Sydney Girls High School

2006

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2006 HSC Examination Paper in this subject.

Candidate Number

General Instructions

- Reading Time - 5 mins
- Working time - 3 hours
- Attempt **ALL** questions
- **ALL** questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Total marks – 120
Attempt Questions 1 – 10
All questions are of equal value

Start each question on a NEW page

Marks

Question 1 (12 marks)

a) Write 5 245 000 in scientific notation correct to two significant figures — 2

b) Factorise $x^2 + xy - 6y^2$ — 2

c) Differentiate $2x^{-2} + x^2$ with respect to x — 2

d) Increase \$800 by 15% and reduce the resulting amount by 10% — 2

e) Solve $|x-4| \le 7$ — 2

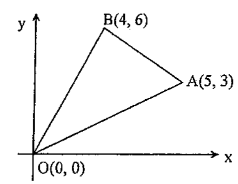
f) Solve $\frac{x-4}{2} - \frac{x+1}{3} = 6$ — 2

Question 2 (12 marks)

a) Find a primitive of $2 + \frac{1}{x}$ — 2

b) For what values of k does $x^2 - 2kx + 9 = 0$ have no real roots? — 3

c) The diagram shows triangle OAB with co ordinates as shown.



i.) Calculate the exact length of interval OA — 1

ii.) Find the gradient of OA — 1

iii.) Find the angle that OA makes with the positive X axis to the nearest degree — 1

iv.) Find the equation of OA in general form — 2

v.) Calculate the perpendicular distance of B from OA and hence the exact area of triangle OAB — 2

Question 3 (12 marks) **Marks**

a) Solve $\sin x = \frac{\sqrt{3}}{2}$ for $0^\circ \le x \le 360^\circ$ **2**

b) Find the equation of the tangent to $y = e^{2x} - 3$ at the point where $x = 0$ **2**

c) Differentiate the following with respect to x:

i.) $\frac{x^2+3x}{2x-1}$ **2**

ii.) $(\log_e x)^3$ **2**

d) Find $\int 3xe^{x^2}\,dx$ **2**

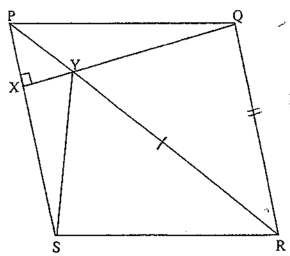
e) Find the equation of the parabola with vertex (1, 2) and focus (1, 4) **2**

Question 4 (12 marks)

a) Solve $5^x = 32$ correct to two decimal places **1**

b) Find two integers a and b such that $\frac{2}{2+\sqrt{3}} = a - \sqrt{b}$ **2**

c) PQRS is a rhombus. QX is perpendicular to PS and meets PR at Y. Copy or trace the diagram.



i.) Why does $\angle SRP = \angle QRP$ **1**

ii.) Prove that triangle SYR is congruent to triangle QYR **3**

iii.) Show that $\angle RQY = 90^\circ$ **1**

iv.) Find the size of $\angle YSR$ **1**

d) The table below gives the values of $f(t)$ for $0 \le t \le 2$ **3**

t	0	0.5	1	1.5	2
$f(t)$	0	0.31	0.39	0.42	0.35

Use Simpson's Rule with four sub-intervals to evaluate:

$\int_0^2 f(t)\,dt$ correct to three decimal places

Question 5 (12 marks) **Marks**

a) A number plate consists of any two digits (0 – 9) followed by any two letters (A – Z) followed by any two digits (0 – 9)

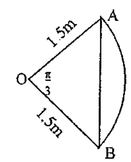
i.) How many different number plates are possible? **1**

ii.) What is the probability that a number plate chosen at random would start with zero six? **1**

iii.) What is the probability that a number plate chosen at random would contain the letters Y and R (in any order) **1**

iv.) What is the probability that a number plate chosen at random does not contain the letters Y and R (any order) **1**

b) OAB is a sector of a circle centre O and radius 1.5 metres. The angle at the centre is $\frac{\pi}{3}$ radians



i.) Find the exact length of arc AB **1**

ii.) Find the length of interval AB **1**

iii.) Find the area of sector OAB **2**

c) Given the function $f(x) = x^3 - 3x + 1$

i.) Find the coordinates of the stationary points on the curve and determine their nature **3**

ii.) Sketch the curve **1**

Marks

Question 6 (12 marks)

a) Find $\sum_{n=1}^{100} 2n+1$ — 2

b) Given $\log_a 2 = 0.4$, $\log_a 5 = 0.8$ find:

i.) $\log_a 10$ — 1

ii.) $\log_a \sqrt{10}$ — 1

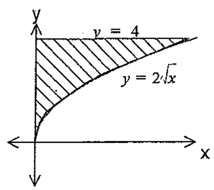
iii.) $\log_a 2a$ — 1

c) A lolly jar contains 30 jelly babies of which 10 are orange, 10 are green and 10 are red. Three are chosen at random.

