SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
TRIAL EXAMINATIONS 2004

FORM VI

MATHEMATICS

Examination date

Wednesday 4th August 2004

Time allowed

3 hours (plus 5 minutes reading time)

Instructions

All ten questions may be attempted.
All ten questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your candidate number clearly on each booklet.
Hand in the ten questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

Checklist

SGS booklets: 10 per boy. A total of 1250 booklets should be sufficient.
Candidature: 109 boys.

Examiner

PKH

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\sqrt{\dfrac{3{\cdot}4^4}{15{\cdot}6 \times 12{\cdot}8}}$, correct to three significant figures. 2

(b) Differentiate $5x^2 - \cos x$. 2

(c) Find a primitive of $x^3 + 5$. 2

(d) A sector of a circle subtends an angle of 40° at the centre of the circle. If the radius of the circle is 9 units, find the area of the sector. 2

(e) (i) Solve $|x - 4| \leq 1$. 1

(ii) Graph your solution on a number line. 1

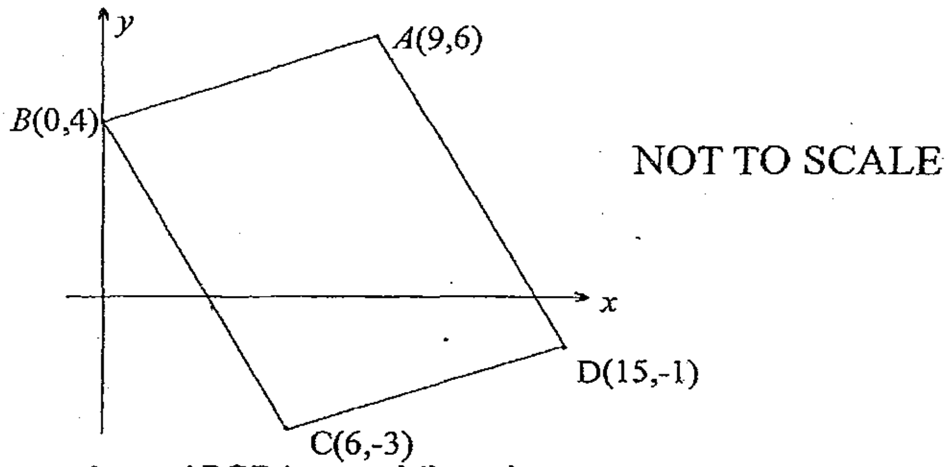
(f) Express $\dfrac{3}{3 - \sqrt{5}}$ with a rational denominator. 2

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Arthur invests \$20 000 in a term deposit at 6% pa compounded annually. How much will the investment be worth after four years? Give your answer correct to the nearest cent. 2

(b) Find the equation of the tangent to the curve $y = e^{2x}$ at the point where $x = 0$. 3

(c)



In the diagram above, $ABCD$ is a quadrilateral:

(i) Find the midpoint of the diagonal AC. 1

(ii) Find the midpoint of the diagonal BD. 1

(iii) Find the gradient of BD. 1

(iv) Show that AC is perpendicular to BD. 2

(v) What shape best describes quadrilateral $ABCD$? Give reasons. 2

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following functions:

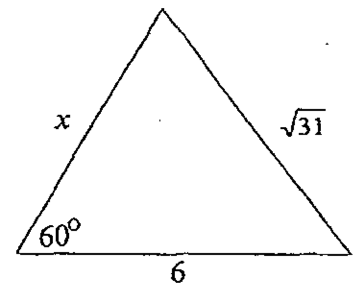
(i) $y = \cos(2x + 1)$ 2

(ii) $y = \dfrac{x}{\log_e x}$ 2

(iii) $y = x^2 \tan x$ 2

(b) Sketch the parabola $x^2 = -4(y + 2)$, showing its vertex and directrix. 3

(c) 3



In the diagram above, use the cosine rule to find the possible values of x.

