2-y^2+2x-2y$ **2**

Question 2 (12 Marks) – Use a SEPARATE writing booklet



In the above diagram the equation of the line OM is $y = 2x$. The line MP is perpendicular to OM and P is the point (10, 0).

a. COPY the above diagram into your answer booklet. **1**

b. Find the gradient of the line MP **2**

c. Show that the equation of the line MP is $x + 2y - 10 = 0$ **2**

d. Find the coordinates of Q, where the line MP cuts the y – axis **1**

e. Show that the equation of the circle with diameter OP is $(x - 5)^2 + y^2 = 25$. Draw this circle on your diagram. **2**

f. Find the coordinates of the point M and show that it lies on the circumference of the circle in (e). **2**

g. On your diagram SHADE in the region satisfying these three inequalities: $(x-5)^2 + y^2 \leq 25$, $y \leq 3$, $x \geq 0$. **2**

Question 3 (12 Marks) – Use a SEPARATE writing booklet

a. Differentiate:

i. $y = x^3 + 2x^2 - 7$ **1**

ii. $y = x^2 e^x$ **2**

iii. $y = \ln(x^2 - 1)$ **2**

b. Evaluate:

i. $\int e^{2x} dx$ **1**

ii. $\int_1^2 \left(x^2 + \frac{1}{x} \right) dx$ (leaving your answer correct to two decimal places.) **2**

iii. $\int \frac{1}{(4x-1)^3} dx$ **2**

iv. $\int \frac{2x}{4x^2+1} dx$ **2**

Question 4 (12 Marks) – Use a SEPARATE writing booklet

a. For the parabola $y = x^2 + 4$

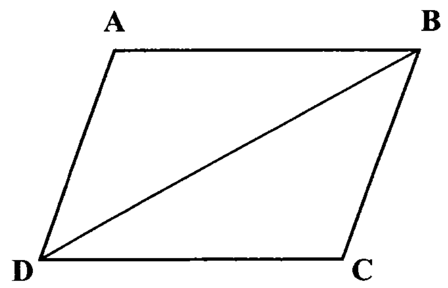
i. Find the coordinates of the focus, the coordinates of the vertex and the equation of the directrix of the parabola. **2**

ii. Graph this parabola in the domain $-4 \le x \le 3$ **1**

iii. Find the minimum and maximum values for the function in the domain given in **ii.** **2**

iv. The region in the first quadrant, bounded by the curve, the y-axis, and the line $y = 8$, is rotated about the y – axis. Calculate the volume of revolution formed, leaving your answer in terms of π . **3**

b. ABCD is a rhombus



(diagram not to scale)

i. COPY the diagram and mark a point R on the diagonal BD. **1**

ii. Prove that the triangles ARD and CRD are congruent. **2**

iii. Hence show that AR = RC. **1**

Question 5 (12 Marks) – Use a SEPARATE writing booklet

a. For the curve $y = \frac{2}{3}x^3 - 8x + 1$

i. Find the coordinates of the stationary points and determine their nature. **4**

ii. Determine any points of inflexion. **2**

iii. SKETCH the curve, showing all important features. **2**

iv. State the domain of x for which the function is increasing. **1**

b. Find the values of m for which the equation

$$x^2 + (m-2)x + 4 = 0 \text{ has}$$

i. Equal roots. **2**

ii. No real roots. **1**

Question 6 (12 Marks) – Use a SEPARATE writing booklet

a. Let α and β be the roots of the equation $5x^2 - 3x - 8 = 0$. Find the value of:

i. $\alpha + \beta$ **1**

ii. $\alpha\beta$ **1**

iii. $\alpha^2\beta + \alpha\beta^2$ **1**

iv. $\dfrac{2}{\alpha} + \dfrac{2}{\beta}$ **2**

b. Brett is given two tickets in a raffle in which 100 tickets are sold. If there are three prizes drawn find the probability that Brett:

i. wins first prize; **1**

ii. wins at least one prize (in simplest fraction form); **2**

iii. wins exactly one prize (correct to 4 significant figures). **2**

c. Graph the solution of $3x \le 15 \le -7x$ on a number line. **2**

Question 7 (12 Marks) – Use a SEPARATE writing booklet

a. In an arithmetic series the addition of the 21^{st} and 51^{st} terms is equal to the 60^{th} term.