i.) What is the probability that all three are orange? — 1

ii.) What is the probability that there is one of each colour? — 2

d) The shaded region in the diagram below is bounded by the curve $y = 2\sqrt{x}$, the Y-axis and the line y = 4. Calculate the volume of the solid of revolution formed when this region is rotated about the Y-axis — 2



e) Given the geometric sequence $1, -3, 9, -27, \ldots\ldots\ldots, T_n, \ldots\ldots\ldots$ find the smallest value of n such that $|T_n| > 1\,000\,000$ — 2

Marks

Question 7 (12 marks)

a) Given the curves $y = x^3$ and $y = x^3 + x^2 - 3x - 4$, for what value of x do the curves have the same gradient — 2

b) The base QR of equilateral triangle PQR is produced to S so that $QS = 2QR$

i.) Use the information given to draw a diagram — 1

ii.) If PQ is equal to 3 units find the exact value of PS — 2

c) A loan of $100 000 is borrowed at 12% interest per annum. The money is to be paid back in equal monthly instalments over 4 years. At the end of each month interest is added to the principle before the monthly installment is deducted. Let the amount of each monthly payment be P dollars and the amount owing after n payments be A_n

i.) Show that the amount owing after one payment is $A_1 = 100000(1.01) - P$ — 1

ii.) Show that after n payments the amount owing is $A_n = 100000(1.01)^n - P(1 + 1.01 + 1.01^2 + \ldots.1.01^{n-1})$ — 2

iii.) Hence calculate the amount of each monthly installment — 2

d) Prove that the line $3x + 4y - 30 = 0$ is a tangent to the circle $x^2 + y^2 = 36$ — 2

Question 8 (12 marks)

a) Sketch the graph of $y = \sqrt{3 - x^2}$ — 1

b) The gradient function of a curve is $5 - 2x$ and the curve passes through the point with co-ordinates (-2, 6). Find the equation of the curve. — 2

c) Given the function $f(x) = 1 + 2\cos x$:

i.) State the range of $f(x)$ — 1

ii.) Sketch $f(x)$ for $0 \le x \le 2\pi$ — 2

d) Show that $\frac{x+1}{x-1} = 1 + \frac{2}{x-1}$, hence find $\int \frac{x+1}{x-1} dx$ — 3

e) Simplify $\frac{\sin\theta}{1-\cos\theta} - \frac{\sin\theta}{1+\cos\theta}$ — 3

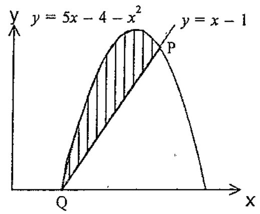
Question 9 (12 marks) **Marks**

a) Kelly invests \$50 into a superannuation fund at the beginning of each month for twenty years. Interest is paid at the rate of 6% pa compounded monthly.

i.) After twenty years, what will be the value of the first \$50 investment. **2**

ii.) Find the total value of her investment at the end of twenty years? **2**

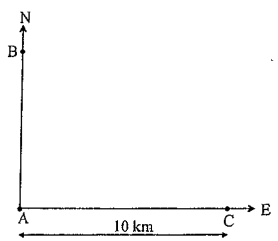
b) The diagram shows the parabola $y = 5x - 4 - x^2$ and the line $y = x - 1$



i.) Find the co-ordinates of P and Q the points where the two graphs intersect **2**

ii.) Calculate the area enclosed by the two curves **2**

c) A bushwalker sets of from the point A walking north towards B at 4km/h. At the same instant a second bushwalker leaves C, 10 km east of A and walks directly towards A at 3km/h.



i.) Show that after t hours their distance apart (D) is given by $D^2 = 25t^2 - 60t + 100$ **2**

