SYDNEY
BOYS
HIGH
SCHOOL

2018

YEAR 12
THSC
ASSESSMENT
TASK #4

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–16, show ALL relevant mathematical reasoning and/or calculations

Total Marks: 100

Section I – 10 marks (pages 3–6)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 8–19)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Examiner:
E. Choy

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the function $f(x) = \dfrac{x+2}{\sqrt{3-x}}$

Which expression represents the largest possible domain for $f(x)$?

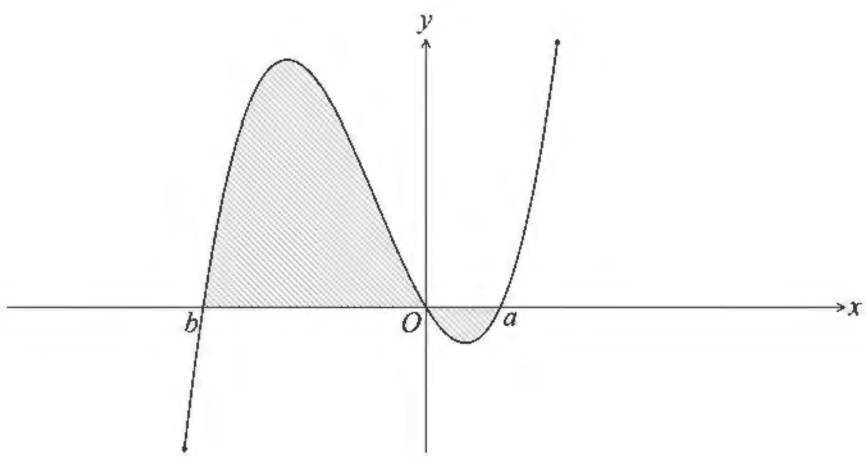
A. $x > 3$

B. $x \geq 3$

C. $x < 3$

D. $x \leq 3$

2 Which of the following correctly finds the shaded area in this diagram?



A. $\displaystyle\int_a^b f(x)\,dx$

B. $\left|\displaystyle\int_b^a f(x)\,dx\right|$

C. $\left|\displaystyle\int_0^a f(x)\,dx\right| + \displaystyle\int_b^0 f(x)\,dx$

D. $\displaystyle\int_0^a f(x)\,dx + \left|\displaystyle\int_b^0 f(x)\,dx\right|$

3 A parabola has a focus (–3, 0) and a directrix $x = 1$. What is the equation of the parabola?

A. $y^2 = 16(x+3)$

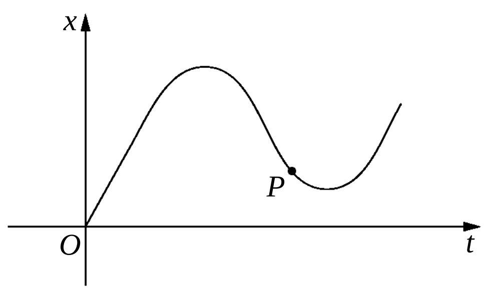
B. $y^2 = -16(x+3)$

C. $y^2 = 8(x+1)$

D. $y^2 = -8(x+1)$

4 The diagram shows the displacement, x metres, of a moving object at time t seconds.

Which statement describes the motion of the particle at the point P?



A. Velocity is negative and acceleration is positive.

B. Velocity is negative and acceleration is negative.

C. Velocity is positive and acceleration is negative.

D. Velocity is positive and acceleration is positive.

5 What is the derivative of $\left(e^{3x}+1\right)^{-2}$?

A. $-2e^{3x}(e^{3x}+1)^{-3}$

B. $-2e^{3x}(e^{3x}+1)^{-1}$

C. $-6e^{3x}(e^{3x}+1)^{-3}$

D. $-6e^{3x}(e^{3x}+1)^{-1}$

6 For $k \neq 0$, what is the limiting sum of the geometric series:

$$k + \frac{k}{1+k^2} + \frac{k}{(1+k^2)^2} + \frac{k}{(1+k^2)^3} + \ldots ?$$

A. $\dfrac{1}{1+k^2}$

B. $\dfrac{k^2}{1+k^2}$

C. $\dfrac{1+k^2}{k}$

D. $\dfrac{1+k^2}{k^2}$

7 What is the value of $\displaystyle\int_1^4 \left| x-3 \right| dx$?

A. 1.5

B. –1.5

C. 2.5

D. –2.5

8 p is an integer chosen at random from the set {5, 7, 9, 11}

q is an integer chosen at random from the set {2, 6, 10, 14, 18}

What is the probability that $p + q = 23$?

A. 0.1

B. 0.2

C. 0.3

D. 0.4

9 The quadratic equation $3x^2 - 5x + 2 = 0$ has roots α and β. Which of the following is false?

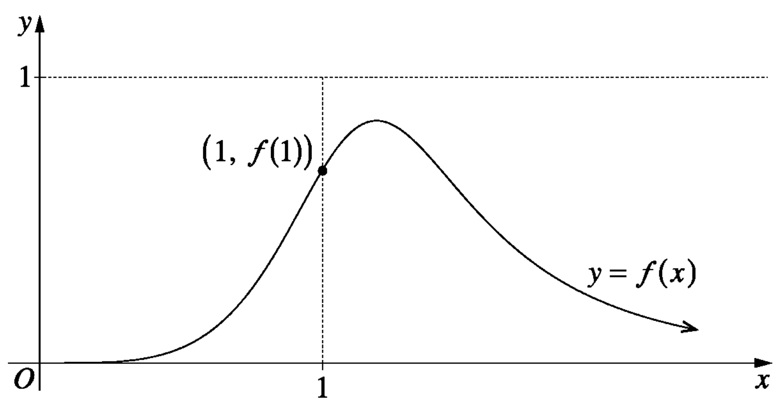
A. $3\alpha\beta = 2$

B. $\alpha + \beta = \dfrac{5}{3}$

C. $\alpha^2\beta + \alpha\beta^2 = \dfrac{10}{3}$

D. $\alpha^2 + \beta^2 = \dfrac{13}{9}$

10 The diagram shows the curve of $y = f(x)$, where $f(0) = f'(0) = 0$.



Which of the following statements is true?

A. $f(1) < 1 < f'(1) < f''(1)$

B. $f'(1) < f(1) < 1 < f''(1)$

C. $f''(1) < f'(1) < f(1) < 1$

D. $f''(1) < f(1) < 1 < f'(1)$

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $|7-3x| \geq 2$ and graph your solution on a number line **2**

(b) If $\cos\theta = \frac{3}{5}$ and $\tan\theta < 0$, find the exact value of $\operatorname{cosec}\theta$. **1**

(c) Differentiate

(i) $x^3 \ln x$ **2**

(ii) $\dfrac{\sqrt{x}}{4x-3}$ **2**

(d) Solve $e^{\ln x^3} = 27$ **1**

(e) Evaluate $\displaystyle\int_0^{\ln 4} e^{3x}\, dx$ **2**

Question 11 continues on page 9

(f) Simplify $\dfrac{\sin(\pi-\theta)\cos(\pi-\theta)}{\sin\left(\dfrac{\pi}{2}-\theta\right)\cos\left(\dfrac{\pi}{2}-\theta\right)}$. **2**

(g) The region enclosed between the curve $y=\sqrt{x}+\dfrac{1}{\sqrt{x}}$, the x-axis and the lines $x = 5$ and $x = 11$ is rotated about the x-axis. **2**

Calculate the volume of the solid generated.

(h) Simplify $\dfrac{6^x-3^x}{2^{x+1}-2}$ **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

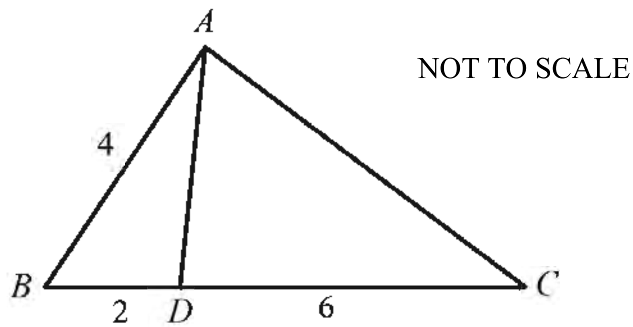
(a) (i) Find an expression for the discriminant of the quadratic expression **1**

$$x^2 + 6x + k + 8.$$

(ii) For what value(s) of k is the line $y = 4x + k$ a tangent to the parabola **2**

$$y = -8 - 2x - x^2.$$

(b) In the diagram below, prove that $\angle BDA = \angle BAC$. **3**



(c) P and Q are midpoints of the sides JK and JL respectively of the triangle JKL. **3**

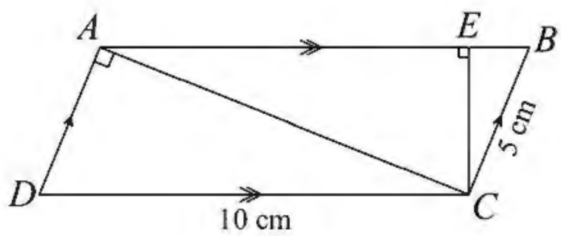
PQ is produced to R so that $PQ = QR$.

Prove that $RL = \dfrac{1}{2} JK$.

