THE SCOTS COLLEGE

YEAR 12 MATHEMATICS

HSC TRIAL

AUGUST 2008

General Instructions

- All questions are of equal value
- 5 minutes reading time
- Working time - 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- Start a new booklet for each question
- All necessary working should be shown in every question
- Standard Integrals Table is attached

TOTAL MARKS: 120

WEIGHTING: 40 %

QUESTION 1 [12 MARKS] — MARKS

a. Simplify $\sqrt{45}-\sqrt{20}$ [2]

b. Solve for x where $3x^2=4x$ [2]

c. A vertical 1 metre rule casts a shadow 180 centimetres long. Find the angle of elevation of the sun at that time to the nearest degree. [2]

d. Given $S=\pi\sqrt{\dfrac{4-x^3}{1+x^2}}$ and $x=0.859$ evaluate S to three significant figures. [2]

e. Solve for x given that $e^{x^2-x-2}=1$ [2]

f. Solve and graph the solution set of the inequality $|x-3|\le 1$ [2]

QUESTION 2 [12 MARKS] **START A NEW BOOKLET** **MARKS**

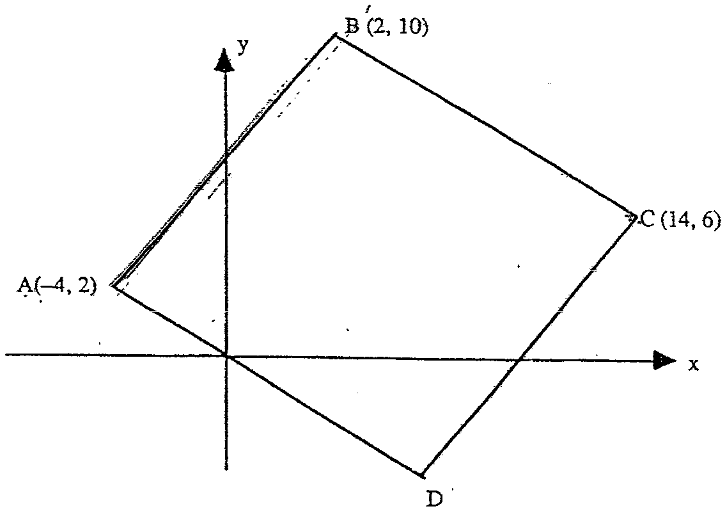


FIGURE NOT TO SCALE

In the diagram, A = (–4,2), B=(2,10), C= (14,6) and ABCD is a parallelogram.

a. Find the length of AB + DC. [2]

b. Find the mid-point of the diagonal AC. [1]

c. Using the fact that the diagonals of a parallelogram bisect each other, or otherwise, find the co-ordinates of point D. [2]

d. Find the gradient of line BC. [1]

e. Show that the side AD does **NOT** pass through the origin (0, 0). [2]

f. You are given that the equation of line AB is $3y = 4x + 22$. Find the shortest distance from C to line AB and deduce the area of ABCD. [4]

QUESTION 3 [12 MARKS] **START A NEW BOOKLET** **MARKS**

a. Differentiate the following with respect to x:

(i) $x^3 - \frac{1}{x}$ [1]

(ii) $\cos(1-2x)$ [1]

(iii) $(x+3)\ln(x+3)$ [2]

b. Find:

(i) $\int e^{3x-2}\,dx$ [1]

(ii) $\int \frac{x}{x^2+3}\,dx$ [1]

(iii) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x\,dx$ [2]

c. The probability of the Lions defeating the Tigers in any one game is 0.6 and the probability of the Tigers defeating the Lions is 0.3.