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find:

(i) $\displaystyle\int_0^{\ln 2} e^{2x}\,dx$ 2

(ii) $\displaystyle\int \frac{2x}{x^2 + 5}\,dx$ 1

(b) Consider the arithmetic series

$$5 + 12 + 19 + \cdots + 292.$$

(i) How many terms in the series? 1

(ii) Find the sum of the series. 2

(c) A particle moves in a straight line. At time t seconds, its displacement x metres from the origin is given by

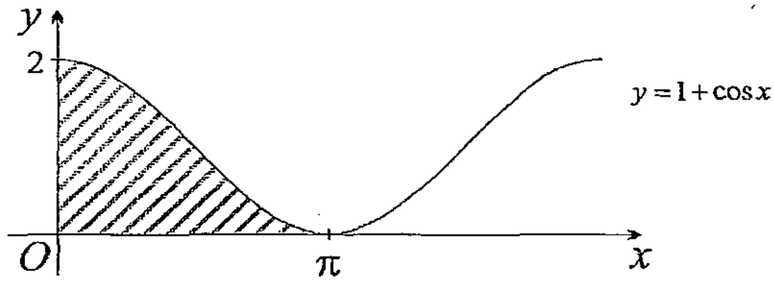
$$x = 8t - 2t^2.$$

(i) Sketch the graph of x as a function of t, showing the vertex. 2

(ii) Find the distance the particle travels in the first three seconds. 2

Exam continues overleaf ...

(d) **2**

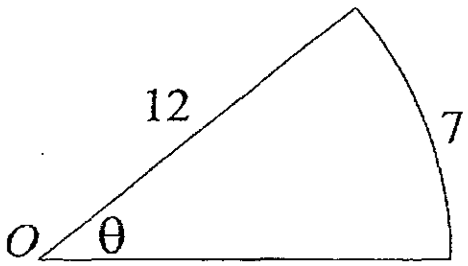


Find the shaded area in the diagram above.

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

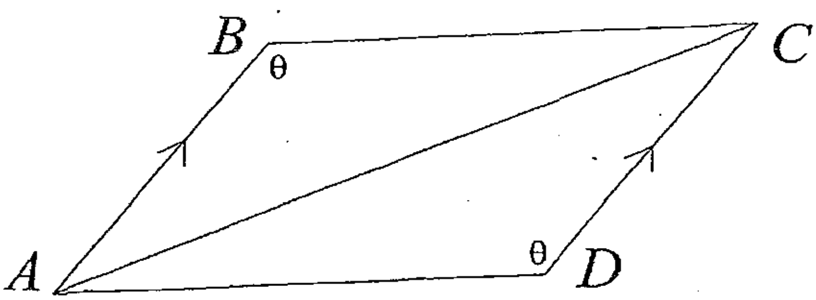
(a) Sketch the function $y = -2\sin x$, for $0 \le x \le 2\pi$. **2**

(b) **2**



The sector drawn above has its centre at O. Find the size of angle θ, correct to the nearest degree.

(c)



In the diagram above, $AB \parallel DC$ and $\angle B = \angle D = \theta$.

(i) Prove that $\triangle ABC \equiv \triangle CDA$ **2**

(ii) Prove that the quadrilateral $ABCD$ is a parallelogram. **3**

(d) (i) Make x^2 the subject of $y = \sqrt{x - 1}$. **1**

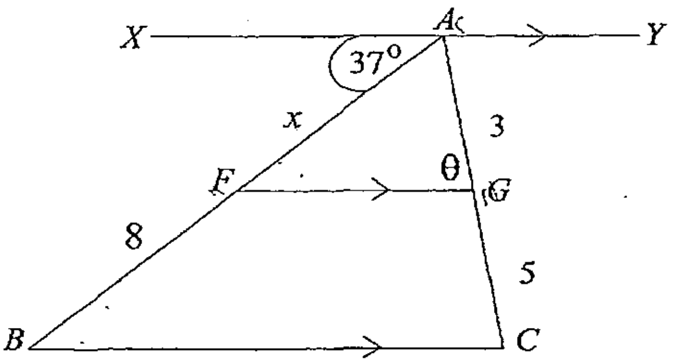
(ii) Find the volume formed when the region between the curve $y = \sqrt{x-1}$ and the y-axis, from $y = 0$ to $y = 3$, is rotated about the y-axis. **2**

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) The derivative of a function is given by $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$. Given that $y = -2$ when $x = 4$, find y as a function of x. **2**

(b)



(i) Find the value of x in the diagram above. **2**

(ii) Find θ, correct to the nearest degree. **2**

(c) If $\log_a 5 = x$ and $\log_a 2 = y$, find $\log_a 400$ in terms of x and y. **2**

(d) (i) Copy and complete the table below for $y = \sqrt{2 + e^x}$, calculating each value correct to three decimal places. **2**

x	0	1	2
y			

(ii) Use Simpson's rule with three function values to approximate $\int_0^1 \sqrt{2 + e^x}\, dx$. Give your answer correct to two decimal places. **2**

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $y = x^4 - 4x^3 + 3$.