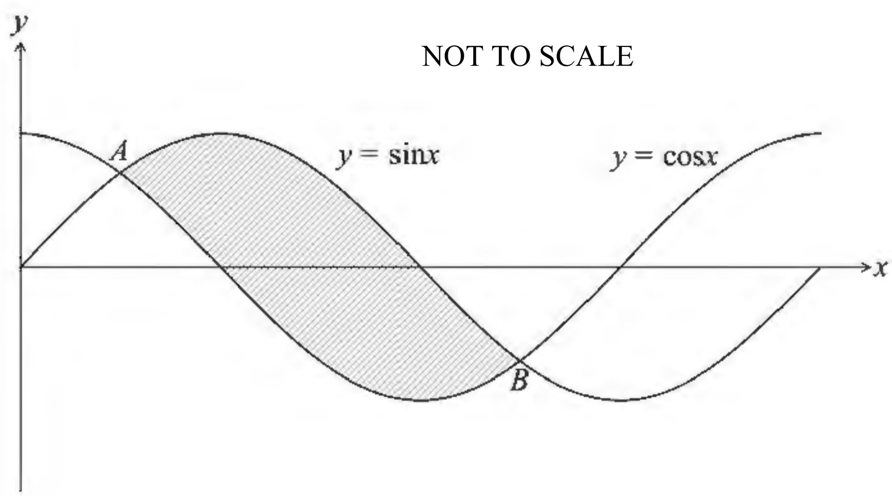
Question 12 continues on page 11

(d) In the figure below, $ABCD$ is a parallelogram with $AC \perp AD$. **2**
Also E is on AB such that $EC \perp AB$.
If $BC = 5$ cm and $CD = 10$ cm, show that $CE = \dfrac{5\sqrt{3}}{2}$ cm.

NOT TO SCALE



(e) The diagram below shows the graphs of $y = \sin x$ and $y = \cos x$ for $0 \le x \le 2\pi$.
They intersect at A and B.



(i) Find the coordinates of A and B. **2**

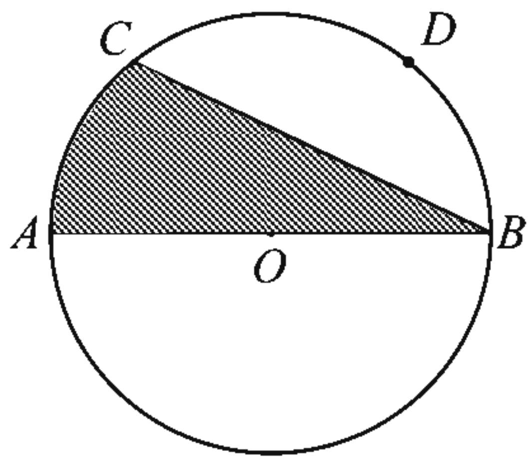
(ii) Hence find the shaded area enclosed between the two curves. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram below, O is the centre of the circle and AB is a diameter of length 4 cm. C is a point on the circumference of the circle such that $BC = 2\sqrt{3}$ cm.

NOT TO SCALE



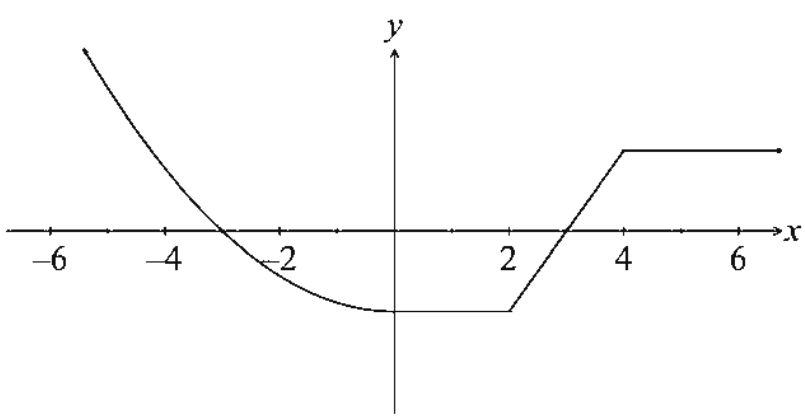
Note that $\angle ACB = \dfrac{\pi}{2}$.

(i) Find $\angle ABC$ in terms of π. **2**

(ii) Find the area of the shaded region in terms of π. **2**

(iii) Hence, find the area of the minor segment BDC in terms of π. **2**

(b) The diagram below shows the graph of $y = f'(x)$, for some function $f(x)$.



Copy the diagram above into your answer booklet.

Given that $f(0) = 0$, sketch the curve of $y = f(x)$ on the same diagram. **3**

Question 13 continues on page 13

(c) $\$P$ is deposited in a bank at the interest of r % p.a. compounded annually.
At the end of each year, one third of the amount in the account, including the principal and interest, is drawn out and the remainder is re-deposited at the same rate.
Let $\$Q_1$, $\$Q_2$, $\$Q_3$, … denote respectively the money drawn out at the end of the first year, second year, third year and so on.

(i) Show that $Q_2 = \frac{2}{9}\left(1+\frac{r}{100}\right)^2 P$. **2**

(ii) Q_1, Q_2, Q_3, … form a geometric series. **1**
Find the common ratio in terms of r.

(iii) Suppose $Q_3 = \frac{27}{128}P$, find the value of r. **2**

(iv) If $P = 10\ 000$, find $Q_1 + Q_2 + \ldots. + Q_{10}$. **1**

Give your answer correct to the nearest integer.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose that $y = e^{kx}$

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ **2**

(ii) Find the value of k such that $y = 2\dfrac{dy}{dx} - \dfrac{d^2y}{dx^2}$ **1**

(b) Let $f(x) = x^2 - \ln(2x-1)$.

(i) Show that the domain of $f(x)$ is $x > \dfrac{1}{2}$ **1**

(ii) Given that $f'(x) = 2x - \dfrac{2}{2x-1}$, find $f''(x)$. **2**

(iii) Find the coordinates of any stationary points in the domain. **2**

(iv) Hence, find the minimum value of the function. **2**

(c) Consider the geometric series:

$$\sin^2\theta + \sin^2\theta\cos^2\theta + \sin^2\theta\cos^4\theta + \ldots$$

where $0 < \theta < \dfrac{\pi}{2}$.

(i) Show that the sum, S_n, of the first n terms is given by $S_n = 1 - \cos^{2n}\theta$ **2**

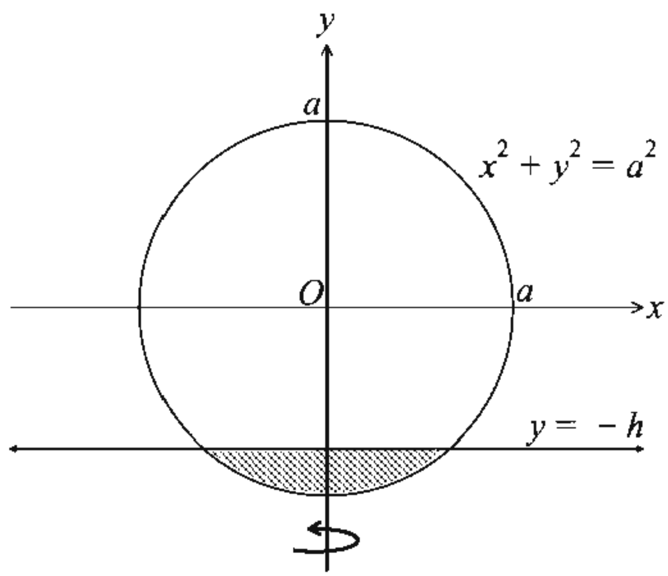
(ii) Explain why this series has a limiting sum. **1**

(iii) Let S be the limiting sum. Show that $S - S_n = \cos^{2n}\theta$ **2**

Question 15 (15 marks) Use a SEPARATE writing booklet

(a) The shaded region enclosed by the circle $x^2 + y^2 = a^2$ and the line $y = -h$ is rotated about the y-axis as shown below.

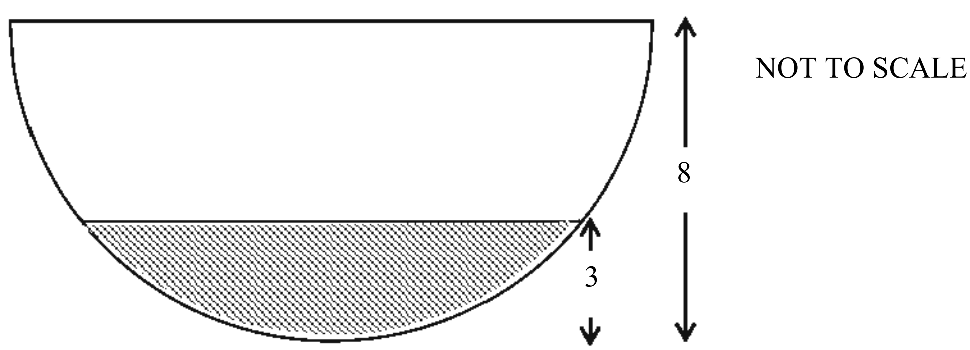
Note that $0 \le h \le a$.



(i) Show that the volume of the solid of revolution is **3**

$$\frac{(2a^3 - 3a^2h + h^3)\pi}{3} \text{ cubic units.}$$

(ii) A bowl is generated by revolving the lower half of $x^2 + y^2 = 64$ about the y-axis. **2**
The bowl contains water of depth 3 units as shown in the figure below.



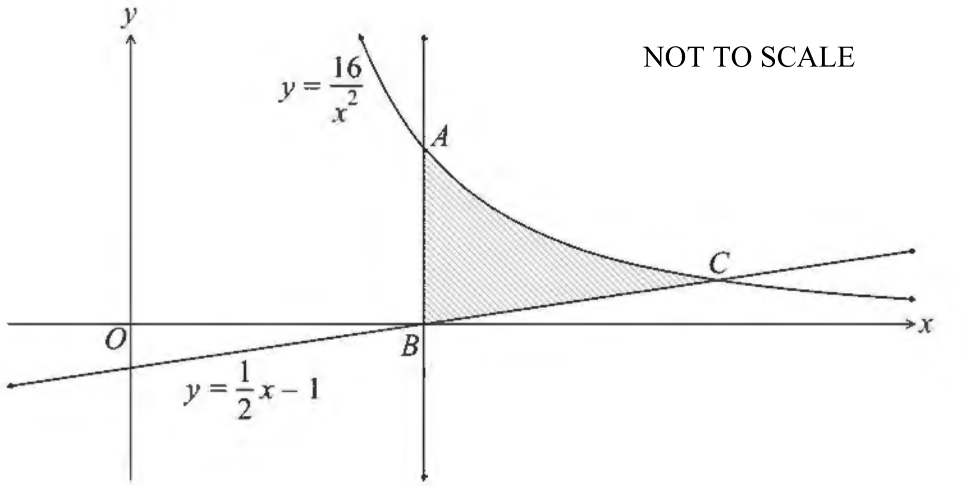
Using part (i), find the volume of the water in the bowl.