i. Find the value of the 12^{th} term. **2**

ii. Show that the sum of the first 12 terms is six times the first term. **2**

b. $1 + p + q + \ldots$ is an arithmetic series. $1 + q + p + \ldots$ is a geometric series and $p \neq q$.

i. Find the value of p and q. **4**

ii. Find the sum of the first 100 terms of the arithmetic series. **2**

iii. Explain why the geometric series has a limiting sum and find this limit. **2**

Question 8 (12 Marks) – Use a SEPARATE writing booklet

a. A population of rats is given by $P = P_0e^{kt}$, where k is a constant, t is the time in weeks and P_0 is the population when $t = 0$.

i. Given that 20 rats increase to 100 rats after 6 weeks, calculate the value of k, to 3 decimal places. **2**

ii. How many rats will there be after 10 weeks? **2**

iii. After how many weeks will there be 500 rats? **2**

b. Consider the function given by $y = \log_{10}x$.

i. Copy and complete the table, to 3 decimal places, in your examination booklet. **2**

x	1	2	3	4	5
y	0				

ii. Apply the trapezoidal rule with 4 subintervals to find an approximation, to 2 decimal places, of $\int_1^5 \log_{10}x \; dx$. **4**

Question 9 (12 Marks) – Use a SEPARATE writing booklet

a. The acceleration in ms^{-2} of a particle moving in a straight line is given by $a = -2$. If initially the particle has a velocity of $6ms^{-1}$ and starts $16m$ to the right of a fixed point P, find:

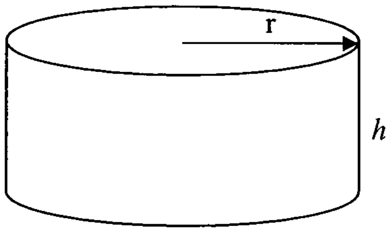
i. the velocity and displacement of the particle in terms of t. **2**

ii. the time(s) when the displacement is $9m$ to the right of P. **1**

iii. the displacement when the particle is at rest. **1**

iv. The distance travelled in the first five seconds. **2**

b.



The surface area of a cylinder is given by the formula $S = 2\pi r(r + h)$. If the cylinder above has a surface area of $160cm^2$, then:

i. Show that the volume is given by $V = 80r - \pi r^3$. **2**

ii. Find the value of r, to 2 decimal places, that gives the maximum volume. **3**

iii. Find the maximum volume correct, to 1 decimal place. **1**

Question 10 (12 Marks) – Use a SEPARATE writing booklet

a. **i.** On the same diagram, sketch the curve $y = \sin \pi x$ and the line $y = x$ in the domain $-1 \leq x \leq 1$. **2**

ii. Use the diagram to find the number of solutions of the equation $\sin \pi x - x = 0$. **1**

b. A couple wishing to buy a holiday house on the central coast of NSW require a loan of $180 000. The loan is to be repaid over 25 years in equal monthly instalments. The interest is 12%p.a. compounded monthly on the balance owing.

i. Show that the size of each monthly repayment to the nearest dollar is $1896. **3**

ii. Find the total interest which will be paid. **2**

iii. Calculate the equivalent simple interest rate on the loan per annum. **2**

iv. How much is still owed on the property, to the nearest dollar after 7 years? **2**

END OF PAPER

THE SCOTS COLLEGE – TRIAL HSC SOLUTIONS 2003

Question 1.