(i) Find the stationary points and determine their nature. **4**

(ii) The curve has a point of inflexion where the tangent is not horizontal. Find the coordinates of this point. **2**

(iii) Sketch the function, showing all stationary points and points of inflexion. **2**

(b) For what values of k is the quadratic **4**

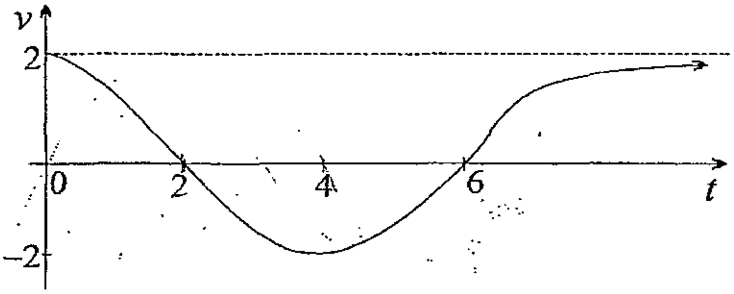
$$kx^2 - 2x\sqrt{6} + k + 1$$

positive definite?

Exam continues overleaf ...

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a)



A particle is moving in a straight line with displacement measured from the origin. The graph drawn above shows the particle's velocity at time t. The particle is initially at the origin.

(i) When does the particle return to the origin? **1**

(ii) Draw a graph of the displacement x against the time t. **2**

(b) The population P of a growing town satisfies the equation

$$P = P_0 e^{kt}$$

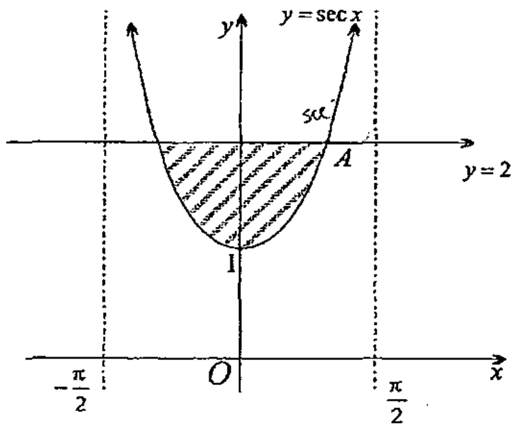
where t is time in years.

The initial population is 22 000 and five years later the population is 27 000.

(i) Find P_0 and k. **3**

(ii) When does the population reach 35 000? Give your answer correct to three significant figures. **2**

(c)



(i) Find the x coordinate of point A in the diagram above. **1**

(ii) Find the volume formed when the shaded area in the diagram above is rotated about the x axis. **3**

Exam continues next page ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) A vessel initially contains 100 litres. It is being emptied, and the rate of change of volume is

$$\frac{dV}{dt} = -\left(2 + \frac{20}{t+1}\right)$$

where V is the volume in the vessel in litres after t minutes.

(i) What is the initial rate $\frac{dV}{dt}$? **1**

(ii) Find how many litres remain in the vessel after five minutes. **3**

(b) A person borrows \$250 000 from a bank at a reducible interest rate of 6% per annum, compounded monthly. The loan is to be repaid in equal monthly installments.

Let $\$M$ be the monthly payment. Let A_n be the amount owing at the end n months when the nth payment has just been made.

The loan must be paid off after twenty years.

(i) Show that the amount owing after three months is given by **2**

$$A_3 = 250\,000 \times (1{\cdot}005)^3 - M(1 + 1{\cdot}005 + 1{\cdot}005^2).$$

(ii) Explain why $A_{240} = 0$. **1**

(iii) Find the value of M. **3**

(iv) Suppose now that the person elects to pay \$2000 per month instead of the amount calculated in part (iii). How much more quickly would the person pay off the loan? **2**

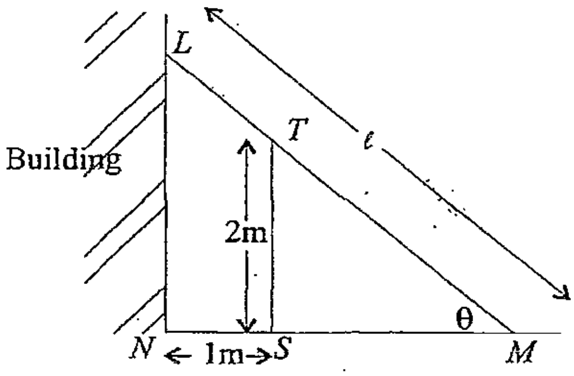
Exam continues overleaf ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Show that the graph of $y = x^{\frac{2}{3}}$ is concave down for all values of x except for $x = 0$. [2]