Question 15 continues on page 17

(a) (iii) The water in the bowl is now heated. At time t seconds after heating the rate of evaporation is $\left(\frac{11}{2}+\frac{t}{3}\right)\pi$ cubic units/s. **3**

Find the time required to evaporate all the water in the bowl.

(b)



In the diagram above, the straight line $y=\frac{x}{2}-1$ intersects the curve $y=\frac{16}{x^2}$ at the point C (4, 1) and cuts the x-axis at B.

The point A is a point on $y=\frac{16}{x^2}$ such that AB is parallel to the y-axis.

(i) Write down the inequalities needed to define the shaded region. **2**

(ii) Calculate the area of the shaded region. **2**

(c) In a lucky dip, there are twelve identical envelopes of which only three contain prizes.

(i) Show that if one were to purchase two envelopes the probability of not getting a prize would be $\frac{6}{11}$. **2**

(ii) What is the probability of getting at least one prize in this case? **1**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) The section of the curve $y = \ln(x+1)$ from $x = 0$ to $x = 2$ is rotated about the x-axis. **3**

Use Simpson's rule with three function values to approximate the volume of this solid of revolution.

Give your answer correct to two decimal places.

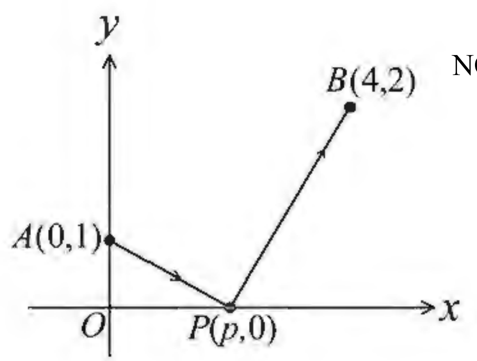
(b) The roots of the quadratic equation $x^2 + (k+4)x + 5k = 0$ are α and β. **3**

Given that $k \neq 0$, show that the quadratic equation with roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is

$$5kx^2 - (k^2 - 2k + 16)x + 5k = 0 .$$

Question 16 continues on page 19

(c)



NOT TO SCALE

A particle travels from a fixed point $A(0, 1)$ to a variable point $P(p, 0)$, where

$0 < p < 4$ on the positive side of the x-axis and finally to another fixed point $B(4, 2)$.

The particle travels along straight paths as shown in the above figure.

Let S be the total distance travelled by the particle from A to B via P.

(i) Find an expression for S in terms of p. **1**

(ii) Show that $\dfrac{dS}{dp} = \dfrac{p}{\sqrt{p^2+1}} + \dfrac{p-4}{\sqrt{(p-4)^2+4}}$ **1**

(iii) Solve $\dfrac{dS}{dp} = 0$. **3**

(iv) What is the minimum distance travelled from A to B via P? **2**

(v) The position of P can also be found by a purely geometrical construction. Describe this construction and use it to verify the position of P found above. **2**

End of paper

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Mathematics

SUGGESTED SOLUTIONS

MC QUICK ANSWERS

1	C
2	C
3	D
4	A
5	C
6	C
7	C
8	A
9	C
10	D

SECTION I MULTIPLE CHOICE SOLUTIONS

1 Consider the function $f(x)=\dfrac{x+2}{\sqrt{3-x}}$

Which expression represents the largest possible domain for $f(x)$?

A. $x>3$

B. $x\geq 3$

C. $x<3$

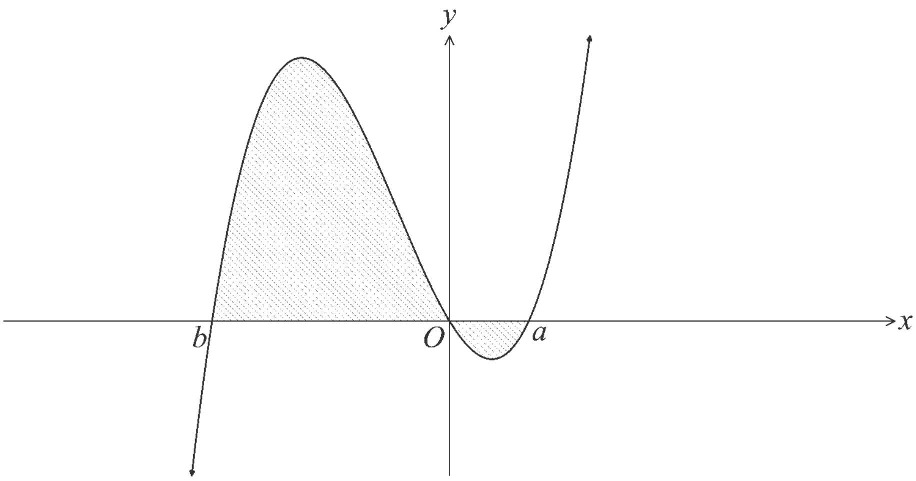
D. $x\leq 3$

$3-x>0$

$\therefore x<3$

A	10
B	4
C	154
D	22

2 Which of the following correctly finds the shaded area in this diagram?



A. $\int_a^b f(x)dx$

B. $\left|\int_b^a f(x)dx\right|$

C. $\left|\int_0^a f(x)dx\right|+\int_b^0 f(x)dx$

D. $\int_0^a f(x)dx+\left|\int_b^0 f(x)dx\right|$

Note $\int_0^a f(x)\,dx<0$

Option A will also a negative answer as $b<a$.

Options B and D will give the same answer.

A	0
B	1
C	183
D	6

3 A parabola has a focus (–3, 0) and a directrix $x = 1$. What is the equation of the parabola?

A. $y^2 = 16(x+3)$

B. $y^2 = -16(x+3)$

C. $y^2 = 8(x+1)$

D. $y^2 = -8(x+1)$

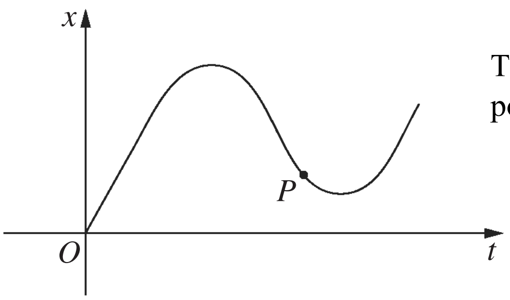
The distance between the focus and the directrix is $2a$.
i.e. $2a = 4 \Rightarrow a = 2$
This eliminates options A and B.
Alternatively the formula:
$(y-k)^2 = \pm 4a(x-h)$ is for a vertex at (h, k)

With the directrix to the right of the focus then C is eliminated.

A	7
B	26
C	16
D	141

4 The diagram shows the displacement, x metres, of a moving object at time t seconds.

Which statement describes the motion of the particle at the point P?



The gradient at P is negative and the concavity is positive.

A. Velocity is negative and acceleration is positive.

B. Velocity is negative and acceleration is negative.

C. Velocity is positive and acceleration is negative.

D. Velocity is positive and acceleration is positive.

A	142
B	40
C	7
D	0

Someone left this question blank

5 What is the derivative of $\left(e^{3x}+1\right)^{-2}$?

A. $-2e^{3x}(e^{3x}+1)^{-3}$

B. $-2e^{3x}(e^{3x}+1)^{-1}$

C. $-6e^{3x}(e^{3x}+1)^{-3}$

D. $-6e^{3x}(e^{3x}+1)^{-1}$

Chain rule:

$$\frac{d}{dx}\left[(e^{3x}+1)^{-2}\right]=-2(e^{3x}+1)^{-2-1}\times(3e^{3x})$$

A	0
B	2
C	176
D	12

6 For $k \neq 0$, what is the limiting sum of the geometric series:

$$k+\frac{k}{1+k^2}+\frac{k}{(1+k^2)^2}+\frac{k}{(1+k^2)^3}+\ldots?$$

A. $\frac{1}{1+k^2}$

B. $\frac{k^2}{1+k^2}$

C. $\frac{1+k^2}{k}$

D. $\frac{1+k^2}{k^2}$

$$a=k;\ r=\frac{1}{1+k^2}$$

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{k}{1-\frac{1}{1+k^2}}\times\frac{1+k^2}{1+k^2} \\ &= \frac{k(1+k^2)}{1+k^2-1} \\ &= \frac{1+k^2}{k} \end{aligned}$$

A	5
B	7
C	160
D	18

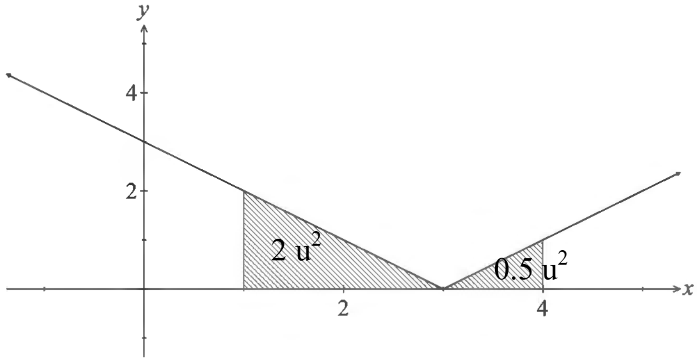
7 What is the value of $\int_1^4 |x-3|\,dx$?

A. 1.5

B. −1.5

C. 2.5

D. −2.5



$\int_1^4 |x-3|\,dx$ = sum of area of shaded triangles above.

Options B and D are eliminated as the graph is above the *x*-axis.