(i) What is the probability of a draw in any one game? [1]

If the teams play two games, find the probability that:

(ii) There are two draws. [1]

(iii) The Lions win more games than the Tigers. [2]

QUESTION 4 [12 MARKS] START A NEW BOOKLET MARKS

a. The graph given by $y=|2x+k|$ where k is a constant, passes through the point (2,3). Find the possible values of k. [2]

b. **(i)** Find the co-ordinates of the vertex and focus and the equation of the directrix for the parabola $x^2-8y+24=0$. [3]

(ii) Sketch on a neat number plane the locus of points which lie 1 unit from the interval $y=0,\ 2\le x\le 4$. [2]

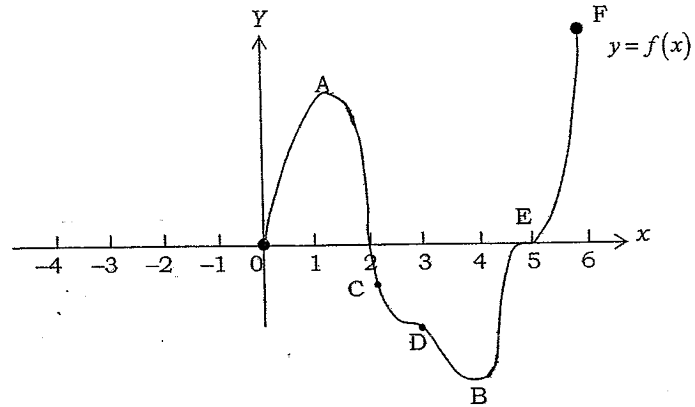
c. In a plague of mice, the number of mice N, at time t days, is given by $N=N_0e^{kt}$, where k is a constant, and N_0 is the number of mice present when $t=0$.

(i) Show that N satisfies the equation $\frac{dN}{dt}=kN$. [1]

(ii) It is estimated that the number of mice increased from 10^6 to 2×10^6 in 30 days. Evaluate k correct to 3 significant figures. [2]

(iii) How many days will it take for the number of mice to increase from 10^6 to 10^7? [2]

QUESTION 5 [12 MARKS] START A NEW BOOKLET MARKS



a. Shown above is a function $y=f(x)$ continuous over its domain $0\le x\le 6$.

(i) Name two stationary points. [1]

(ii) Name three points of inflexion. [1]

(iii) Which point represents the absolute maximum value of $f(x)$? [1]

(iv) Which point represents the absolute minimum value of $f(x)$? [1]

b. If $G(x)=x^3-6x^2$

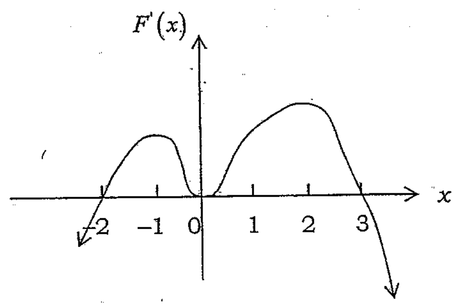
(i) Find $G'(2)$ [1]

(ii) Hence, determine where $y=G(x)$ is increasing. [2]

(iii) Find $G''(x)$ [1]

(iv) Hence, determine where $y=G(x)$ is concave downwards. [1]

c. Copy the graph of $y=F'(x)$ shown below and sketch a possible graph of $y=F(x)$ on the same number plane. [3]



QUESTION 6 [12 MARKS] START A NEW BOOKLET

MARKS

a. **(i)** Show that $3t^2-2t+3$ is a positive definite quadratic by examining the discriminant. [1]

(ii) Find $F(t)$ given that $F'(t)=3t^2-2t+3$ and $F(1)=3$. [1]

(iii) Find where the curve $y=F(t)$ meets the t axis. [1]

(iv) Use the result in part **(i)** to explain why the curve $y=F(t)$ meets the t axis in only one point. [1]

b. **(i)** Sketch the curves $y=2\cos x$ and $y=\cos x$ on the same set of axes for $0\le x\le\frac{\pi}{2}$. [2]

(ii) Find the area bounded by the curves and the y axis. [2]

(iii) Show the volume of the solid generated when this region is rotated about the x axis is given by $V=3\pi\int_0^{\frac{\pi}{2}}\cos^2 x\,dx$. [2]

(iv) Copy and complete the table of values set out below. [2]

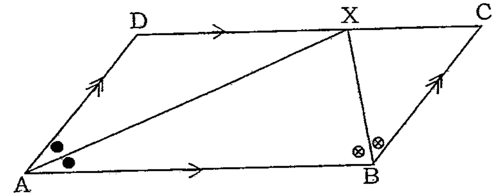
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\cos^2 x$			

Use Simpson's rule and three function values to show $V=\frac{3\pi^2}{4}$ cubic units approximately.