(ii) Solve $x^{\frac{2}{3}} \leq \frac{x}{2}$. [2]

(b)



The diagram above shows an extension ladder LM of variable length ℓ. The ladder leans against the wall of a building. It also touches the top of the fence ST, which is 2 metres high and stands 1 metre from the wall.

Let θ be the angle between the ladder and the horizontal ground.

(i) Show that $\ell = \dfrac{2}{\sin\theta} + \dfrac{1}{\cos\theta}$. [2]

(ii) Show that the stationary point of the graph of ℓ occurs when $\tan\theta = \sqrt[3]{2}$. [3]

(iii) For safety reasons, θ must lie between 55° and 70°. Find the minimum length of ℓ. Justify your answer. [3]

END OF EXAMINATION

SOLUTION TO SGS TRIAL 2004 Mathematics

Question One (12 marks)

(a) $\sqrt{\dfrac{3.4^4}{15.6 \times 12.8}} = 0.818$ ✓ answer ✓ correct rounding

(b) $y = 5x^2 - \cos x$

$y' = 10x + \sin x$ ✓✓ one for each

(c) A primitive of $x^3 + 5$

is $\dfrac{x^4}{4} + 5x$ ✓✓ one for each

(d) $\dfrac{40^\circ}{360^\circ} = \dfrac{1}{9}$

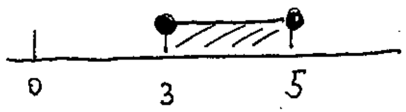
Area of a sector $= \frac{1}{9}\pi r^2$ ✓ one for $40^\circ = \frac{\pi}{9}$

$= \frac{1}{9}\pi \times 9^2$

$= 9\pi$ units ✓

(e) (i) $|x - 4| \leq 1$

$3 \leq x \leq 5$ ✓

(ii) 

✓

(f) $\dfrac{3}{3-\sqrt{5}} \times \dfrac{3+\sqrt{5}}{3+\sqrt{5}}$ ✓

$= \dfrac{9 + 3\sqrt{5}}{\quad}$ ✓

Question Two (12 marks)

(a) $P_n = P\left(1 + \frac{r}{100}\right)^n$

$P_4 = 20000 \times (1.06)^4$ ✓

$= \$25249.54$ ✓

(b) $y = e^{2x}$

$y' = 2e^{2x}$ ✓

When $x = 0$, gradient $= 2e^0 = 2$ ✓

$y = e^0 = 1$

Equ of tangent is $y - y_1 = m(x - x_1)$

$y - 1 = 2(x - 0)$ ✓

$y = 2x + 1$

(c) (i) midpoint of AC $= \left(\frac{9+6}{2}, \frac{6+-3}{2}\right)$ ✓

$= (7\frac{1}{2}, 1\frac{1}{2})$

(ii) midpoint of BD $= \left(\frac{0+15}{2}, \frac{4+-1}{2}\right)$ ✓

$= (7\frac{1}{2}, 1\frac{1}{2})$

(iii) $m_{BD} = \dfrac{4--1}{0-15} = -\frac{1}{3}$ ✓

(iv) $m_{AC} = \dfrac{6--3}{9-6} = 3$ ✓

$m_{AC} \times m_{BD} = 3 \times -\frac{1}{3} = -1$ ✓

$\therefore$ AC $\perp$ BD ✓

(v) ABCD is rhombus ✓

Question Three (12 marks)