A	84
B	16
C	89
D	1

8 p is an integer chosen at random from the set {5, 7, 9, 11}

q is an integer chosen at random from the set {2, 6, 10, 14, 18}

What is the probability that $p + q = 23$?

A. 0.1

B. 0.2

C. 0.3

D. 0.4

The only options are (5, 18) and (9, 14).

There are $4 \times 5 = 20$ options

$$P(p+q=23)=\frac{2}{20}=0.1$$

A	173
B	16
C	1
D	0

9 The quadratic equation $3x^2 - 5x + 2 = 0$ has roots α and β. Which of the following is false?

A. $3\alpha\beta = 2$

B. $\alpha + \beta = \frac{5}{3}$

C. $\alpha^2\beta + \alpha\beta^2 = \frac{10}{3}$

D. $\alpha^2 + \beta^2 = \frac{13}{9}$

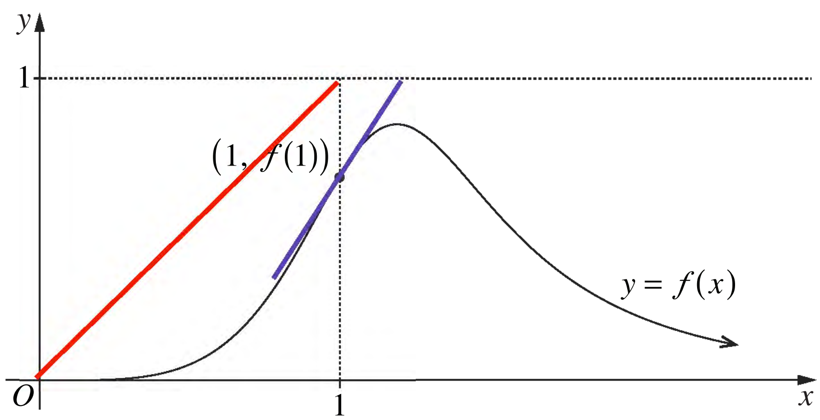
$$\alpha + \beta = \frac{5}{3};\ \alpha\beta = \frac{2}{3}$$

$$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha + \beta) = \frac{2}{3} \times \frac{5}{3}$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{5}{3}\right)^2 - 2 \times \frac{2}{3} = \frac{13}{9}$$

A	1
B	1
C	161
D	27

10 The diagram shows the curve of $y = f(x)$, where $f(0) = f'(0) = 0$.



Which of the following statements is true?

A. $f(1) < 1 < f'(1) < f''(1)$

B. $f'(1) < f(1) < 1 < f''(1)$

C. $f''(1) < f'(1) < f(1) < 1$

D. $f''(1) < f(1) < 1 < f'(1)$

A	14
B	15
C	51
D	110

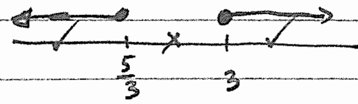
The (red) diagonal of the square defines a gradient of 1 with the given scale.
The (blue) tangent is steeper that this diagonal and so $f'(1) > 1$.
At $x = 1$, the curve is concave down and so $f''(1) < 0$.

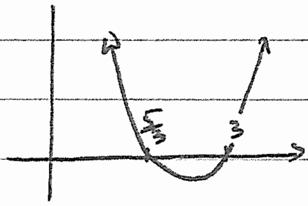
Question 11

Alternate

Solve $7-3x=2$ | $3x-7=2$
$3x=5$ | $3x=9$
$x=\frac{5}{3}$ | $x=3$

Then test





a) $|7-3x| \geq 2$

$(7-3x)^2 \geq 2^2$

$9x^2-42x+49 \geq 4$

$9x^2-42x+45 \geq 0$

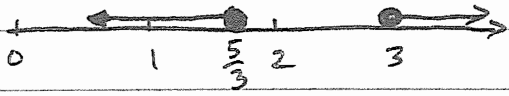
$3x^2-14x+15 \geq 0$

$(3x-5)(x-3) \geq 0$

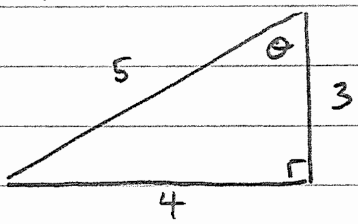
$x \leq \frac{5}{3}$ or $x \geq 3$

No number line $-\frac{1}{2}$

$\frac{5}{3} \leq x \leq 3$ 1



b)



By Pythagoras' Thm, opp $=4$

For the triangle, $\operatorname{cosec}\theta = \frac{hyp}{opp}$
$= \frac{5}{4}$

Since $\cos\theta$ is positive, and $\tan\theta$ is negative, θ is in the 4th quadrant, where $\operatorname{cosec}\theta$ is negative.

$\therefore \quad \operatorname{cosec}\theta = \frac{-5}{4}$ $\quad$ $\frac{1}{2}$ For $\operatorname{cosec} C = \frac{5}{4}$ (positive). No marks for $\frac{-5}{3}$, $\frac{-4}{5}$ etc

c) i) let $u=x^3$ $\quad v=\ln(x)$

$u'=3x^2$ $\quad v'=\frac{1}{x}$

$\frac{d}{dx}(x^3\ln x) = u'v+v'u$

$= 3x^2\ln(x)+x^2$

Can be factorised to $x^2(3\ln(x)+1)$ $\quad$ 2 Correct answer

1 Significant correct working

c) ii) Let $u = \sqrt{x} = x^{\frac{1}{2}}$ $\quad v = 4x-3$

$u' = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad v' = 4$

$$\frac{d}{dx}\left(\frac{\sqrt{x}}{4x-3}\right) = \frac{u'v - v'u}{v^2}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(4x-3) - 4x^{\frac{1}{2}}}{(4x-3)^2}$$

② For correct answer, simplified/unsimplified

① For significant working

Can be simplified to

$$\frac{2x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}} - 4x^{\frac{1}{2}}}{(4x-3)^2} = \frac{-\frac{3}{2}x^{-\frac{1}{2}} - 2x^{\frac{1}{2}}}{(4x-3)^2}$$

$$= -\frac{3+4x}{2\sqrt{x}(4x-3)^2}$$

Some minor subsequent errors ignored, but be careful!

d) $e^{\ln(x^3)} = 27$

$\ln(27) = \ln(x^3)$

$x^3 = 27$

$x = 3$ ① correct

e) $$\int_0^{\ln(4)} e^{3x}\,dx = \left[\frac{1}{3}e^{3x}\right]_0^{\ln(4)}$$

$$= \frac{1}{3}e^{3\ln(4)} - \frac{1}{3}e^0$$

$$= \frac{1}{3}\left(e^{\ln(4)}\right)^3 - \frac{1}{3}$$

$$= \frac{64}{3} - \frac{1}{3}$$

$= 21$ ② for correct answer

① for some correct working

Common mistakes: not evaluating $e^{\ln 4} = 4$ or $e^{3\ln 4} = 64$,

fraction errors 😐,

thinking e^0 can be ignored.

f) $\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

$$\frac{\sin(\pi-\theta)\cos(\pi-\theta)}{\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{\sin\theta \times -\cos\theta}{\cos\theta \times \sin\theta}$$

Bald answer only given 1

$= -1$, as long as $\sin\theta \neq 0$ and $\cos\theta \neq 0$

i.e. $\theta \neq \frac{k\pi}{2}$, where $k \in \mathbb{Z}$

Not needed, but technically ~~technically~~ more correct

g) $$V = \pi\int_5^{11} y^2\,dx$$

$$= \pi\int_5^{11}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \pi\int_5^{11}\left(x+2+\frac{1}{x}\right)dx$$

$$= \pi\left[\frac{x^2}{2}+2x+\ln(x)\right]_5^{11}$$

$$= \pi\left(\frac{11^2}{2}+2\times 11+\ln(11)-\frac{5^2}{2}-5\times 2-\ln(5)\right)$$

$$= \pi\left(60+\ln\left(\frac{11}{5}\right)\right)$$

$$= 60\pi+\pi\ln\left(\frac{11}{5}\right)$$

≈ 190.97

Common mistakes:
Not squaring $\sqrt{x}+\frac{1}{\sqrt{x}}$, incorrectly squaring $(\sqrt{x}+\frac{1}{\sqrt{x}})$, in particular forgetting the "2" term. evaluating the definite integral incorrectly

2 For correct answer (decimal accepted, though less correct)

$1\frac{1}{2}$ only missing π

Up to 1 significant working, if the error doesn't make the question too easy.

h) $$\frac{6^x-3^x}{2^{x+1}-2} = \frac{3^x.2^x-3^x}{2.2^x-2}$$

$$= \frac{3^x(2^x-1)}{2(2^x-1)} = \frac{3^x}{2}$$

1 for correct answer.

No half-marks awarded

NO 1/2 MARKS

Question 12

(a) (i) $x^2 + 6x + k + 8$

$$\Delta = b^2 - 4ac$$
$$= 6^2 - 4(1)(k+8)$$
$$= 36 - 4k - 32$$
$$= 4 - 4k$$

[1]

Generally well done.
✓ [1] for the correct expression of the discriminant.

(ii) $y = -8 - 2x - x^2$ (1)
$y = 4x + k$ (2)

(1) = (2): $4x + k = -8 - 2x - x^2$
$$x^2 + 6x + k + 8 = 0$$

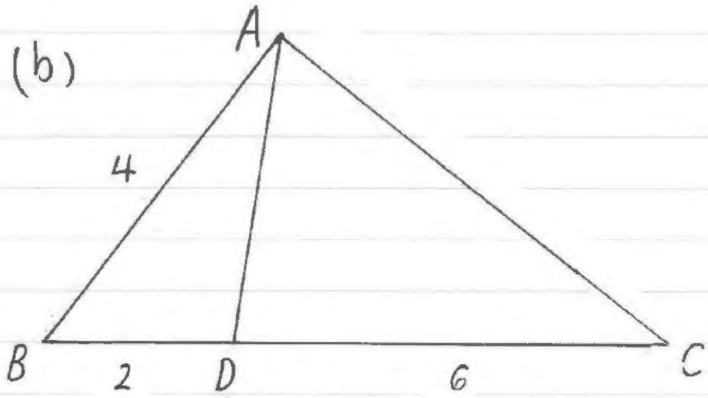
$\Delta = 0 \Rightarrow$ 1 real root.

