SYDNEY GRAMMAR SCHOOL

2009 Trial Examination

FORM VI
MATHEMATICS 2 UNIT

Tuesday 11th August 2009

General Instructions
- Reading time — 5 minutes
- Writing time — 3 Hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 120
- All ten questions may be attempted.
- All ten questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the ten questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper insisde your answer booklet for Question 1.

6F: SJE	6G: FMW	6H: BDD	6P: KWM
6Q: JMR	6R: LYL	6S: RCF	

Checklist
- SGS booklets — 10 per boy
- Candidature — 101 boys

Examiner
KWM

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $2^x = \dfrac{1}{16}$. **1**

(b) Factorise $x^3 + 27$. **1**

(c) Simplify $\dfrac{x}{2} - \dfrac{x-1}{3}$. **2**

(d) Find a primitive for $\sqrt{x}$. **1**

(e) Evaluate $\displaystyle\sum_{k=1}^{3} k^2$. **1**

(f) Solve $\sin\alpha = -\frac{1}{2}$, for $0 \le \alpha \le 2\pi$. **2**

(g) Solve $|x-2| < 5$. **2**

(h) Given $\dfrac{2}{\sqrt{5}+2} = p\sqrt{5} - q$, find p and q. **2**

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Consider the series $32 + 36 + 40 + \cdots + 92$.

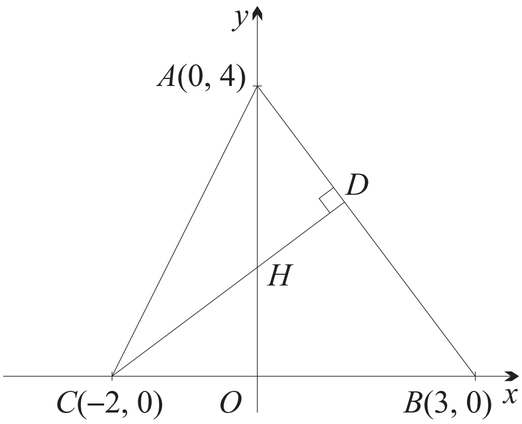
(i) Show that the series is arithmetic. **1**

(ii) How many terms are there in the series? **1**

(iii) Find the sum of the series. **1**

(b) Gillian deposits \$12 000 in a fixed term investment account earning 6% p.a. compounded **monthly**. Calculate the value of her investment after five years. Give your answer correct to the nearest cent. **2**

(c)



The triangle above has vertices $A(0,4)$, $B(3,0)$ and $C(-2,0)$. AO and CD are the altitudes drawn from vertices A and C respectively.

(i) Find the gradient of the side AB. **1**

(ii) Show that the side AB has equation $4x + 3y - 12 = 0$. **1**

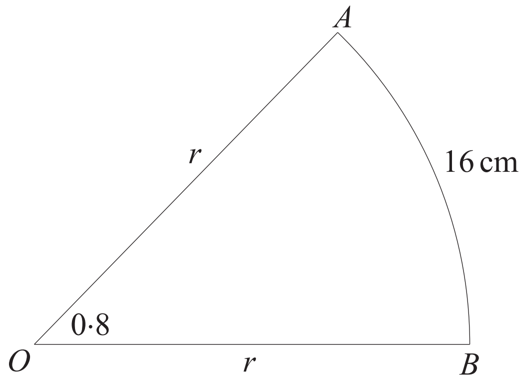
(iii) Calculate the perpendicular distance from the point $C(-2,0)$ to the side AB. **2**

(iv) Find the equation of the altitude CD. **2**

(v) Hence find the coordinates of the point H, the point of intersection of the altitudes AO and CD. **1**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a)

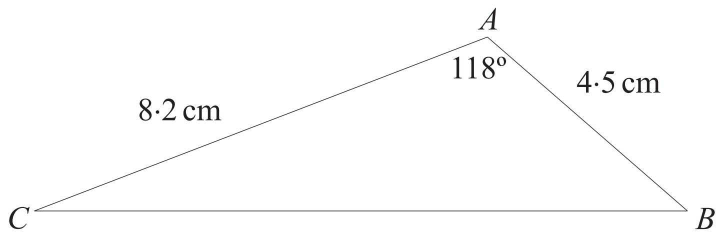


The diagram shows a sector of a circle. The arc AB is 16 cm, the radius is r cm and $\angle AOB = 0{\cdot}8$ radians.