QUESTION 7 [12 MARKS] START A NEW BOOKLET

MARKS

a.



NOT TO SCALE

ABCD is a parallelogram. XA bisects ∠DAB and XB bisects ∠ABC

(i) Copy the diagram into your booklet.

(ii) Prove that ΔADX is isosceles. [2]

(iii) Deduce that X is the midpoint of DC. [2]

(iv) Find the measure of ∠AXB. [2]

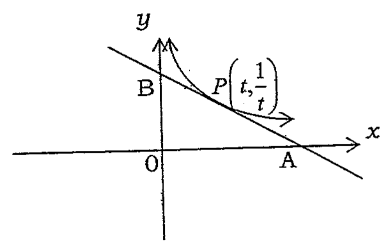
b. The point $P\left(t,\frac{1}{t}\right)$ lies on the hyperbola $y=\frac{1}{x}$.

(i) Given $\frac{dy}{dx}=-\frac{1}{x^2}$ write down the gradient at $x=t$. [1]

(ii) Hence show the equation of the tangent at the point P is $x+t^2y=2t$. [2]

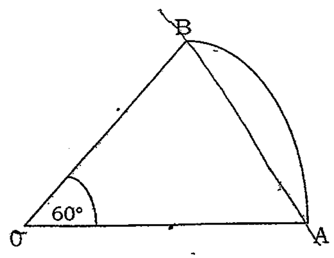
(iii) Find the coordinates of the points A and B where the tangent at P crosses the x and y axes respectively. [2]

(iv) Show the area of the triangle A0B is a constant and so independent of the position of P. [1]



QUESTION 8 [12 MARKS] **START A NEW BOOKLET** **MARKS**

a.



The length of the arc AB of the sector A0B is 2π metres.

(i) Show the length of 0A is 6 metres. [2]

(ii) Calculate the exact area of the sector A0B. [1]

(iii) Calculate the exact area of the segment bounded by the chord AB and the arc AB. [2]

b. The position of a particle at time t seconds is x metres from the origin 0 and $x = 2t^3 - 9t^2 + 12t + 6$.

(i) Find the initial displacement of the particle. [1]

(ii) Find when the particle is at rest. [2]

(iii) Find when acceleration is zero and the velocity at that time. [2]

(iv) How far does the particle travel during the second second? [2]

QUESTION 9 [12 MARKS] **START A NEW BOOKLET** **MARKS**

a. Given an arithmetic series S = 81 + 77 + 73 .... find:

(i) The thirty first term. [1]

(ii) The sum of the first thirty one terms. [2]

(iii) The sum of n terms is denoted by S_n. Find the least value of n for which S_n is negative. [2]

b. Tom borrows $1,000,000 from Nodoc Loan Company. He is to repay it over 10 years in equal monthly instalments. These repayments are due at the end of each month. The interest is calculated on the balance owing at the start of each month and is charged at 18%p.a.

Set $$M$ be the monthly repayment and $$A_n$ be the amount owing after n months.

(i) What interest is charged for the first month? [1]

(ii) Show that $A_1 = 1.015 \times 10^6 - M$
and $A_2 = 1.015^2 \times 10^6 - 1.015M - M$ [2]

(iii) Write down a similar expression to those in **(ii)** for A_n. [1]

(iv) Show that $A_n = 1.015^n \times 10^6 - \frac{M(1.015^n - 1)}{0.015}$ [1]

(v) Find $$M$, correct to the nearest dollar. [2]

QUESTION 10 [12 MARKS] START A NEW BOOKLET

MARKS

a.