(a) (i) $y = \cos(2x+1)$

$y' = -\sin(2x+1) \times 2$
$= -2\sin(2x+1)$ ✓✓

(ii) $y = \dfrac{x}{\log_e x}$

$y' = \dfrac{\log_e x \times 1 - x \times \frac{1}{x}}{(\log_e x)^2}$ ✓✓

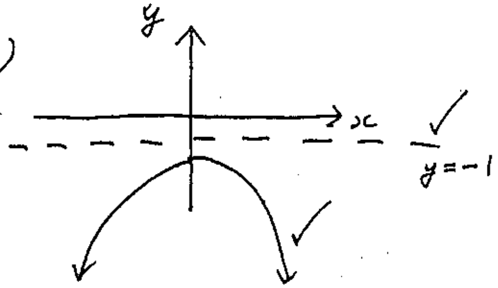
$= \dfrac{\log_e x - 1}{(\log_e x)^2}$

(iii) $y = x^2 \tan x$

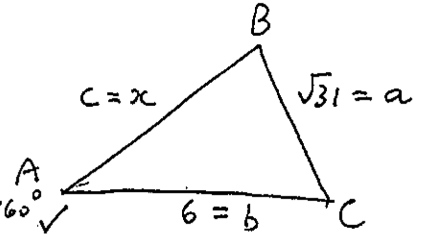
$y' = x^2 \sec^2 x + 2x \tan x$
✓ ✓

(b) $x^2 = -4(1)(y+2)$

$V = (0, -2)$ ✓



(c)



$a^2 = b^2 + c^2 - 2bc\cos A$

$31 = 36 + x^2 - 2 \times 6 \times x \times \cos 60°$ ✓

$31 = 36 + x^2 - 2 \times 6x \times \frac{1}{2}$

$31 = 36 + x^2 - 6x$

$x^2 - 6x + 5 = 0$ ✓

$(x-5)(x-1) = 0$

$x = 5$ or $x = 1$ ✓

Question 4 (12 marks)

(a) (i) $\int_0^{\ln 2} e^{2x}\,dx$

$= \left[\frac{1}{2} e^{2x}\right]_0^{\ln 2}$

$= \frac{1}{2} e^{2\ln 2} - \frac{1}{2} e^0$

$= \frac{1}{2} e^{\ln 4} - \frac{1}{2}$

$= 2 - \frac{1}{2} \doteq 1\frac{1}{2}$
✓

(ii) $\int \dfrac{2x}{x^2+5}\,dx$

$= \ln(x^2+5) + c$ ✓ ✓

(d) $A = \int_0^{\pi} 1 + \cos x\, d$[…]

$= \left[x + \sin x\right]_0^{\pi}$ ✓

$= \pi + \sin\pi$

$= \pi$ units^2 ✓

(b) $5 + 12 + 19 + \ldots 292$

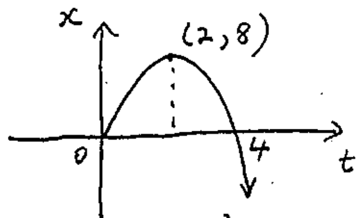
(i) # terms $= \dfrac{292-5}{7} + 1 = 42$ ✓

(ii) $S_n = \frac{n}{2}(a + l)$ ✓

$S_{42} = 21(5 + 292)$ ✓

$= 6237$

(c) (i)



✓ shape

✓ vertex

$x = 8t - 2t^2$
$= 2t(4-t)$

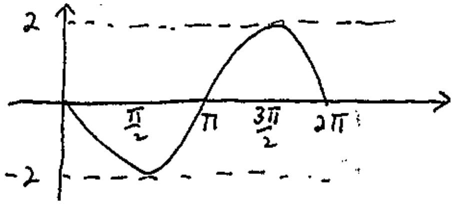
V: $t = 2$, $x = 8$

(ii) When $t = 3$, $x = 24 - 18 = 6$ ✓ (2 metr[…] from[…]

Distance covered $= 8 + 2$
$= 10$ metres ✓

Question 5 (12 marks)

(a) $y = -2\sin x$ for $0 \le x \le 2\pi$



✓ shape and amplitude

✓ period.

(b) $l = r\theta$

$\frac{7}{12} = \theta$ ✓

$\theta = \left(\frac{7}{12} \times \frac{180}{\pi}\right)$ degrees

$\theta = 33°$ ✓

(c) (i) AC is common

$\angle BAC = \angle ACD$ (alt ∠'s, AB ∥ CD)

$\angle ABC = \angle ADC$ (given) ✓

∴ $\Delta ABC \equiv \Delta CDA$ (AAS test) ✓

(ii) $\angle BCA = \angle DAC$ (matching sides of congruent Δ's) ✓

But these are alternate angles

∴ BC ∥ AD ✓

∴ ABCD is a parallelogram since two pairs of opposite sides are parallel ✓

(d) (i) $y = \sqrt{x-1}$

$y^2 = x - 1$

$(y^2+1)^2 = x^2$ ✓

$V = \pi \int_0^3 y^4 + 2y^2 + 1 \, dy$ ✓

$= \pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y\right]_0^3$ ✓

Question 6 (12 marks)