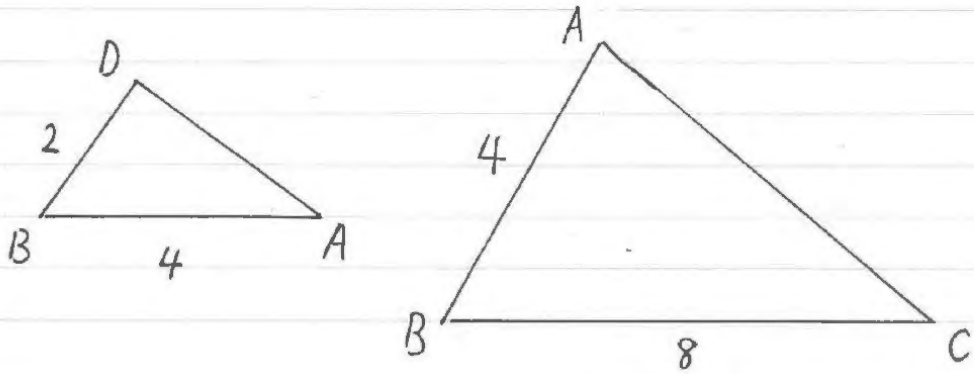
$$\therefore\ 4 - 4k = 0$$
$$4k = 4$$
$$k = 1$$

[2]

Generally well done. However, some students found the value of *k*, but differentiating and finding the equation of the tangent.
✓ [1] for recognising that for line to be a tangent, the discriminant of the equated expression is equal to 0 for one real solution.
✓ [1] correctly determining *k*.

(b)





In ΔDAB and ΔACB,

$\angle ABC$ is common

$\frac{BD}{AB} = \frac{2}{4} = \frac{1}{2}$ (In the same ratio).

$\frac{AB}{BC} = \frac{4}{8} = \frac{1}{2}$

$\therefore \Delta DAB \;|||\; \Delta ACB$ (two pairs of matching sides are in proportion and the included angles are equal).

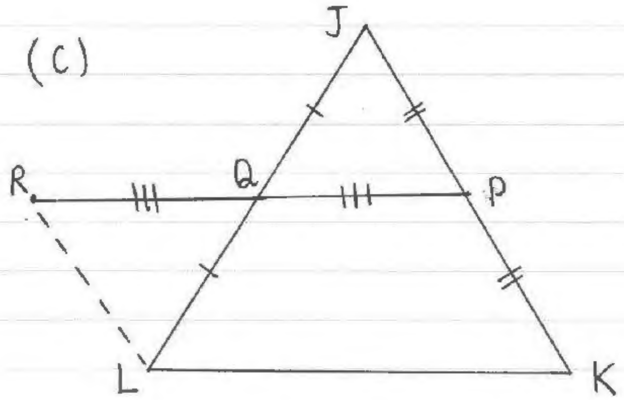
$\therefore \angle BDA = \angle BAC$ (matching angles of similar triangles).

[3]

Poorly done. Student MUST state in which triangle they are working in.

- ✓ [1] the correct connection of matching sides and matching angles, giving reasons.
- ✓ [1] the correct expression of the concluding statement about the similar triangles.
- ✓ [1] for correctly stating that the 2 angle are equal and the reasoning.

(c)



(NTS)

In ΔJPQ and ΔQRL

$JQ = QL$ (given, as Q is the midpoint of JL).

$\angle RQL = \angle JQP$ (vertically opposite angles)

$PQ = QR$ (given)

$\therefore \Delta JPQ \equiv \Delta QRL$ (SAS)

$\therefore RL = JP$ (corresponding side in congruent triangles).

$2JP = JK$ (P is the midpoint of JK).

$\therefore 2RL = JK$

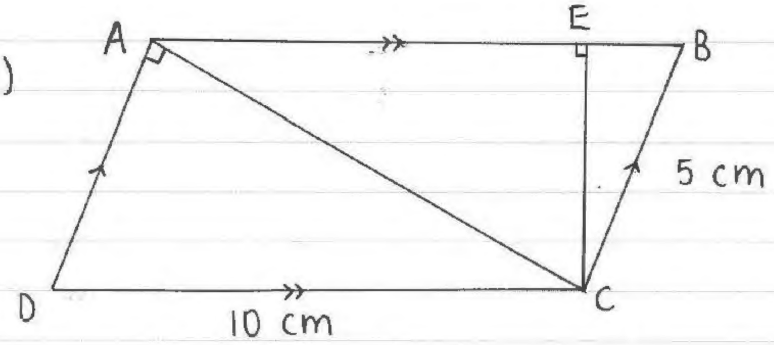
$RL = \frac{1}{2} JK$

[3]

Poorly done. Student MUST state in which triangle they are working in.

- ✓ [1] the correct connection of matching sides and matching angle, giving reasons.
- ✓ [1] the correct expression of the concluding statement about the congruent triangles.
- ✓ [1] for correctly stating that the 2 sides are equal and giving reasons.

(d)



$AC^2 = 10^2 - 5^2$ (opposite sides of a parallelogram are equal).

$AC = \sqrt{100 - 25}$
$= \sqrt{75}$
$= 5\sqrt{3}$

Area of ΔADC = Area of ΔABC.

$\frac{1}{2} \times 5 \times 5\sqrt{3} = \frac{1}{2} \times 10 \times CE$

$\frac{25\sqrt{3}}{2} = 5CE$

$\therefore CE = \frac{5\sqrt{3}}{2}$ [2]

$\boxed{\text{OR}}$ In ΔACD: $\sin \angle ACD = \frac{5}{10}$

$\therefore \angle ACD = \sin^{-1}\left(\frac{1}{2}\right)$
$= 30°$

$\angle ACD = \angle CAE$ (alternate angles, $AB \parallel CD$).

$\therefore \sin 30° = \frac{EC}{5\sqrt{3}}$

$EC = 5\sqrt{3} . \sin 30°$
$= \frac{5\sqrt{3}}{2}$

Generally well done. Student did this question in a variety of different way, such as using trigonometry, with the areas and with similar triangles.
✓ [2] correct showing the exact value of *CE*.

(e) (i) $y = \sin x \quad (1) \qquad 0 \leq x \leq 2\pi.$

$y = \cos x \quad (2)$

$(1) = (2): \quad \sin x = \cos x$

$$\frac{\sin x}{\cos x} = \frac{\cos x}{\cos x}$$

$$\tan x = 1$$

$$\therefore \quad x = \tan^{-1}(1)$$

$$= \frac{\pi}{4}, \frac{5\pi}{4}$$

when $x = \frac{\pi}{4}$, $y = \sin \frac{\pi}{4}$

$$= \frac{1}{\sqrt{2}} \qquad \therefore A\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$$

when $x = \frac{5\pi}{4}$, $y = \sin \frac{5\pi}{4}$

$$= -\frac{1}{\sqrt{2}} \qquad \therefore B\left(\frac{5\pi}{4}, \frac{-1}{\sqrt{2}}\right)$$

[2]

(ii) $A = \int_{\pi/4}^{5\pi/4} \sin x - \cos x \, dx$

$$= \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

$$= \left(-\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right)\right) - \left(-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right)$$

$$= \left(-\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{4}{\sqrt{2}} \quad u^2$$

$$= 2\sqrt{2} \quad u^2$$

[2]

Generally well done. However, many student forgot that coordinates means a *x*-value and a *y*-value and only provided the *x* values.

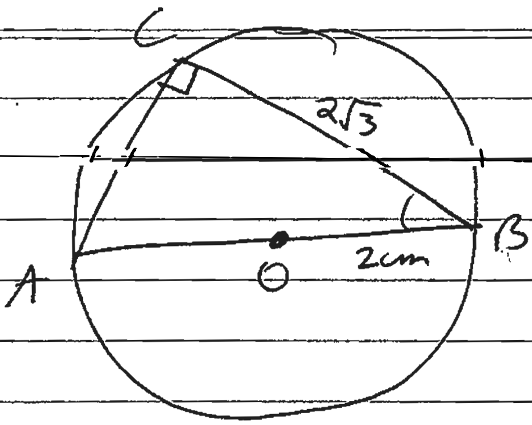
In part (i):

- ✓ [1] both of the correct *x* values.
- ✓ [1] both of the correct *y* values.

In part (ii):

- ✓ [1] for correctly identifying that you need to subtract the lower curve from the top curve and for the correct integration of the curves.
- ✓ [1] for the correct finding the area.

13 (a)



(i) $\cancel{\cos \angle ABC = \frac{2\sqrt{3}}{4}}$

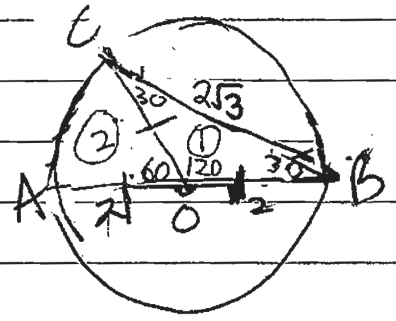
$\cos \angle ABC = \frac{\sqrt{3}}{2}$

$\Rightarrow \angle ABC = \frac{\pi}{6}$

2

This question was done well.

(ii)



$A_{shaded} = A_1 + A_2 = \frac{1}{2}ab\sin\theta + \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 4 \sin\frac{2\pi}{3} + \frac{1}{2} 2^2 \frac{\pi}{3}$

$= 2\sin\frac{2\pi}{3} + \frac{2\pi}{3}$

$= 2 \times \frac{\sqrt{3}}{2} + \frac{2\pi}{3}$

$= \left(\sqrt{3} + \frac{2\pi}{3}\right) u^2$ 2

Many students incorrectly treated this as a sector of a circle with angle 30 degrees.But then the radii are different lengths so it is NOT a sector.