(i) For which values of x does the Geometric series $S = e^x + e^{2x} + e^{3x} + \dots$ have a limiting sum? [1]

(ii) Allowing that the series has a limiting sum, find it. [1]

(iii) Find the value of x when the limiting sum (S_∞) equals 2. [2]

b.

(i) Show that $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}$ and π are the solutions of the equation $2\sin^2 x - \sin x = 0,\ 0 \le x \le \pi$. [2]

(ii) Given $y = 2\sin^2 x - \sin x$ show that $\frac{dy}{dx} = \cos x(4\sin x - 1)$. [1]

(iii) Show that the curve $y = 2\sin^2 x - \sin x$ has turning points at $x = 0.25$, $x = \frac{\pi}{2}$ and $x = 2.89$. [2]

(iv) Find the co-ordinates of these turning points to two decimal place accuracy and sketch $y = 2\sin^2 x - \sin x,\ 0 \le x \le \pi$ on a neat number plane. [2]

(v) Write down the maximum value of $2\sin^2 x - \sin x,\ 0 \le x \le \pi$. [1]

Q3 (a) (i) $\frac{d(x^3 - x^{-1})}{dx} = 3x^2 + x^{-2}$ ✓

(ii) $\frac{d(\cos(1-2x))}{dx} = -2x - \sin(1-2x)$

$= 2\sin(1-2x)$ ✓

(iii) $y = (x+3)\ln(x+3)$

$\frac{dy}{dx} = (x+3) \times \frac{1}{x+3} + \ln(x+3) \times 1$ ✓

$= 1 + \ln(x+3)$ ✓

(b) (i) $\int e^{3x-2}\,dx = \frac{1}{3} e^{3x-2} + c$ ✓

(ii) $\frac{1}{2}\int \frac{2x}{x^2+3}\,dx = \frac{1}{2}\ln(x^2+3) + c$ ✓

(iii) $= \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= -\frac{1}{2}\left[\cos 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ ✓

$= -\frac{1}{2}\left[\left(\cos\frac{2\pi}{3}\right) - \left(\cos\frac{\pi}{3}\right)\right]$

$= -\frac{1}{2}\left[-\frac{1}{2} - \left(\frac{1}{2}\right)\right]$

$= -\frac{1}{2}\left[-1\right]$ ✓

$= \frac{1}{2}$

(c) (i) P(DRAW) $= 1 - (0.6 + 0.3)$

$= 0.1$ ✓

(ii) P(TWO DRAWS) $= 0.1 \times 0.1$

$= 0.01$ ✓

(iii) $=$ P(WIN DRAW) $+$ P(WIN WIN) ✓

$= 0.6 \times 0.1 + 0.6 \times 0.6$

$= 0.06 + 0.36$

$= 0.42$ ✓

7mm A4

SOLUTIONS.

Q1. (a) $= 3\sqrt{5} - 2\sqrt{5}$

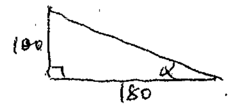
$= \sqrt{5}$ ✓✓

(b) $3x^2 - 4x = 0$

$x(3x-4) = 0$ ✓

$x = 0$ or $\frac{4}{3}$ ✓

(c)



$\tan\alpha = \frac{100}{180}$ ✓

$\alpha = 29.054\ldots$

$\alpha = 29^\circ$ ✓

(d) $S = \pi\sqrt{\frac{4-(0.859)^3}{1+(0.859)^2}}$ ✓

$= 4.3722\ldots$

$= 4.37$ ✓

(e) $x^2 - x - 2 = 0$ ✓

$(x-2)(x+1) = 0$

$x = -1$ or 2 ✓

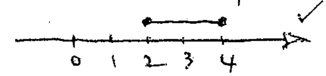
(f) $x - 3 \le 1$ or $-(x-3) \le 1$

$x \le 4$ or $x - 3 \ge -1$

$x \ge 2$ ✓

Solution $2 \le x \le 4$ ✓

Graph ✓



Q2

(i) $AB = \sqrt{(2--4)^2 + (10-2)^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$= 10$ ✓