(i) Find the value of r. **1**

(ii) Calculate the area of the sector. **2**

(b)



The diagram above shows $\triangle ABC$ where $AB = 4{\cdot}5$ cm, $AC = 8{\cdot}2$ cm and $\angle CAB = 118°$.

(i) Find the length of side BC, correct to the nearest millimetre. **2**

(ii) Calculate the area of $\triangle ABC$ in cm^2, correct to one decimal place. **1**

(c) Differentiate the following functions:

(i) $y = \dfrac{1}{x}$ **1**

(ii) $y = \tan 2x$ **1**

(iii) $y = xe^x$ **2**

(iv) $y = \dfrac{\log_e x}{x}$ **2**

Exam continues next page ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify $2\log_3 6 - \log_3 4$. **2**

(b) Find:

(i) $\displaystyle\int_0^{\ln 3} e^x\,dx$ **2**

(ii) $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$ **2**

(c) Find the equation of the tangent to the curve $y = \cos(\pi - x)$ at the point where $x = \frac{\pi}{3}$. **3**

(d) Consider the parabola $x^2 - 2x + 4y + 9 = 0$.

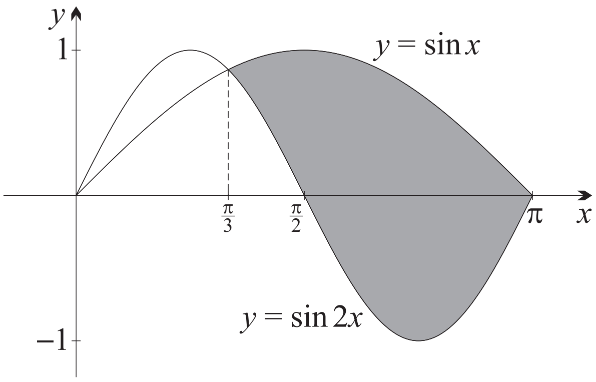
(i) Express the equation in the form $(x-h)^2 = -4a(y-k)$. **1**

(ii) Find the coordinates of the focus. **1**

(iii) Write down the equation of the directrix. **1**

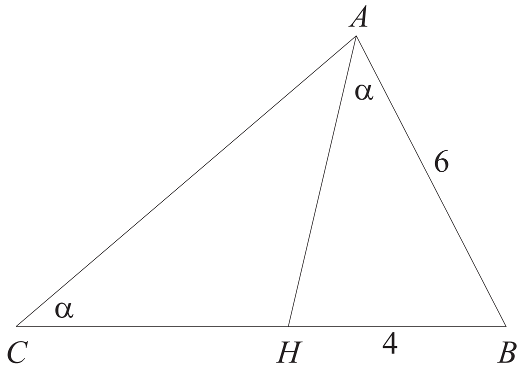
QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the curves $y = \sin 2x$ and $y = \sin x$ for $0 \leq x \leq \pi$, intersecting at $x = 0$, $x = \frac{\pi}{3}$ and $x = \pi$. Find the exact area of the shaded region bounded by the two curves. **3**

(b)



In the diagram above, $\angle BCA = \angle BAH = \alpha$, $AB = 6$ and $BH = 4$.

(i) Show that $\triangle ABC \,|||\, \triangle HBA$. **2**

(ii) Hence, or otherwise, find the length HC. **2**

(c) Consider the quadratic equation $x^2 - 2kx + (8k - 15) = 0$.

(i) Find the discriminant and write it in simplest form. **1**

(ii) For what values of k does the equation have real roots? **2**

(iii) If three times the sum of the roots is equal to twice the product of the roots, find the value of k. **2**

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $h(x) = \sqrt{x^2 - 1}$.

(i) Show that $h(x)$ is an even function. **1**

(ii) Find the domain of $h(x)$. **1**

(b) Consider the function $y = (x+1)^3(x-3)$.