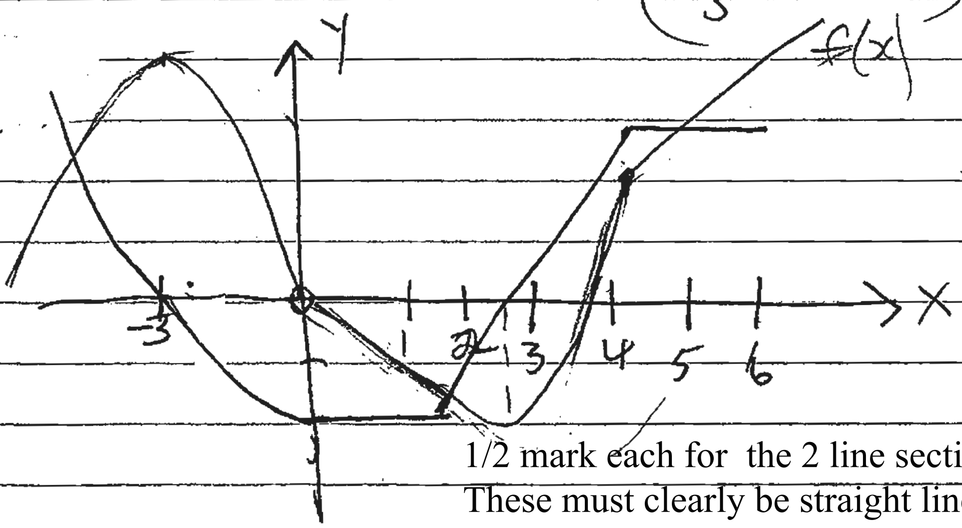
(iii) $A_{\text{minor segment}}$ = Semi-circle $- A_{shaded}$

Marks were given if students proceeded correctly with their answer from (ii).

$= \frac{1}{2}\pi \times 4 - \left(\sqrt{3} + \frac{2\pi}{3}\right)$

$= 2\pi - \frac{2\pi}{3} - \sqrt{3}$

$= \left(\frac{4\pi}{3} - \sqrt{3}\right)$ units^2 2



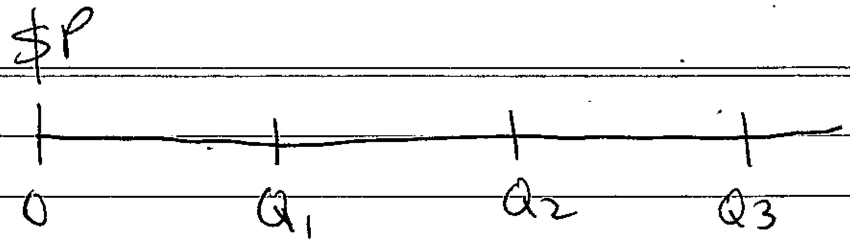
max

min

straight lines

1/2 mark each for the 2 line sections.
These must clearly be straight lines.

13(c)



Let A_n be an amount in account at time n years — $r\%$ pa. where r is a decimal.

(i) $Q_1 = \frac{1}{3}P(1+r)$

$$A_1 = P(1+r) - Q_1 = \frac{2}{3}P(1+r)$$

Then $Q_2 = \frac{1}{3}A_1(1+r) = \frac{1}{3}\left(\frac{2}{3}P(1+r)\right)(1+r)$ 2

$$= \frac{2}{9}P(1+r)^2$$

This part was done well.

$$\left(\text{or} = \frac{2}{9}P\left(1+\frac{r}{100}\right)^2\right)$$

Then $A_2 = A_1(1+r) - Q_2$

$$= \frac{2}{3}P(1+r)^2 - \frac{2}{9}P(1+r)^2$$

$$= \frac{4}{9}P(1+r)^2$$

$$Q_3 = \frac{1}{3}A_2(1+r) = \frac{4}{27}P(1+r)^3$$

(ii) $Q_1, Q_2, Q_3, \ldots = \frac{1}{3}P(1+r), \frac{2}{9}P(1+r)^2, \frac{4}{27}P(1+r)^3, \ldots$

GP with $a = \frac{1}{3}P(1+r)$

$CR = \frac{2}{3}(1+r)$

$\therefore$ Common ratio $= \frac{2}{3}(1+r)$ 1

Again, most students did this well.

13(c) cont.

(iii) $Q_3 = \frac{27}{128}P = \frac{4}{27}P(1+r)^3$

$\Rightarrow \frac{4}{27}(1+r)^3 = \frac{27}{128}$

$(1+r)^3 = \frac{729}{512}$

~~Those who did (ii) correctly generally~~
did this part correctly.

$1+r = \frac{9}{8}$

$r = \frac{1}{8} = 0{\cdot}125$ or $12{\cdot}5\%$

2

(iv) $P = 10000$

Find $Q_1 + Q_2 + \ldots + Q_{10}$.

$n = 10$, $a = \frac{10000}{3}(1{\cdot}125)$

C. Ratio $= \frac{2}{3}(1{\cdot}125) = 0{\cdot}75$

Then $S_{10} = \frac{10000}{3}(1{\cdot}125)\left(\frac{0{\cdot}75^{10}-1}{-0{\cdot}25}\right)$

$= \$14155{\cdot}30$

$= \$14155$

1

Many students mistakenly used the value of r from (iii) as the common ratio in the GP.

Question 14

a) i) $y = e^{kx}$

$\frac{dy}{dx} = ke^{kx}$ (1)

$\frac{d^2y}{dx^2} = k^2e^{kx}$ (1)

Marker's comment:

- Most common error by candidates in this question is finding the second derivative and treating k as a variable rather than as a constant.

(ii) $y = 2\frac{dy}{dx} - \frac{d^2y}{dx^2}$

$e^{kx} = 2ke^{kx} - k^2e^{kx}$

$k^2e^{kx} - 2ke^{kx} + e^{kx} = 0$

$e^{kx}(k^2 - 2k + 1) = 0$

$e^{kx} \neq 0$

$\therefore k^2 - 2k + 1 = 0$

$(k-1)^2 = 0$

$k = 1$ (1)

Marker's comments:

- Candidates whom did not get the correct answer for i), did not get the answer for this part.
- Significant number of candidates cannot factorise $k^2 - 2k + 1$ correctly.
- Significant number of candidates did not sub e^{kx} into y and instead let y = 0 which led to the wrong answer.

b) i) $f(x) = x^2 - \ln(2x-1)$

Domain for x^2 is all real x

Domain for $\ln(2x-1)$: $2x - 1 > 0$

$2x > 1$

$x > 1/2$ (1)

$\therefore$ Domain for $f(x)$: $x > 1/2$

Marker's Comment:

- Candidates did not get full marks by just substituting a number from the domain. Need to prove for all cases.

(ii) $f'(x) = 2x - \frac{2}{2x-1}$

$= 2x - 2(2x-1)^{-1}$

$\therefore f''(x) = 2 - 2 \times -1 \times 2 \times (2x-1)^{-2}$ 1

$= 2 + \frac{4}{(2x-1)^2}$

1

Marker's Comments:

- This was poorly done question.
- Common errors included:
  i) Cant differentiate $\frac{2}{2x-1}$
  ii) Didn't take account of the 2 negatives when differentiating $-\frac{2}{2x-1}$.
  iii) Did integration rather than differentiation.

b)(iii) let $f'(x) = 0$ to find stationary pts

$0 = 2x - \frac{2}{2x-1}$

$0 = \frac{2x(2x-1) - 2}{2x-1}$

$0 = \frac{4x^2 - 2x - 2}{2x-1}$ $x \neq \frac{1}{2}$

$\therefore 4x^2 - 2x - 2 = 0$

$2(2x^2 - x - 1) = 0$

$2(2x+1)(x-1) = 0$

$\therefore x = -\frac{1}{2}$ and $x = 1$ 1

But from i), $x > \frac{1}{2}$

$\therefore x = 1$ and, $y = 1^2 - \ln(2\times1-1)$ 1

$\therefore (1,1)$ is the stationary pt

Marker's comment:

- Most common errors were:
  i) Incorrectly factorising $4x^2 - 2x - 2$, particularly when taking the 2 out.
  ii) Not stating why $x = -\frac{1}{2}$ is not a valid solution (i.e. outside of the domain.)
  iii) Not finding the y –value for the x – value as we need a **point**.

(iv) At $(1,1)$

$$f''(1) = 2 + \frac{4}{(2\times1-1)^2}$$
$$= 2 + 4$$
$$= 6$$
$$> 0$$

$\therefore$ $(1,1)$ is a minimum turning pt (1)

As $x \to \frac{1}{2}^+$ $f(x) \to \infty^+$ as $-\ln(2x-1)$ approaches ∞)

and

$x \to \infty^+$ $f(x) \to \infty$ as x^2 approaches ∞

$\therefore$ Minimum value $= 1$ (1)

Marker's comments

- Candidates whom did not write the ***y*** value as the minimum value were penalised.
- Candidates need to prove (1,1) is a minimum POINT.

(c) (i) $\sin^2\theta + \sin^2\theta\cos^2\theta + \sin^2\theta\cos^4\theta + \dots$

$a = \sin^2\theta \quad r = \cos^2\theta$ (1)

$$\therefore S_n = \frac{a(1-r^n)}{1-r}$$
$$= \frac{\sin^2\theta\,(1-(\cos^2\theta)^n)}{1-\cos^2\theta}$$ (1)
$$= \frac{\sin^2\theta}{\sin^2\theta}\,(1-(\cos^2\theta)^n)$$
$$= 1-\cos^{2n}\theta$$

Marker's comment

Mostly well done by candidates.

(ii) Since $0 < \theta < \frac{\pi}{2}$

$0 < r = \cos^2\theta < 1$

$\therefore |r| < 1$ (1)

$\therefore$ Series has a limiting sum.