Since $CD = AB = 10$

$AB + DC = 20$ ✓

(ii) Midpoint is $\left(\frac{14-4}{2}, \frac{2+6}{2}\right)$

or $(5, 4)$ ✓

(iii) Let coords of D be (x_0, y_0)

So $\frac{x_0+2}{2} = 5$ and $\frac{y_0+10}{2} = 4$ ✓

$x_0 + 2 = 10$, $y_0 + 10 = 8$

$x_0 = 8$, $y_0 = -2$

$D(8, -2)$ ✓

(iv) Grad. $BC = \frac{6-10}{14-2}$

$= -\frac{4}{12}$

$= -\frac{1}{3}$ ✓

(v) Grad $AD = -\frac{1}{3}$ ✓

Grad $AO = \frac{2-0}{-4-0}$ ✓

$= -\frac{1}{2}$

Grad AD ≠ Grad AO

so A, O and D are NOT collinear

(vi) Equat AB is $4x - 3y + 22 = 0$

Distance from C is $p = \left|\frac{4\times14 - 3\times6 + 22}{\sqrt{4^2+3^2}}\right|$

$p = \left|\frac{56 - 18 + 22}{\sqrt{16+9}}\right|$

$= \left|\frac{60}{5}\right| = 12$ ✓

Area is Base × Height ✓

$= AB \times p$

$= 10 \times 12$ ✓

$= 120\ u^2$

Q3 (a) (i). ALLOW ANY TWO OF A, B AND E. ✓

(ii) C, D AND E. ✓

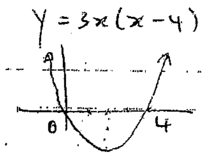
(iii) F ✓

(iv) B ✓

(b) $G(x) = x^3 - 6x^2$

(i) $G'(x) = 3x^2 - 12x$ ✓
$G'(2) = 12 - 24 = -12$

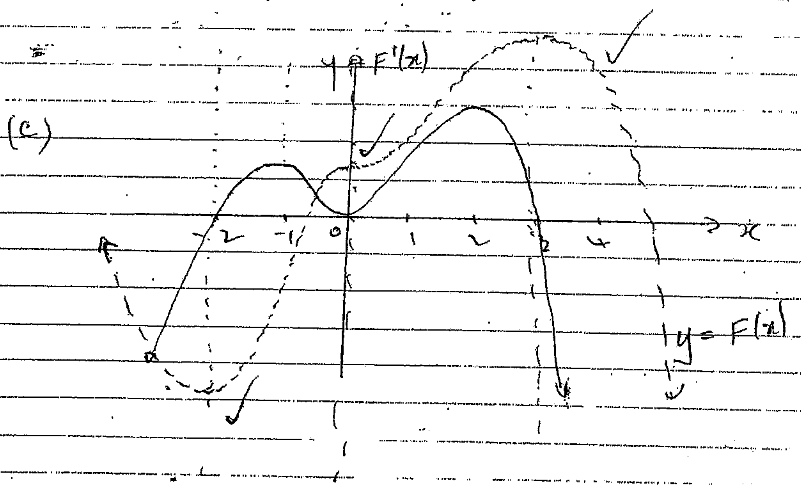
(ii) $G(x)$ increases where $3x(x-4) > 0$
$x < 0$ OR $x > 4$ ✓✓



(iii) $G''(x) = 6x - 12$ ✓

(iv) $G(x)$ is concave down where $G''(x) < 0$
So $6x - 12 < 0$
$x < 2$ ✓

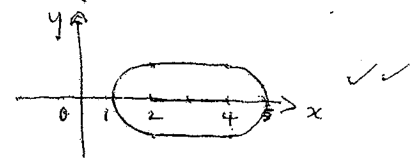
(c)



Q4 (a) Since (2,3) satisfies
$3 = |2(2) + k|$ ✓
$|k + 4| = 3$
$k = -7$ or -1. ✓

(b). (i) $x^2 = 8(y-3)$
Vertex (0,3) focal length $a = 2$ ✓
Focus (0, 5) ✓
Directrix $y = 1$ ✓