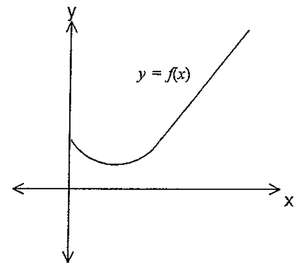
ii.) Find the value of t for which their distance apart, D is a minimum **2**

Question 10 (12 marks) **Marks**

a) For what value of n does $\dfrac{6^{2n} \times 9^{2n-1}}{4^n} = 1$? **2**

b) Show that the curve $y = \sqrt{2x - 1}$ has no stationary points **2**

c) The graph of $y = f(x)$ ~~is~~ consists of a section of a parabola and a line segment. **2**



Copy the graph onto your exam paper and sketch (either on or below $y = f(x)$) the graph of $y = f'(x)$

d) The cost of running a car at an average speed of S kilometre per hour is given by the formula $C = \left(\dfrac{S^2}{80} + 100\right)$ cents per hour. Find the average speed at which the cost of a 500 kilometre trip is a minimum **3**

e) Solve the equation $x(x+3)(x+1)(x+2) - 120 = 0$ **3**

$\alpha\beta\chi\delta\varepsilon\phi\gamma\eta\iota\kappa\lambda\mu\nu o\pi\varpi\theta\vartheta\rho\sigma\varsigma\tau\upsilon\omega\xi\psi$

Question One

a) 5.2×10^{6}

b) $x^2 + xy - 6y^2$
$= (x+3y)(x-2y)$

c) $\frac{d}{dx}(2x^{-2} + x^2)$
$= -4x^{-3} + 2x$

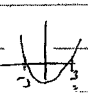
d) $\$800 \times 1.15 \times 0.9$
$= \$828$

e) $|x-4| \le 7$
$-7 \le x-4 \le 7$
$-3 \le x \le 11$

f) $\frac{x-4}{2} - \frac{x+1}{3} = 6$
$3(x-4) - 2(x+1) = 36$
$3x - 12 - 2x - 2 = 36$
$x - 14 = 36$
$x = 50$

Question Two

a) $2x + \log_e x + C$

b) $x^2 - 2kx + 9 = 0$
for no real roots
$b^2 - 4ac < 0$
$(-2k)^2 - 4(1)(9) < 0$
$4k^2 - 36 < 0$
$k^2 - 9 < 0$
$(k-3)(k+3) < 0$

$-3 < k < 3$

c) i) $d = \sqrt{5^2 + 3^2}$
$= \sqrt{34}$

ii) $m = \frac{3}{5}$

iii) $31°$

iv) $y = \frac{3}{5}x$
$5y = 3x$
$3x - 5y = 0$

v) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ B(4,6)
$= \frac{|3(4) - 5(6) + 0|}{\sqrt{3^2+5^2}}$
$= \frac{18}{\sqrt{34}}$
$A = \frac{1}{2} \times \sqrt{34} \times \frac{18}{\sqrt{34}}$
$= 9 \text{ units}^2$

Question Three

a) $\sin x = \frac{\sqrt{3}}{2}$
$x = 60°, 120°$

b) $y = e^{2x} - 3$
$\frac{dy}{dx} = 2e^{2x}$
when $x = 0$, $m = 2$, $y = -2$
$y + 2 = 2(x - 0)$
$y = 2x - 2$

c) i) $y = \frac{x^2 + 3x}{2x - 1}$
$= \frac{(2x-1)(2x+3) - (x^2+3x)(2)}{(2x-1)^2}$
$= \frac{4x^2 + 4x - 3 - 2x^2 - 6x}{(2x-1)^2}$
$= \frac{2x^2 - 2x - 3}{(2x-1)^2}$

ii) $y = (\log_e x)^3$
$\frac{dy}{dx} = 3(\log_e x)^2 \cdot \frac{1}{x}$
$= \frac{3}{x}(\log_e x)^2$

d) $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$
$\therefore \int 2xe^{x^2}\,dx = e^{x^2}$
$\therefore \int 3xe^{x^2}\,dx = \frac{3}{2}e^{x^2} + C$

e) vertex $(1,2)$, focus $(1,4)$
$(x-p)^2 = 4a(y-q)$ $a = 2$
$(x-1)^2 = 8(y-2)$

Question Four

a) $5^x = 32$
$\log 5^x = \log 32$
$x = \frac{\log 32}{\log 5}$
$= 2.15$

b) $\frac{2}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{4-2\sqrt{3}}{1}$
$= 4 - \sqrt{12}$
$a = 4, b = 12$