Marker's comment

- Candidates need to state that limiting exists ONLY if $|r| < 1$ or they list a domain that STRICTLY belongs to the domain to get the 1 mark.
- $r < 1$ does not mean the series will converge, i.e. contains $r \leq -1$ which means the series wont converge.

(iii) $S = \dfrac{a}{1-r} = \dfrac{\sin^2\theta}{1-\cos^2\theta}$

$= \dfrac{\sin^2\theta}{\sin^2\theta}$

$= 1$ 1

$\therefore$ LHS $= S - S_n$

$= 1 - (1-\cos^{2n}\theta)$ 1

$= \cos^{2n}\theta$

$=$ RHS

Marker's comment

Mostly well done by candidates.

2018 Trial HSC: Mathematics – Q15

Part A, Subsection I	3 marks
Rotation is about the y-axis, so: $V = \pi \int_a^b x^2\, dy$ $= \pi \int_{-a}^{-h} a^2 - y^2\, dy$ $= \pi \left[a^2 y - \frac{y^3}{3}\right]_{-a}^{-h}$ $= \pi\left(a^2 \times (-h) - \frac{(-h)^3}{3}\right) - \pi\left(a^2 \times (-a) - \frac{(-a)^3}{3}\right)$ $= \pi\left(-a^2 h + \frac{h^3}{3} + a^3 - \frac{a^3}{3}\right)$ $= \pi\left(-a^2 h + \frac{h^3}{3} + \frac{2a^3}{3}\right)$ $= \pi\left(\frac{-3a^2 h + h^3 + 2a^3}{3}\right)$ $= \frac{(2a^3 - 3a^2 h + h^3)\pi}{3}\ u^3$	1 mark awarded for deriving the correct integral expression: $V = \pi \int_{-a}^{-h} a^2 - y^2\, dy$ 1 mark awarded for correctly integrating the expression to: $V = \pi \left[a^2 y - \frac{y^3}{3}\right]_{-a}^{-h}$ 1 mark awarded for correctly substituting the limits in and showing some form of working out when building towards the required final expression. ~ Overall, candidates struggled with this question. Common errors: • Using the incorrect limits in the integral. • Rearranging the equation of the circle incorrectly. Many candidates also skipped too many steps in their working out after integration, leading to errors such as forgetting to carry the minus sign through. Clever candidates calculated the volume of an equivalent solid of revolution that had easier limits. However, they should explicitly state its equivalence to the solid of revolution described in the question.
Part A, Subsection II	**2 marks**
Comparing the figure with the graph from Part A, Subsection I: $a = 8$ $\quad h = 8 - 3 = 5$ Substituting the values in: $V = \frac{\pi\left((2 \times 8^3) - (3 \times 8^2 \times 5) + (5^3)\right)}{3}$ $= 63\pi\ u^3$	1 mark awarded for finding the correct values for a and h. 1 mark awarded for correctly evaluating the volume. ~ Overall, candidates did reasonably well on this question. Common errors: • Mistaking $h = 3$ instead of $h = 5$. • Not leaving answers in exact form.

Part A, Subsection III:	3 marks

Rate of evaporation is given by:

$$\frac{dV}{dt} = \left(\frac{11}{2} + \frac{t}{3}\right)\pi\, u^3/s$$

$$\therefore V = \int \left(\frac{11}{2} + \frac{t}{3}\right)\pi\, dt$$

$$= \pi \int \frac{11}{2} + \frac{t}{3}\, dt$$

$$= \pi\left(\frac{11t}{2} + \frac{t^2}{2 \times 3}\right) + c$$

$$= \pi\left(\frac{11t}{2} + \frac{t^2}{6}\right) + c$$

At $t = 0$, the volume of water evaporated is 0.

$$0 = \pi\left(\frac{11 \times 0}{2} + \frac{0^2}{6}\right) + c$$

$$c = 0$$

$$\therefore V = \pi\left(\frac{11t}{2} + \frac{t^2}{6}\right)$$

Solve for t when $V = 63\pi$:

$$63\pi = \pi\left(\frac{11t}{2} + \frac{t^2}{6}\right)$$

$$63 = \frac{11t}{2} + \frac{t^2}{6}$$

$$= \frac{33t + t^2}{6}$$

$$378 = 33t + t^2$$

$$t^2 + 33t - 378 = 0$$

Use the quadratic equation:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-33 \pm \sqrt{33^2 - (4 \times 1 \times 378)}}{2 \times 1}$$

$$= \frac{-33 \pm 51}{2}$$

$$= 9\ or - 42$$

But $t \geq 0$

$$\therefore t = 9\ s$$

1 mark awarded for correctly integrating the volume with respect to time to get the expression:

$$V = \pi\left(\frac{11t}{2} + \frac{t^2}{6}\right) + c$$

1 mark awarded for correctly evaluating the constant c and deriving the expression:

$$t^2 + 33t - 378 = 0$$

1 mark awarded for correctly evaluating the time.

~

Overall, candidates struggled with this question.

Common errors:

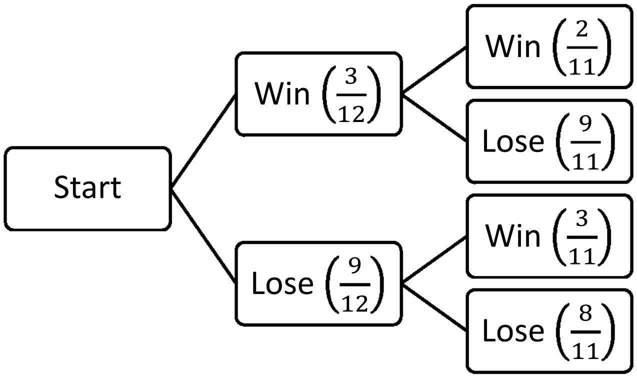
- Equating the rate expression to the volume.
- Attempting to derive an expression for volume from the rate expression without integration.
- Forgetting c after integration.
- Assuming $c = 0$ without sufficient proof.
- Misinterpreting V to be the volume of water remaining instead of the volume of water evaporated at $t = 0$.

Some candidates opted to express the rate as a negative instead such that $V = 63\pi$ at $t = 0$, which is a viable alternative.

Part B, Subsection I:	2 marks
The x-coordinate of Point B can be found by solving the equation $y = \frac{1}{2}x - 1$ at $y = 0$. $$0 = \frac{1}{2}x - 1$$ $$\frac{1}{2}x = 1$$ $$x = 2$$ $$B = (2, 0)$$ Based on the graph: $$x \geq 2$$ $$y \geq \frac{1}{2}x - 1$$ $$y \leq \frac{16}{x^2}$$	1 mark awarded for correctly determining the vertical line inequality by solving the linear equation at $y = 0$. 1 mark awarded for correctly determining the other inequalities. ~ Overall, candidates did reasonably well on this question. Common errors: • Not including the boundaries as a part of the inequalities. • Forgetting to determine the vertical line inequality. Some candidates also failed to use any inequality whatsoever when attempting to answer this question.
Part B, Subsection II:	2 marks
Let: $$f(x) = \frac{16}{x^2} \qquad g(x) = \frac{1}{2}x - 1$$ Then: $$A = \int_a^b \left(f(x) - g(x)\right) dx$$ $$= \int_2^4 \left(\left(\frac{16}{x^2}\right) - \left(\frac{1}{2}x - 1\right)\right) dx$$ $$= \int_2^4 16x^{-2} - \frac{x}{2} + 1\, dx$$ $$= \left[\frac{16x^{-1}}{-1} - \frac{x^2}{2 \times 2} + x\right]_2^4$$ $$= \left[\frac{-16}{x} - \frac{x^2}{4} + x\right]_2^4$$ $$= \left(\frac{-16}{4} - \frac{4^2}{4} + 4\right) - \left(\frac{-16}{2} - \frac{2^2}{4} + 2\right)$$ $$= 3\ u^2$$	1 mark awarded for correctly deriving the integral expression: $$A = \int_2^4 \frac{16}{x^2} - \frac{x}{2} + 1\, dx$$ 1 mark awarded for correctly integrating the expression to find the area. ~ Overall, candidates did reasonably well on this question. Common errors: • Using the incorrect limits in the integral. • Incorrectly integrating the expression. • Incorrectly multiplying out the negatives. Some candidates did not notice that the coordinates of C were given and tried to solve the associated cubic equation. Others attempted to calculate the area by integrating with respect to y, which was much more difficult than integrating with respect to x. Clever candidates noticed that the area under the line $y = \frac{1}{2}x - 1$ was a triangle and used the formula for the area of a triangle instead of integration.

<table>
<tr><td>Part C, Subsection I:</td><td>2 marks</td></tr>
<tr><td>

Based on the question, the following tree diagram can be constructed:



$$\begin{aligned}&P(No\ prize\ at\ all)\\&= P(No\ prize\ in\ first)\\&\qquad\qquad \times P(No\ prize\ in\ second)\\&= \frac{9}{12} \times \frac{8}{11}\\&= \frac{6}{11}\end{aligned}$$

</td><td>

1 mark awarded for showing some form of logical deduction when calculating the probability.

1 mark awarded for correctly substituting the probability values in.

~

Overall, candidates did reasonably well on this question.

Common error:

- Forgetting to reduce the numerator and/or denominator by 1 for the second envelope.

Candidates should note that, as a proof question, full marks cannot be awarded if insufficient/illogical proofs were used to derive the final expression.

</td></tr>
<tr><td>Part C, Subsection II:</td><td>1 mark</td></tr>
<tr><td>

$$\begin{aligned}P(At\ least\ one\ prize) &= 1 - P(No\ prize\ at\ all)\\&= 1 - \frac{6}{11}\\&= \frac{5}{11}\end{aligned}$$

</td><td>

1 mark awarded for correctly determining the probability.

~

Overall, candidates did reasonably well on this question.