(ii)



✓✓

(c). $N = N_0 e^{kt}$

(i) $\frac{dN}{dt} = k(N_0 e^{kt})$
$= kN$ as required ✓

(ii). $2 \times 10^6 = 10^6 e^{30k}$

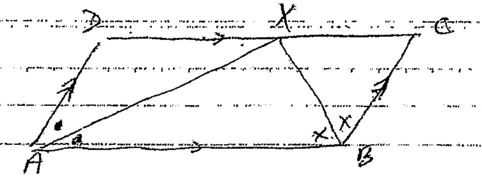
$e^{30k} = 2$ ✓
$30k = \ln 2$
$k = \frac{1}{30} \ln 2$
$k = 0.023104\ldots$ ✓
$k = 0.0231$ to three sig. figs.

(iii) So $10 \times 10^6 = 10^6 e^{kt}$
$e^{kt} = 10$
$kt = \ln 10$ ✓
$t = \frac{\ln 10}{0.0231}$
$= 99.679\ldots$ days. ✓
$\doteqdot 100$

7mm A4

Q7 (a)

(i)



(ii) $\angle DXA = \angle BAX$ (Alternate $\angle$s of $\parallel$ lines) ✓

so $\angle DXA = \angle DAX = \angle BAX$ (Data)

In Δ DAX, $AD = DX$ (Equal sides are opposite equal angles in a Δ) ✓

Δ ADX is isosceles

(iii) By a similar argument Δ BXC is isosceles and $BX = XC$ ✓

but $AD = BC$ (opposite sides of a $\parallel$ogram)

so $DX = XC$ and X is the midpoint of DC. ✓

(iv) $2\angle BAX + 2\angle ABX = \angle BAD + \angle ABC$ ✓

$= 180^\circ$ (Co-interior $\angle$s of $\parallel$ lines) ✓

so $\angle BAX + \angle ABX = 90^\circ$ ✓

and $\angle AXB = 180^\circ - 90^\circ$ (Angle sum of Δ)

$\angle AXB = 90^\circ$

(b) Since $y = x^{-1}$

$\frac{dy}{dx} = -\frac{1}{x^2}$ ✓

At $(t, \frac{1}{t})$ $\frac{dy}{dx} = -\frac{1}{t^2}$ ✓

(i) Equat. tangent is $y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$

$t^2y - t = -x + t$ ✓

$x + t^2y = 2t$

(ii) Cuts $y = 0$ where $x = 2t$ and $x = 0$ where $y = \frac{2}{t}$

$A(2t, 0)$ ✓ $B(0, \frac{2}{t})$ ✓

(iii) Area $\Delta AOB = \frac{1}{2} \times 2t \times \frac{2}{t}$

$= 2\ u^2$ ✓

= a constant which does not depend on t

7mm A4

Q6. (a) (i) $\Delta = (-2)^2 - 4(3)(3)$

$= 4 - 36$

$= -32$ and $a = 3$ so the expression is positive definite. ✓

(ii) $F'(t) = 3t^2 - 2t + 3$

$F(t) = t^3 - t^2 + 3t + c$ Since $F(1) = 3$

$3 = 1 - 1 + 3 + c$ ✓

$c = 0$

$F(t) = t^3 - t^2 + 3t$

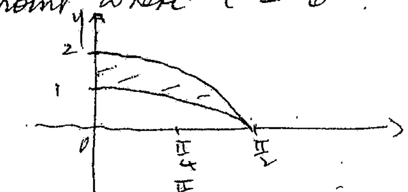
(iii) $F(t) = 0$ where $t(t^2 - t + 3) = 0$ ✓

$t = 0$ or $t^2 - t + 3 = 0$.