c) i) diagonals of a rhombus bisect the angles through which they pass

ii) In Δ's SYR, QYR
YR common
$\angle SRP = \angle QRP$ (above)
$SR = QR$ (equal sides of a rhombus)
$\therefore \Delta SYR \equiv \Delta QYR$ (SAS)

iii) $\angle PXQ = 90°$
$= \angle RQY$ (alt $\angle$'s $PS \parallel QR$)

iv) $\angle YSR = 90°$

d)

t	0	0.5	1	1.5	2
$f(t)$	0	0.31	0.39	0.42	0.35
w	1	4	2	4	1

$I = \frac{h}{3}\sum wf(t)$
$= \frac{0.5}{3}(4.05)$
$= 0.675$

Question Five

a) i) $10 \times 10 \times 26 \times 26 \times 10 \times 10$
$= 6\,760\,000$

ii) $P(06) = \frac{1 \times 1 \times 26 \times 26 \times 10 \times 10}{6\,760\,000}$
$= \frac{67600}{6760000}$
$= \frac{1}{100}$

iii) $P = \left(\frac{10 \times 10 \times 1 \times 1 \times 10 \times 10}{6\,760\,000}\right) \times$
$= \frac{1}{338}$

iv) $P = 1 - \frac{1}{338}$
$= \frac{337}{338}$

b) i) $l = r\theta$
$= \frac{3}{2} \times \frac{\pi}{3}$
$= \frac{\pi}{2}$ m

ii) $(AB)^2 = (1.5)^2 + (1.5)^2 - 2(1.5)(1.5)\cos\frac{\pi}{3}$
$= 1.5$ m (Δ equilateral

iii) $A = \frac{1}{2}r^2\theta$
$= \frac{1}{2}(1.5)^2\left(\frac{\pi}{3}\right)$
$= 1.178 \text{ m}^2$

c) i) $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

for a stny pt $f'(x) = 0$

$3x^2 - 3 = 0$

$(x+1)(x-1) = 0$

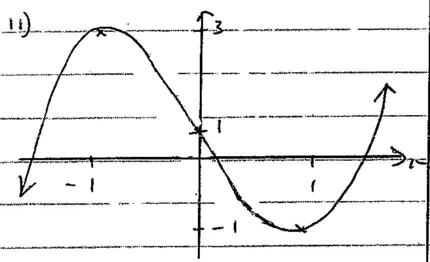
$x = -1 \qquad x = 1$

$y = 3 \qquad y = -1$

$f''(-1) < 0 \qquad f''(1) > 0$

$\therefore$ max , min

ii)



Question Six

a) $\sum_{n=1}^{100} 2n+1 = 3+5+ \dots + 201$

$S_n = \frac{n}{2}[a+d]$

$= 50[3+201]$

$= 10200$

b) $\log_a 2 = 0.4$

$\log_a 5 = 0.8$

i) $\log_a 10 = \log_a (2 \times 5)$

$= \log_a 2 + \log_a 5$

$= 1.2$

ii) $\log_a \sqrt{10} = \log_a 10^{\frac{1}{2}}$

$= \frac{1}{2} \log_a 10$

$= \frac{1}{2} \times 1.2$

$= 0.6$

iii) $\log_a 2a = \log_a 2 + \log_a a$

$= 0.4 + 1$

$= 1.4$

c) i) $P(OOO) = \frac{10}{30} \times \frac{9}{29} \times \frac{8}{28}$

$= \frac{6}{203}$

ii) 3 different colours can be arranged 6 ways

$\therefore P = 6\left(\frac{10}{30} \times \frac{10}{29} \times \frac{10}{28}\right)$

$= \frac{50}{203}$

d) $V = \pi \int_0^4 f(y)^2 \, dy$

$x = \frac{y^2}{4} \Rightarrow x^2 = \frac{y^4}{16}$

$= \pi \int \frac{y^4}{16} \, dy$

$= \frac{\pi}{80} \left[y^5\right]_0^4$

$= \frac{\pi}{80}(12-8)$

$= \frac{64\pi}{5}$ units^3

e) $a = 1, \quad r = -3 \quad n = ?$

$|T_n| = |(1)(-3)^{n-1}|$

$+3^{n-1} > 1\,000\,000$

$n-1 > \frac{\log_e 1\,000\,000}{\log_e 3}$

$n-1 > 12.5$

$n > 13.5$

i.e. T_{14}

Question Seven.

a) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$

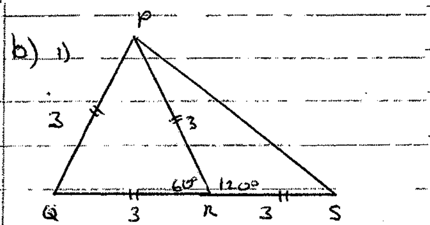
$y = x^3 + x^2 - 3x - 4$

$\frac{dy}{dx} = 3x^2 + 2x - 3$

same gradient if

$2x - 3 = 0 \quad \therefore x = \frac{3}{2}$

b) i)



ii) $PS^2 = 3^2 + 3^2 - 2(3)(3)\cos 120°$

$= \sqrt{27}$

$= 3\sqrt{3}$

c) i) Amount owing is \$100 000 plus interest (12% ÷ 12 months = 1%) minus a payment

$A_1 = 100\,000(1{\cdot}01) - P$

ii) After two payments

$A_2 = A_1 \times 1{\cdot}01 - P$

$= [100\,000(1{\cdot}01) - P](1{\cdot}01) - P$

$= 100\,000(1{\cdot}01)^2 - 1{\cdot}01P - P$

$A_3 = A_2(1{\cdot}01) - P$

$= 100\,000(1{\cdot}01)^3 - 1{\cdot}01^2 P - 1{\cdot}01P - P$

$= 100\,000(1{\cdot}01)^3 - P(1 + 1{\cdot}01 + 1{\cdot}01^2)$

$A_n = 100\,000(1{\cdot}01)^n - P(1 + 1{\cdot}01 + \dots 1{\cdot}01^{n-1})$

iii) when $n = 48$, $A_n = 0$

$0 = 100\,000(1{\cdot}01)^{48} - P(1 + 1{\cdot}01 + \dots 1{\cdot}01^{47})$

$P = \frac{100\,000(1{\cdot}01)^{48}}{1 + 1{\cdot}01 + \dots 1{\cdot}01^{47}}$

$= 100\,000(1{\cdot}01)^{48} \times \frac{0{\cdot}01}{1{\cdot}01^{48} - 1}$

$= \$2633{\cdot}38$

d) $3x + 4y - 30 = 0$

If a tangent then perpendicular distance from $(0,0)$ to line is 6 units

$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

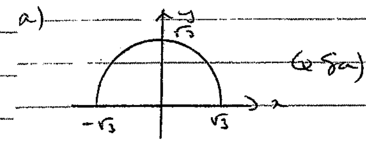
$= \frac{|0 + 0 - 30|}{\sqrt{3^2 + 4^2}}$

$= \frac{30}{5}$

$= 6 \quad \therefore$ a tangent

Question 8

a)



b) $\frac{dy}{dx} = 5 - 2x$

$y = \int 5 - 2x \, dx$

$y = 5x - x^2 + c$

when $x = -2, \quad y = 6$

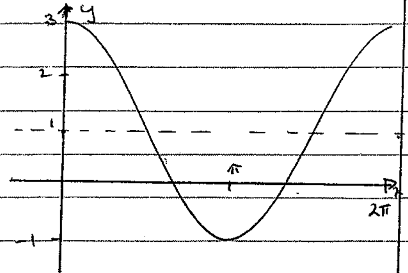
$6 = -10 - 4 + c$

$\therefore c = 20$

$y = 5x - x^2 + 20$

c) $f(x) = 1 + 2\cos x$

i) $-1 \le y \le 3$



d) $\dfrac{x+1}{x-1} = \dfrac{x-1+2}{x-1}$

$= 1 + \dfrac{2}{x-1}$

$\int \dfrac{x+1}{x-1}\,dx$

$\int 1 + \dfrac{2}{x-1}\,dx$

$x + 2\log_e(x-1) + C$

e) $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\sin\theta}{1+\cos\theta}$

$\dfrac{\sin\theta(1+\cos\theta) - \sin\theta(1-\cos\theta)}{1-\cos^2\theta}$

$\dfrac{\cancel{\sin\theta} + \sin\theta\cos\theta - \cancel{\sin\theta} + \sin\theta\cos\theta}{\sin^2\theta}$