Some candidates chose to calculate the probability as a summation instead of using their answer from the previous subsection with mixed results.

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Question 16 SOLUTIONS

(a)	The section of the curve $y = \ln(x+1)$ from $x = 0$ to $x = 2$ is rotated about the x-axis. Use Simpson's rule with three function values to approximate the volume of this solid of revolution. Give your answer correct to two decimal places.	**3**

$$V = \pi \int_0^2 [\ln(x+1)]^2 \, dx$$

$$h = \frac{2-0}{2} = 1$$

x	y	y^2	weight (w)	$w \times y^2$
0	ln 1 = 0	0	1	0
1	ln 2	$(\ln 2)^2$	4	$4(\ln 2)^2$
2	ln 3	$(\ln 3)^2$	1	$(\ln 3)^2$
				$\sum wy^2 \doteqdot 3.1288$

$$\therefore V \doteqdot \pi \times \frac{h}{3} \times \sum wy^2$$

$$= 3.2764\ldots \text{ u}^3$$

$$\doteqdot 3.28 \text{ u}^3$$

Note: You can't just add π to an answer i.e. 1.04π

Comment

On the whole this question was not done well. Students did not read the question.
As a result, a student could score 1 mark for just stating the first integral above.

Students who only worked out the area scored a maximum of 1.
Many students thought that $V = \pi A$ or $V = \pi A^2$, where A is the area.
If the student correctly worked out A this scored a maximum of ½.

The formula for Simpson's rule is on the Reference Sheet, so those with errors with the formula generally scored 0 marks. Students who used the Trapezoidal rule, also did not score well.

Some students incorrectly applied a log rule to $[\ln(x+1)]^2$.
As long as there were no other errors, these students could gain a maximum of 2 marks.

Question 16 SOLUTIONS (continued)

(b) The roots of the quadratic equation $x^2+(k+4)x+5k=0$ are α and β. **3**

Given that $k \neq 0$, show that the quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

$$5kx^2-(k^2-2k+16)x+5k=0.$$

A quadratic equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is $\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$

$$\therefore x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=0$$

$$\therefore x^2-\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)x+1=0$$

$$\therefore x^2-\left(\frac{k^2-2k+16}{5k}\right)x+1=0$$

$$\therefore 5kx^2-(k^2-2k+16)x+5k=0$$

$$\alpha+\beta=-(k+4)$$
$$\alpha\beta=5k$$

$$\begin{aligned}\alpha^2+\beta^2&=(\alpha+\beta)^2-2\alpha\beta\\&=(k+4)^2-10k\\&=k^2+8k+16-10k\\&=k^2-2k+16\end{aligned}$$

ALTERNATIVE 1

Consider $5kx^2-(k^2-2k+16)x+5k=0$:

Sum of roots $=\dfrac{k^2-2k+16}{5k}$

Product of roots $=\dfrac{5k}{5k}=1$

Now:

$$\begin{aligned}\frac{\alpha}{\beta}+\frac{\beta}{\alpha}&=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\\&=\frac{(k+4)^2-10k}{5k}\\&=\frac{k^2-2k+16}{5k}\end{aligned}$$

$$\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1$$

$\therefore$ the roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

ALTERNATIVE 2

Let the equation with roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ be

$x^2+bx+c=0$.

Now $b=-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$ and $c=\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1$

$$\therefore x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0$$

The proof now follows the one above.

Question 16 SOLUTIONS (continued)

See over for Marking comments.

Comment

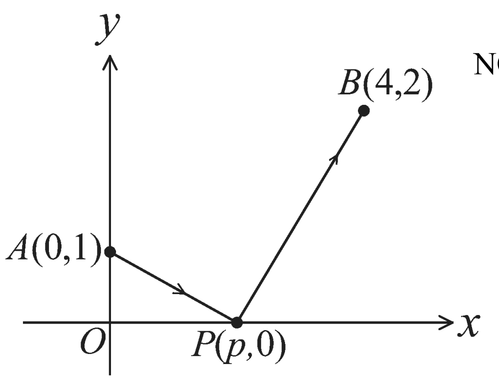
Many students' logic was confusing as to what they were proving and what they had assumed.

This question was not done well as many students used $-\frac{b}{a}$ as an abbreviation for the 'sum of roots' and $\frac{c}{a}$ for the 'product of the roots'. Students should not do this. This wasn't penalised unless they used it again for a different equation.

Similarly, many students used a, b and c without defining them. This was penalised.

Question 16 SOLUTIONS (continued)

(c)



NOT TO SCALE

A particle travels from a fixed point $A(0, 1)$ to a variable point $P(p, 0)$, where $0 < p < 4$ on the positive side of the x-axis and finally to another fixed point $B(4, 2)$.

The particle travels along straight paths as shown in the above figure.

Let S be the total distance travelled by the particle from A to B via P.

(i) Find an expression for S in terms of p. **1**

$$S = \sqrt{(p-0)^2 + (0-1)^2} + \sqrt{(p-4)^2 + (0-2)^2}$$
$$= \sqrt{p^2+1} + \sqrt{(p-4)^2+4}$$

Comment

This was generally done well.

(ii) Show that $\dfrac{dS}{dp} = \dfrac{p}{\sqrt{p^2+1}} + \dfrac{p-4}{\sqrt{(p-4)^2+4}}$. **1**

$$S = (p^2+1)^{\frac{1}{2}} + \left[(p-4)^2+4\right]^{\frac{1}{2}}$$
$$\frac{dS}{dp} = \frac{1}{2}(p^2+1)^{-\frac{1}{2}} \times 2p + \frac{1}{2}\left[(p-4)^2+4\right]^{-\frac{1}{2}} \times 2(p-4)$$
$$= \frac{p}{\sqrt{p^2+1}} + \frac{p-4}{\sqrt{(p-4)^2+4}}$$

Comment

This was generally done well, though for 1 mark many students over did their working.

Question 16 SOLUTIONS (continued)

(iii) Solve $\dfrac{dS}{dp} = 0$. **3**

$$\frac{dS}{dp} = 0 \Rightarrow \frac{p}{\sqrt{p^2+1}} + \frac{p-4}{\sqrt{(p-4)^2+4}} = 0$$

$$\therefore \frac{p}{\sqrt{p^2+1}} = -\frac{p-4}{\sqrt{(p-4)^2+4}}$$

$$\therefore \frac{p^2}{p^2+1} = \frac{(p-4)^2}{(p-4)^2+4}$$

$$\therefore p^2\left[(p-4)^2+4\right] = (p-4)^2\left(p^2+1\right)$$

$$\therefore p^2(p-4)^2 + 4p^2 = p^2(p-4)^2 + (p-4)^2$$

$$\therefore 4p^2 = (p-4)^2$$

$$\therefore 2p = p-4 \text{ OR } 2p = -(p-4)$$

$$\therefore p = -4 \text{ OR } p = \frac{4}{3}$$

$$\therefore p = \frac{4}{3} \qquad \left[0 < p < 4\right]$$

Comment

Students who 'miraculously' answered $p = \frac{4}{3}$ were heavily penalised.

Many students couldn't get past the first two lines above (or equivalent).

To score 1 mark, students had to successfully demonstrate their ability to square both sides of line 2 above (or equivalent). Too many students think that $(a+b)^2 = a^2 + b^2$

Question 16 SOLUTIONS (continued)

(iv)	What is the minimum distance?	**2**

p	1	$\frac{4}{3}$	2
$\frac{dS}{dp}$	-0.12	0	0.19

The minimum value of S is when $p = \frac{4}{3}$

$$
\begin{aligned}
S_{\min} &= \sqrt{\left(\frac{4}{3}\right)^2 + 1} + \sqrt{\left(\frac{4}{3} - 4\right)^2 + 4} \\
&= \frac{5}{3} + \frac{10}{3} \\
&= 5
\end{aligned}
$$

Comment

If students only found the minimum value of S they could only score 1 mark.

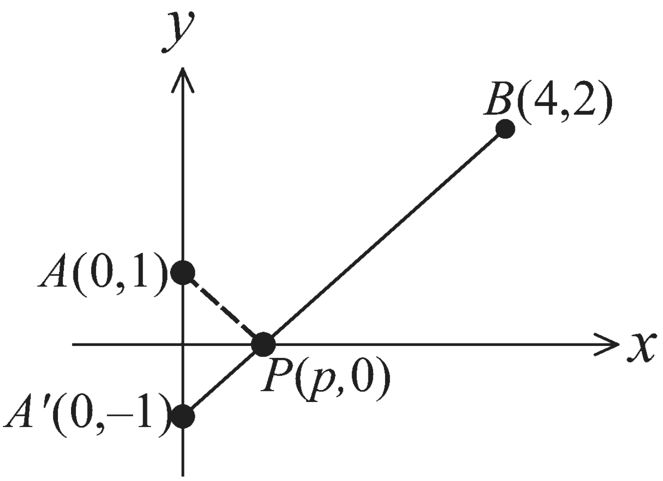
Students who didn't demonstrate numbers (or equivalent) in the table above were penalised.

Students who 'miraculously' found the answer $S = 5$, scored 0 marks.

Question 16 SOLUTIONS (continued)

(v)	The position of P can also be found by a purely geometrical construction. Describe this construction and use it to verify the position of P found above.	**2**

The shortest distance between any two points is the straight-line distance between them.



Reflect A in the x-axis to get A'.
The shortest distance between A' and B is $A'B$.

As $AP = A'P$ then the shortest distance travelled from A to B via P is $A'PB$.

$$\begin{aligned} A'B &= \sqrt{4^2 + (2+1)^2} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

Note: The equation of $A'B$ is $y = \frac{3}{4}x - 1$.

As P is the x-intercept then $p = \frac{4}{3}$.

Comment

Some students realised that for the minimum distance that $|m_{AP}| = |m_{PB}|$. Without a reason, they scored a maximum of 1 mark.
Many students could have made more effective use of their time (and hence maximised their marks) by not having attempted this problem and spent the time checking earlier work.