(a) $\frac{dy}{dx} = x^{-\frac{1}{2}}$

$y = 2x^{\frac{1}{2}} + C$ ✓

When $x = 4$, $y = -2$: $-2 = 2\sqrt{4} + C$

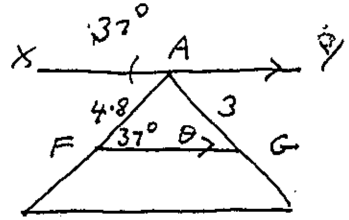
$C = -6$

$y = 2\sqrt{x} - 6$ ✓

(b) (i) $\frac{x}{3} = \frac{8}{5}$ ✓

$x = \frac{24}{5}$

$x = 4.8$ ✓



$\angle AFG = 37°$ (alt ∠'s, XY ∥ FG)

$\frac{\sin\theta}{4.8} = \frac{\sin 37°}{3}$ ✓

$\theta = 74°$ ✓

(c) $\log_a 400$

$= \log_a 25 + \log_a 16$ ✓ 'splitting logs'

$= 2\log_a 5 + 4\log_a 2$

$= 2x + 4y$ ✓ correct answer

(d) (i)

x	0	1	2
y	1.732	2.172	3.064

✓ one correct

✓✓ three correct

(ii) $A \doteq \frac{h}{6}\left[f(a) + 4 \times f\left(\frac{a+b}{2}\right) + f(b)\right]$ ✓

$= \frac{2-0}{6}\left[1.732 + 4 \times 2.172 + 3.064\right]$ ✓ (Follow through

Question 7 (12 marks)

(a) (i) $y = x^4 - 4x^3 + 3$

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x = 12x(x-2)$

Stat pts where $y' = 0$

$4x^3 - 12x^2 = 0$

$4x^2(x-3) = 0$

$x = 0$ or $x = 3$ ✓

When $x = 3$, $y'' = 108 - 72 > 0$ ✓

$\therefore$ Local minimum at $(3, -24)$

When $x = 0$, $y'' = 0$ (test inconclusive) ✓

When $x = -1$, $y' = -16 < 0$

$x = 1$ $y' = -8 < 0$ ✓

$\therefore$ Horizontal point of inflexion at $(0, 0)$

(ii) Possible pts of inflexion where $y'' = 0$

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0$ or $x = 2$ ✓

At $x = 0$ there is a horizontal pt of inflexion

Consider $x = 1.9$, $y'' = 12(1.9)(-0.1) < 0$

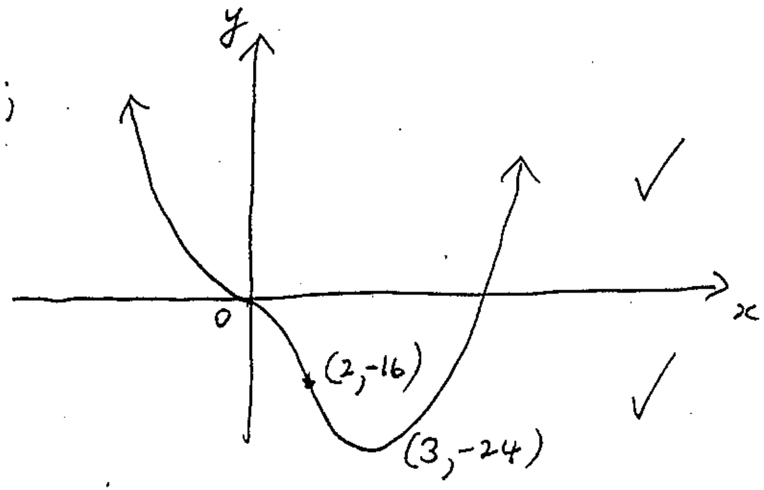
$x = 2.1$, $y'' = 12(2.1)(0.1) > 0$ ✓

Change in concavity

There is a point of inflexion at $(2, -16)$

[illegible]

(iii)