(a) $\dfrac{90.88}{0.0625} = 1454.08$

$= \underline{1454.1}$ (1dp) ②

(b) $3(t-1) - 2(t+2) = 24$

$3t - 3 - 2t - 4 = 24$

$t - 7 = 24$

$\therefore \underline{t = 31}$. ②

(c) $\dfrac{\sqrt{2}}{\sqrt{2}-1} \times \dfrac{\sqrt{2}+1}{\sqrt{2}+1}$

$= \dfrac{2+\sqrt{2}}{2-1}$

$= \underline{2+\sqrt{2}}$. ②

(d) $x^2 - 3x - 4 < 0$

$(x-4)(x+1) < 0$

$\underline{-1 < x < 4}$ ②

(e) $\int \dfrac{1}{x-2}\,dx = \underline{\log_e(x-2) + c}$ ②

(f) $x^2 - y^2 + 2x - 2y$

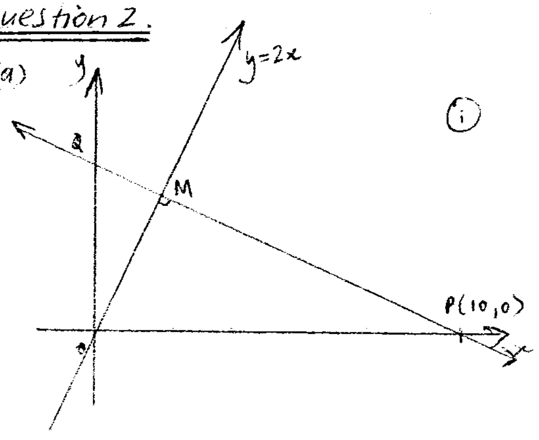
$= (x-y)(x+y) + 2(x-y)$

$= (x-y)[x+y+2]$

$= \underline{(x-y)(x+y+2)}$. ②

Question 2.

(a)



①

(b) MP ⊥ $y = 2x$ (given)

$\therefore$ m of MP $= \underline{\dfrac{-1}{2}}$ $(m_1 m_2 = -1)$. ②

(c) $m = \dfrac{-1}{2}$, $P(10,0)$

using $y - y_1 = m(x - x_1)$

$y - 0 = \dfrac{-1}{2}(x - 10)$

$2y = -x + 10$

$\therefore \underline{x + 2y - 10 = 0}$ as required ②

(d) coordinates of $Q(0, y)$

$\therefore 0 + 2y = 10$

$y = 5$

then $\underline{Q(0,5)}$. ①

(e) $O(0,0)$, $P(10,0)$

Midpoint of OP has coordinates $(5,0)$ which represents the centre of the circle with radius 5 units.

So the equation becomes:

$(x-5)^2 + (y-0)^2 = 5^2$

$\therefore \underline{(x-5)^2 + y^2 = 25}$ as reqd

(see diagram for sketch)

(f) $y = 2x$ — (1)

$x + 2y - 10 = 0$ — (2) sub (1) into (2).

then $x + 2(2x) - 10 = 0$

$x + 4x - 10 = 0$

$\therefore x = 2$. when $x = 2$, $y = 2 \times 2 = 4$.

$\therefore M(2,4)$

using the equation of the circle

$(2-5)^2 + 4^2 = 5^2$

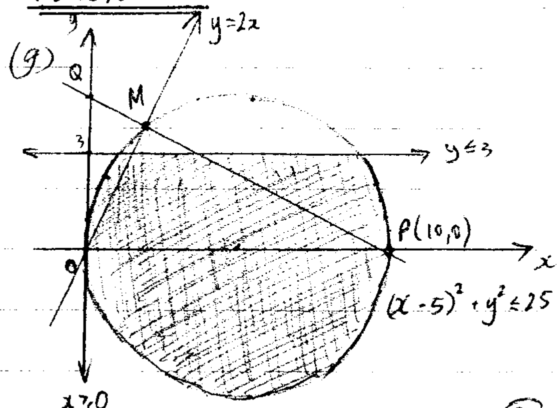
$9 + 16 = 25$

LHS = RHS

$\therefore$ M lies on the circumference of $\underline{(x-5)^2 + y^2 = 25}$ ②

Question 2

(g)