So $t = 0$

(iv) Since $F'(t) > 0$ for all t it follows that $F(t)$ increases over its domain and so $F(t)$ meets the t axis at the point where $t = 0$. ✓

(b) (i)



✓✓

(ii) Area is $\int_0^{\frac{\pi}{2}} 2\cos x - \cos x\ dx$

$= \int_0^{\frac{\pi}{2}} \cos x\ dx$

$= [+\sin x]_0^{\frac{\pi}{2}}$

$= (\sin\frac{\pi}{2}) - (\sin 0)$

$= 1$ ✓✓

(iii) $V = \pi\int_0^{\frac{\pi}{2}} (2\cos x)^2\,dx - \pi\int_0^{\frac{\pi}{2}} (\cos x)^2\,dx$

$= \pi\int_0^{\frac{\pi}{2}} 4\cos^2 x - \cos^2 x\ dx$ ✓✓

$= 3\pi\int_0^{\frac{\pi}{2}} \cos^2 x\ dx$

(iv)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
$\cos^2 x$	1	$\frac{1}{2}$	0

✓

$V = 3\pi \cdot \frac{\frac{\pi}{2} - 0}{6}\{1 + 0 + 4 \times \frac{1}{2}\}$

$= 3\pi \cdot \frac{\pi}{12}[3]$ ✓

$= \frac{9\pi^2}{} = \frac{3\pi^2}{}$

Q9 (a). $S = 81 + 77 + 73 + \ldots$

(i) $a = 81$, $d = -4$, $n = 31$

So $T_{31} = 81 + (31-1) \times -4$
$= 81 + 30 \times -4$
$= 81 - 120$
$= -39$ ✓

(ii) $S_{31} = \frac{31}{2}[81 + -39]$ ✓
$= \frac{31}{2}[42]$
$= 31 \times 21$
$= 651$ ✓ 4371 (1 mark)

(iii) SOLVE $S_n < 0$ FOR LEAST n.

$\frac{n}{2}[2a + (n-1)d] < 0$

$\frac{n}{2}[162 + (n-1) \times -4] < 0$

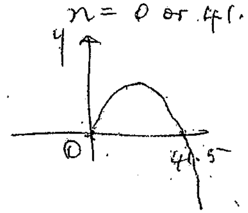
$n[162 - 4n + 4] < 0$ ✓

$n(166 - 4n) < 0$

$n < 0$ or $n > 41.5$

So $n = 42$. ✓

Consider $y = n(166 - 4n)$.
$y = 0$, $n = 0$ or $\frac{166}{4}$
$n = 0$ or 41.5



Q8.

(a). (i) $l = r\theta$ $60^\circ = \frac{60}{1} \times \frac{\pi}{180}{}^c = \frac{\pi}{3}{}^c$ ✓

so $2\pi = r \times \frac{\pi}{3}$

$r = 2\pi \times \frac{3}{\pi}$
$= 6$ m. ✓

(ii) Area $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 36 \times \frac{\pi}{3}$ ✓
$= 6\pi\ u^2$

(iii) Area segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
$= \frac{1}{2}(36)\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$ ✓
$= 18\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ ✓
$= 3(2\pi - 3\sqrt{3})$

(b) $x = 2t^3 - 9t^2 + 12t + 6$

(i) When $t = 0$, $x = 6$ m. ✓

(ii) now $v = 6t^2 - 18t + 12$ ✓
$v = 0$ when $6(t^2 - 3t + 2) = 0$
$(t-1)(t-2) = 0$
$t = 1$ or 2 seconds

(iii) $a = 12t - 18$
$a = 0$ when $12t - 18 = 0$
$t = \frac{3}{2}$ sec ✓
and $v = 6\left(\frac{9}{4}\right) - 18 \times \frac{3}{2} + 12$
$= \frac{27}{2} - \frac{54}{2} + 12$ ✓
$= -\frac{3}{2}$ m/s.

(iv) When $t = 1$, $x = 2 - 9 + 12 + 6$
$= 11$ m

When $t = 2$, $x = 16 - 36 + 24 + 6$
$= 10$

Distance travelled is $|10 - 11| = 1$ m.

OR Distance is $\left|\int_1^2 6t^2 - 18t + 12\, dt\right|$ ✓✓
$= \left|\left[2t^3 - 9t^2 + 12t\right]_1^2\right|$
$= |(4) - (5)|$