✓ ✓

(b) Consider the quadratic

$Kx^2 - 2x\sqrt{6} + (K+1)$

This is positive definite when

$a > 0$ and $b^2 - 4ac < 0$ ✓

ie. $K > 0$ and $(2\sqrt{6})^2 - 4K(K+1) < 0$ ✓

$24 - 4K^2 - 4K < 0$

$K^2 + K - 6 > 0$

$(K+3)(K-2) > 0$

$K > 0$ **and** $K < -3$ or $K \quad 2$ ✓

The quadratic is positive definite

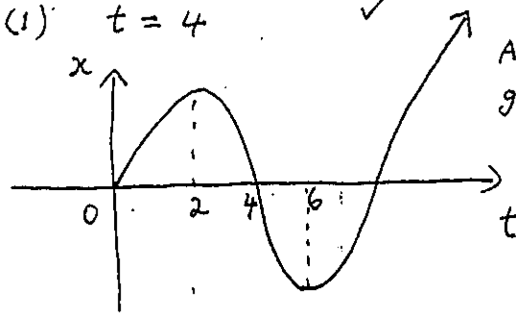
when $K > 2$ ✓

(9)

Question 8 (12 marks)

(a) (i) $t = 4$ ✓

(ii)



As $t \to \infty$, gradient $\to 2$ ✓ ✓

(b) (i) $P_0 = 22000$ ✓

$P = P_0 e^{kt}$

$t = 5$, $P = 27000$

$27000 = 22000 e^{5k}$ ✓

$k = \frac{1}{5} \ln \frac{27}{22}$ ✓

$\doteqdot 0.04095 \ldots$

(ii) $P = 35000$, $t = ?$

$35000 = 22000 e^{kt}$ ✓

$\frac{35}{22} = e^{kt}$

$t = \frac{1}{k} \ln \frac{35}{22}$ ✓

$= 11.3$ years

(c) (i) Pts of intersection where $\sec x = 2$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ ✓

(ii) $V = 2 \left\{ \int_0^{\frac{\pi}{3}} 2^2 \, dx - \int_0^{\frac{\pi}{3}} \sec^2 x \, dx \right\}$ ✓

$= 2 \left\{ \left[4x \right]_0^{\frac{\pi}{3}} - \left[\tan x \right]_0^{\frac{\pi}{3}} \right\}$ ✓

$= 2 \left\{ \frac{4\pi}{3} - \sqrt{3} \right\}$ ✓ $= \left(\frac{8\pi}{3} - 2\sqrt{3} \right)$ units^3

Question 9 (12 marks

(a) $\frac{dV}{dt} = -2 - \frac{20}{t+1}$

(i) When $t = 0$, $\frac{dV}{dt} = -2 - \frac{20}{1}$ ✓

$= -22$ l/min.

(ii) $\frac{dV}{dt} = -2 - \frac{20}{t+1}$

$V = -2t - 20 \ln(t+1) + C$ ✓

When $t = 0$, $V = 100$: $100 = 0 - 20 \ln 0 + C$

$C = 100$

$V = -2t - 20 \ln(t+1) + 100$ ✓

When $t = 5$, $V = -10 - 20 \ln 6 + 100$

$\doteqdot 54.2$ litres ✓

(b) (i) $A_1 = 250\,000 \times 1.005 - M$ ✓

$A_2 = \left((250\,000 \times 1.005) - M \right) 1.005 - M$

$= 250000 \times (1.005)^2 - 1.005M - M$

$= 250000 \times (1.005)^2 - M(1 + 1.005)$ ✓

$A_3 = 250000 \times (1.005)^3 - M(1 + 1.005 + 1.00$

(ii) Loan is paid of over 240 months = 20y ✓

(iii) $0 = 250\,000 \times (1.005)^{240} - M(1 + 1.005 \ldots + 1.00$

$250000 \times (1.005)^{240} = M \left[\frac{a(r^n - 1)}{r - 1} \right]$

$250\,000 \times (1.005)^{240} = M\left(\frac{(1.005)^{240} - 1}{1.005 - 1}\right)$ ✓ ✓

$M = \$1791.08$ ✓

(iv) Suppose $M = \$2000$ we need to find n so that

$250000 \times (1.005)^{n} = 2000\left(\frac{(1.005)^{n} - 1}{0.005}\right)$ ✓

$250000 \times 0.005 \times (1.005)^{n} = 2000 \times 1.005^{n} - 2000$

$1250 \times (1.005)^{n} = 2000 \times 1.005^{n} - 2000$

$(1.005)^{n}(2000 - 1250) = 2000$

$(1.005)^{n} = \frac{8}{3}$

$n \ln(1.005) = \ln \frac{8}{3}$

$n = \frac{\ln \frac{8}{3}}{\ln 1.005}$ ✓

$= 196.65$ months

Loan is paid off approximately 43 months earlier.