(2)

Question 3

(a)(i) $y' = 3x^2 + 4x$ (1)

(ii) $u = x^2 \quad v = e^x$

$u' = 2x \quad v' = e^x$

$y' = x^2e^x + 2xe^x$

$= xe^x(x+2)$ (2)

(iii) $y' = \dfrac{2x}{x^2-1}$

$= \dfrac{2x}{(x-1)(x+1)}$ (2)

(b)(i) $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$ (1)

(ii) $\int_1^2 \left(x^2 + \frac{1}{x}\right) dx$

$= \left[\frac{x^3}{3} + \log_e x\right]_1^2$

$= \left(\frac{8}{3} + \log_e 2\right) - \left(\frac{1}{3} + \log_e 1\right)$

$= \frac{7}{3} + \log_e 2$

$\doteqdot 2.63$ (2dp). (2)

(iii) $\int (4x-1)^{-3}\,dx$

$= \dfrac{(4x-1)^{-2}}{-2.4}$

$= \dfrac{-1}{8(4x-1)^2} + C$ (2)

(iv) $\int \dfrac{2x}{4x^2+1}\,dx$

$= \frac{1}{4}\log_e(4x^2+1) + C$ (2)

Question 4.

(i) $y = x^2 + 4$

ie $x^2 = y - 4$

$V(0,4)$ coordinates of vertex.

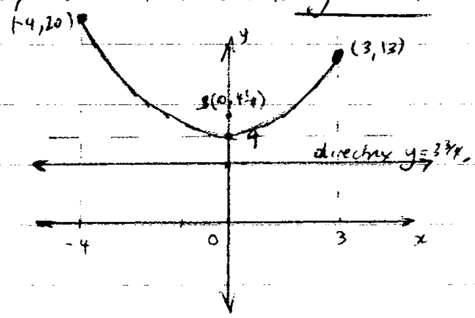
Since $4a = 1$

$\therefore a = 1/4$

So focus is $(0, 4\frac{1}{4})$

and equation of directrix $y = 3\frac{3}{4}$. (2)

(ii)



(1)

(iii) Maximum value when $x = -4$, $y = 20$. (1)

minimum value occurs when $x = 0$, $y = 4$. (1)

(iv) Volume: $V = \pi\int x^2\,dy$

$= \pi\int_4^8 (y-4)\,dy$

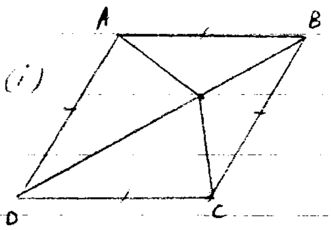
$= \pi\left[\frac{y^2}{2} - 4y\right]_4^8$

$= \pi[(32-32) - (8-16)]$

$V = 8\pi$ units³. (3)

Question 4

(b) (i)



(1)

(ii) $AD = DC$ (equal sides of a rhombus)

$\angle ADB = \angle CDB$ (diagonal bisect opp. ∠s)

DR is common.

$\Delta ARD \equiv \Delta CRD$ (SAS). (2)

(iii) Since

$\Delta ARD \equiv \Delta CRD$

then $AR = RC$ (1)

Question 5

(a) (i) $y = \frac{2}{3}x^3 - 8x + 1$

Stationary points occur when $f'(x) = 0$.

$f'(x) = 2x^2 - 8$

so $2(x^2 - \ ) = 0$

$2(x-2)(x+2) = 0$

$\therefore x = 2, \ y = -9^2/_3$

$x = -2, \ y = 11^2/_3$

When $x = 2$, $f''(2) > 0 \ \therefore$ min.