$\dfrac{2\sin\theta\cos\theta}{\sin^2\theta}$

$\dfrac{2\cos\theta}{\sin\theta}$

$2\cot\theta$

Question Nine

a) $r = 0.005,\ n = 12 \times 20$

$= 240$

i) Value $= \$50(1.005)^{240}$

$= \$165.51$ (2)

ii) $\$50(1.005)^{240} +$

$\$50(1.005)^{239} +$

$\$50(1.005)^{238} + \ldots$

ie $\$50(1.005 + 1.005^2 + \ldots 1.005^{240})$

$= \dfrac{\$50 \times (1.005)(1.005^{240} - 1)}{0.005}$

$= \$23217.55$ (2)

b) $y = 5x - 4 - x^2$

i) $y = x - 1$

pt of intersection

$x - 1 = 5x - 4 - x^2$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$\left.\begin{matrix} x = 1 \\ y = 0 \end{matrix}\right\},\ \left.\begin{matrix} x = 3 \\ y = 2 \end{matrix}\right\}$ (2)

ii) $A = \int_1^3 (5x - 4 - x^2) - (x-1)\,dx$

$= \int_1^3 (4x - 3 - x^2)\,dx$

$= \left[2x^2 - 3x - \dfrac{x^3}{3}\right]_1^3$

$= [18 - 9 - 9] - [2 - 3 - \tfrac{1}{3}]$

$= \tfrac{4}{3}$ units^2 (2)

9c) i) B walks $4t$ km

A walks $10 - 3t$ km

$D^2 = (4t)^2 + (10-3t)^2$

$= 16t^2 + 100 - 60t + 9t^2$

$= 25t^2 - 60t + 100$ (2)

ii) $\dfrac{dD}{dt} = 50t - 60$

$\dfrac{d^2D}{dt^2} = 50 > 0 \therefore$ min

for min $\dfrac{dD}{dt} = 0$

$50t - 60 = 0$

$t = \tfrac{6}{5}$ hr (2)

Question 10

a) $\dfrac{6^{2n} \times 9^{2n-1}}{4^n} = 1$

$\dfrac{\cancel{2^{2n}} \times 3^{2n} \times 3^{4n-2}}{\cancel{2^{2n}}} = 1$

$3^{6n-2} = 1$

$3^{6n-2} = 3^0$

$n = \tfrac{1}{3}$ (2)

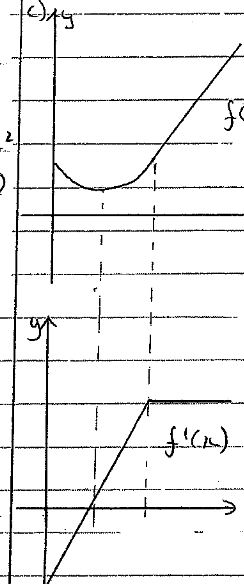
b) $y = (2x-1)^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \tfrac{1}{2}(2x-1)^{-\frac{1}{2}}(2)$

$= \dfrac{1}{\sqrt{2x-1}}$

which cannot equal zero $\therefore$ no stat pts (2)

c)



(2)

d) $C = \left(\dfrac{S^2}{80} + 100\right)$

$S = \dfrac{D}{T} \therefore T = \dfrac{D}{S}$

$= \dfrac{500}{S}$

Cost for trip

$C = \dfrac{500}{S}\left(\dfrac{S^2}{80} + 100\right)$

$= 500\left(\dfrac{S}{80} + 100S^{-1}\right)$

$\dfrac{dC}{dS} = 500\left(\dfrac{1}{80} - \dfrac{100}{S^2}\right)$

for min $\dfrac{dC}{dS} = 0$

$\dfrac{1}{80} - \dfrac{100}{S^2} = 0$

$S^2 = 8000$

$S \doteqdot 89$ km/h (3)

$\dfrac{d^2C}{dS^2} = 500\left(0 + \dfrac{200}{S^3}\right)$

$> 0 \therefore$ min

Q10 e) $x(x+3)(x+1)(x+2) - 120 = 0$

$(x^2+3x)(x^2+3x+2) = 120$

put $m = x^2+3x$

$m(m+2) = 120$

$m^2 + 2m - 120 = 0$

$(m+12)(m-10) = 0$

$m = -12$ or $m = 10$

$x^2+3x+12 = 0$, $x^2+3x-10=0$

$\Delta < 0$ $\quad (x+5)(x-2) = 0$

$\therefore$ no solns $\quad x = -5, x = 2$

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