Question 10

(a) (i) $y = x^{\frac{2}{3}}$

$y' = \frac{2}{3} x^{-\frac{1}{3}}$

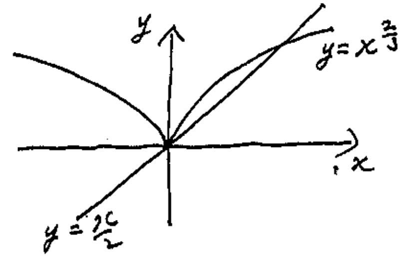
$y'' = -\frac{2}{9} x^{-\frac{4}{3}}$

$y'' = -\frac{2}{9} \sqrt[3]{\frac{1}{x^4}}$

$x^4 > 0$ for $x \neq 0$

$\therefore y'' < 0$ for $x \neq 0$

(ii) Solve $x^{\frac{2}{3}} = \frac{x}{2}$



Cube both sides

$x^2 = \frac{x^3}{8}$

$x^3 - 8x^2 = 0$

$x^2(x - 8) = 0$ ✓ ← Give one mark for sketch OR

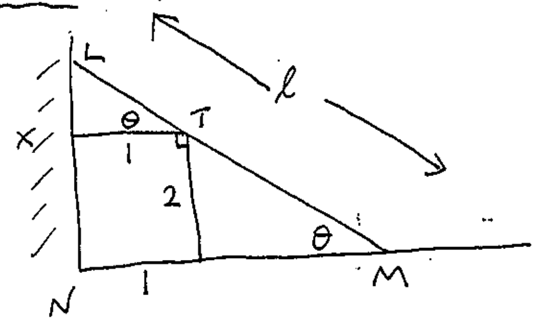
$x = 0$ or $x = 8$

Solution to $x^{\frac{2}{3}} \leq \frac{x}{2}$

is $x \geq 8$ or $x = 0$ ✓

Question 10 (12 marks) cont'd

(b)



(i) Draw XT parallel to NM so that $\angle LTX = \theta$.

$\frac{TL}{TX} = \sec\theta$ ✓ OR $\frac{TM}{2} = \operatorname{cosec}\theta$

$TL = \sec\theta$ $\quad$ $TM = 2\operatorname{cosec}\theta$

$\therefore \quad LM = \sec\theta + 2\operatorname{cosec}\theta$

$l = \frac{1}{\cos\theta} + \frac{2}{\sin\theta}$ ✓

(ii) $l = (\cos\theta)^{-1} + 2(\sin\theta)^{-1}$

$l' = -\frac{-\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$ ✓

Stat pts where $l' = 0$

$\frac{\sin\theta}{\cos^2\theta} = \frac{2\cos\theta}{\sin^2\theta}$ ✓

$\frac{\sin^3\theta}{\cos^3\theta} = 2$

$\tan^3\theta = 2$ ✓

$\tan\theta = \sqrt[3]{2}$

(iii) Minimum l at $\theta = \tan^{-1}\sqrt[3]{2}$ or end points $\theta = 55°$ or $\theta = 70°$

$\theta = \tan^{-1}\sqrt[3]{2}$ ✓
$\doteqdot 51°34'$ which is outside the safety constraint.

Test ✓ $\theta = 55°$ $\quad l = \frac{2}{\sin 55°} + \frac{1}{\cos 55°} \doteqdot 4.18$

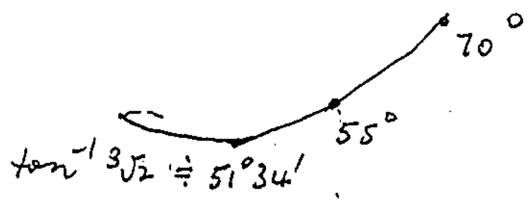
$\theta = 70°$ $\quad l = \frac{2}{\sin 70°} + \frac{1}{\cos 70°} \doteqdot 5.05$

So minimum l when $\theta = 55°$ ✓ and the length is 4.18 m

OR You can show that l is increasing for $\theta > \tan^{-1}\sqrt[3]{2}$

θ	$50°$	$\tan^{-1}\sqrt[3]{2}$	$52°$
l'	$-$	0	+

So the curve looks like



Clearly minimum at $\theta = 55°$