$x = -2$, $f''(2) < 0 \ \therefore$ max

$\therefore (2, -9^2/_3)$ minimum turning point. (2)

$(-2, 11^2/_3)$ maximum turning point. (2)

(ii) Points of inflexion when $f''(x) = 0$

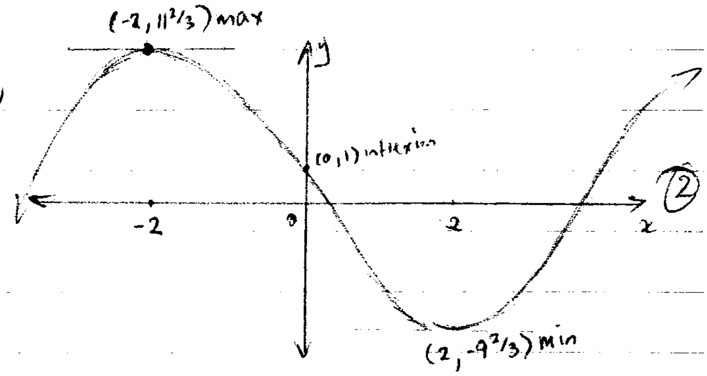
$f''(x) = 4x = 0$

check $x = {}^+0$, $f''(x) > 0$

$x = {}^-0$, $f''(x) < 0$

$\therefore (0, 1)$ is a point of inflexion (2).

(iii)



(2)

(iv) The function is increasing in the domain defined by $x > 2$ and $x < -2$.

b. (i) Equal roots occur when $\Delta = 0$

ie $b^2 - 4ac = 0$

$(m-2)^2 - 4(1)(4) = 0$

$m^2 - 4m - 12 = 0$

$(m-6)(m+2) = 0$

$m = 6, \ m = -2$ (2)

(ii) no real roots $\Delta < 0$

$\therefore m^2 - 4m - 12 < 0$

$-2 < m < 6$ (1)

Question 6.

(a) (i) $\alpha + \beta = \frac{-b}{a}$

$= \frac{--3}{5} = \frac{3}{5}$ (1)

(ii) $\alpha\beta = c/a = -8/5$ (1)

(iii) $\alpha^2\beta + \alpha\beta^2$

$= \alpha\beta(\alpha + \beta)$

$= \frac{-8}{5}\left(\frac{3}{5}\right)$

$= \frac{-24}{25}$ (1)

Question 6

(a)(iv) $\frac{2}{\alpha} + \frac{2}{\beta} = \frac{2(\alpha+\beta)}{\alpha\beta}$

$= \frac{2 \times \left(\frac{3}{5}\right)}{-\left(\frac{8}{5}\right)}$

$= -3/4$.

b. (i) $P(win) = \frac{2}{100} = \frac{1}{50}$ (1)

(ii) $P(\text{at least one win}) = 1 - P(LLL)$

$= 1 - \left(\frac{98}{100} \times \frac{97}{99} \times \frac{96}{98}\right)$

$= 1 - \frac{776}{825}$

$= \frac{49}{825}$ (2)

(iii) $P(WLL) + P(LWL) + P(LLW)$

$= \left(\frac{2}{100} \times \frac{98}{99} \times \frac{97}{98}\right) + \left(\frac{98}{100} \times \frac{2}{99} \times \frac{97}{98}\right) + \left(\frac{98}{100} \times \frac{97}{99} \times \frac{2}{98}\right)$

$= \frac{97}{4950} + \frac{97}{4950} + \frac{97}{4950}$

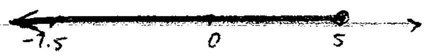
$= \frac{97}{1650} = 0.05879$ (4 s.f).

or 5.879×10^{-2}.

(c) $3x \le 15 \le -7x$

ie $3x \le 15$ or $15 \le -7x$

$x \le 5$ or $x \le -7.5$



Question 7.

(a)(i) $T_{21} = a + 20d$

$T_{51} = a + 50d$

$T_{60} = a + 59d$

$(a + 20d) + (a + 50d) = a + 59d$

$2a + 70d = a + 59d$

$a + 11d = 0$

$\therefore T_{12} = 0$, so 12^{th} term is zero. (2)

(ii) $S_{12} = \frac{n}{2}(a + l)$

$= \frac{12}{2}(a + 0)$

$\therefore S_{12} = 6a$

So the sum of the first 12 terms is 6 times the first term. (2)

(b)(i) $p - 1 = q - p$ & $\frac{q}{1} = \frac{p}{q}$

ie $2p = q + 1$ ie $p = q^2$

Hence $2q^2 = q + 1$

$2q^2 - q - 1 = 0$

$(2q + 1)(q - 1) = 0$

$q = -\frac{1}{2}$ or 1

$p = \frac{1}{4}$ or 1

But $p \ne q$, hence $p = \frac{1}{4}$, $q = -\frac{1}{2}$ (2)

(ii) $1 + \frac{1}{4} - \frac{1}{2} + \ldots$

$a = 1$, $d = -3/4$

$S_{100} = \frac{100}{2}\left[2 + (100-1)(-3/4)\right]$

$S_{100} = -3612.5$. (2)

(iii) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

$r = -\frac{1}{2}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - (-\frac{1}{2})}$

$S_\infty = \frac{2}{3}$ (2)

Question 8

(a)(i) $P = P_0 e^{kt}$

$100 = 20e^{6k}$

$5 = e^{6k}$

$\log_e 5 = 6k \log_e e$

$\therefore k = \frac{1}{6}\log_e 5$

$k = 0.268$ (3dp). (2)

(ii) $P = 20e^{0.268t}$

when $t = 10$, $P = ?$

$P = 20e^{0.268(10)}$

$P = 292.4$

$P \doteqdot 292$

$P \doteqdot 292$ rats after 10 weeks. (2)

(iii) find t, when $P = 500$

$500 = 20e^{0.268t}$

$\log_e \frac{500}{20} = 0.268t \log_e e$

$t = \frac{\log_e 25}{0.268}$

$t = 12$ weeks (2)

(b) (i)

x	1	2	3	4	5
y	0	0.301	0.477	0.602	0.699

(2)

(ii) $h = \frac{5-1}{4} = 1$.

$\therefore \int_1^5 \log_{10} x \, dx \doteqdot \frac{1}{2}[0 + 2(0.301) + 2(0.477) + 2(0.602) + 0.699]$

$\doteqdot \frac{1}{2}(3.459)$

$\doteqdot 1.7295$

$= 1.73$ (2dp) (4)

Question 9.

(a) $a = -2$ @ $t = 0$, $v = 6$, $x = 16$.

(i) $a = -2$

$v = -2t + c$

when $t = 0$, $v = 6$

$\therefore c = 6$

V in terms of time is given by

$V = -2t + 6$ (1)

then $x = -t^2 + 6t + c_1$

when $t = 0$, $x = 16$, $\therefore c_1 = 16$

so $x = -t^2 + 6t + 16$ (1)

(ii) find t, when $x = 9$.

$t^2 - 6t - 7 = 0$

$(t - 7)(t + 1) = 0$

$\therefore t = 7$ seconds when $x = +9$m. (1)

(iii) find x when $v = 0$

$0 = -2t + 6$

$t = 3$

$\therefore x = -t^2 + 6t + 16$

$x = -(3)^2 + 6(3) + 16$

$x = 25$m (1)

(iv) $t = 0$; particle 16m to the right.

$t = 3$; particle 25m to the right.

<changes direction>

$t = 5$; particle 21m to the right.

$\therefore 9 + 4 = 13$m

Question 9

(b) (i) $160 = 2\pi r(r+h)$

$80 = \pi r^2 + \pi r h$

we know $V = \pi r^2 h$

$\therefore h = \dfrac{V}{\pi r^2}$

from above $80 = \pi r^2 + \pi r \times \dfrac{V}{\pi r^2}$

$80 = \pi r^2 + \dfrac{V}{r}$

$80r = \pi r^3 + V$

$\therefore \underline{V = 80r - \pi r^3}$ as reqd.

(ii) $\dfrac{dV}{dr} = 80 - 3\pi r^2$

Stationary points occur $\dfrac{dV}{dr} = 0$

$80 - 3\pi r^2 = 0$

$\pi r^2 = \dfrac{80}{3}$

$r = \pm\sqrt{\dfrac{80}{3\pi}}$

$\therefore$ radius $= 2.91$ cm.

$\dfrac{d^2V}{dr^2} = -6\pi r$

when $r = \sqrt{\dfrac{80}{3\pi}}$ $\dfrac{d^2V}{dr^2} < 0 \therefore$ max.

so the radius of $\underline{2.91\text{ cm}}$ gives a maximum volume.

(iii)

$V = 80r - \pi r^3$

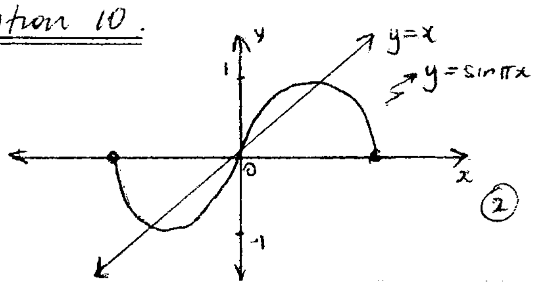
$= 80\left(\sqrt{\dfrac{80}{3\pi}}\right) - \pi\left(\sqrt{\dfrac{80}{3\pi}}\right)^3$

$V = 230.3918305$

$V = \underline{230.4\text{ cm}^3}$

Question 10.

(a)(i)



(2)

(ii) $y = x$ — (1) $\quad y = \sin\pi x$ — (2)

Note: $\sin\pi x - x = 0$ has 3 solutions as their are 3 pts of intersection from the graph (i).

(1)

(b)(i) Let P be the repayments and A be the amount owing.

$A_1 = \$180000 \times 1.01 - P$

$A_2 = \$180000 \times 1.01^2 - P(1 + 1.01)$

$A_3 = \$180000 \times 1.01^3 - P(1 + 1.01 + 1.01^2)$

$\vdots$

$A_{300} = \$180000 \times 1.01^{300} - P(1 + 1.01 + 1.01^2 + \ldots + 1.01^{299})$

but $A_{300} = 0$.

$\therefore P(1 + 1.01 + 1.01^{299}) = \$180000 \times 1.01^{300}$

$P\left(\dfrac{1.01^{300} - 1}{0.01}\right) = \$180000 \times 1.01^{300}$

so $P = \dfrac{0.01 \times \$180000 \times 1.01^{300}}{(1.01^{300} - 1)}$

$\underline{P = \$1896}$ required.

(3)

(ii) Total Repayments $= \$1896 \times 25 \times 12 = \$568800$

$\therefore$ Interest $= \$568800 - \$180000$

$= \underline{\$388800}$.

(iii) $I = \dfrac{P \times R \times T}{100}$ then $388800 = 180000 \times R \times 0.25$

$\underline{R = 8.64\%\text{ pa}}$ (2)

(iv) We know each monthly repayment is \$1896 after 7 years ie 84 repayments

ie $A_{84} = \$180000 \times 1.01^{84} - 1896 \times (1 + \ldots + 1.01^{83})$

$= \$180000 \times 1.01^{84} - 1896\left(\dfrac{1.01^{84} - 1}{0.01}\right)$

$= \underline{\$167455}$.

END OF PAPER.