(i) Use the product rule to show that $\dfrac{dy}{dx} = 4(x+1)^2(x-2)$. **2**

(ii) Find the coordinates of the stationary points and determine their nature. **4**

(iii) The curve has a point of inflexion where the tangent is not horizontal. Find the coordinates of this point. **2**

(iv) Sketch the curve $y = (x+1)^3(x-3)$, showing all the important features. **2**

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) The population P of mosquitoes in a laundry is growing exponentially according to the equation $P = 50e^{kt}$, where t is the time in days after the insects are first counted. After four days the population has doubled.

(i) Find the exact value of the constant k. **2**

(ii) How many mosquitoes will there be after 10 days? **2**

(iii) At what rate is the population increasing after 10 days? **1**

(b) (i) Copy and complete the table correct to four decimal places where necessary for the function $y = \log_e(x+1)$. **1**

x	0	0·5	1	1·5	2
y					

(ii) Use Simpson's rule with 5 function values to find an approximation to $\displaystyle\int_0^2 \log_e(x+1)\,dx$. Write your answer correct to three decimal places. **2**

(iii) Show that $\dfrac{d}{dx}\Big((x+1)\log_e(x+1) - x\Big) = \log_e(x+1)$. **2**

(iv) Hence find the exact value of $\displaystyle\int_0^2 \log_e(x+1)\,dx$, and determine whether or not your approximation in part (ii) is accurate to three decimal places. **2**

Exam continues overleaf ...

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) A swimming pool is being emptied. The volume of water L litres in the pool after t minutes is given by the equation

$$L = 1000(20 - t)^3.$$

(i) Find the rate at which the pool is emptying after 10 minutes. **2**

(ii) When is the pool emptying at a maximum rate? **1**

(b) (i) Expand $\left(\sqrt{3}u - 1\right)\left(u - \sqrt{3}\right)$. **1**

(ii) Hence solve $\sqrt{3}\tan^2\theta - 4\tan\theta + \sqrt{3} = 0$, for $0 \le \theta \le 2\pi$. **2**

(c) A particle moves in a straight line so that after t seconds ($t \ge 0$) its velocity v is given by $v = \left(\dfrac{2}{1+t} - t\right)$ m/s. The displacement of the particle from the origin is given by x metres.

(i) Find the acceleration of the particle when $t = 0$. **2**

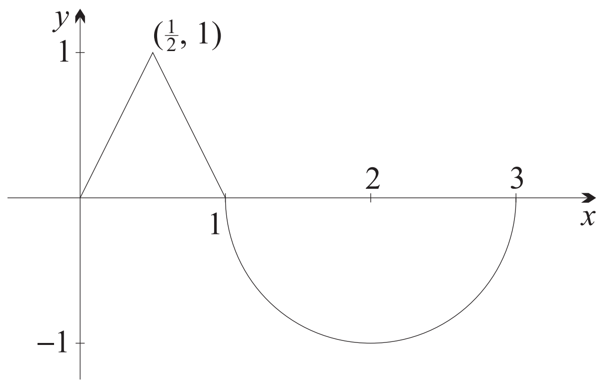
(ii) If the particle is initially at the origin, find the displacement as a function of t. **2**

(iii) When is the particle stationary? **1**

(iv) How far does the particle travel in the first 2 seconds? Give your answer correct to three significant figures. **1**

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a)

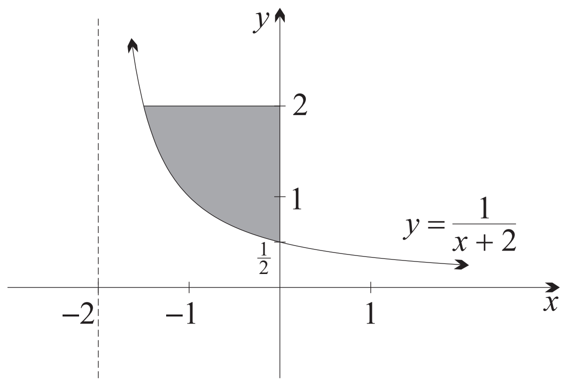


The diagram above shows the function $y = g(x)$ with domain $0 \le x \le 3$. The arc is a semi-circle. Find $\displaystyle\int_0^3 g(x)\,dx$. **2**

Exam continues next page ...

<u>QUESTION NINE</u> (Continued)

(b)

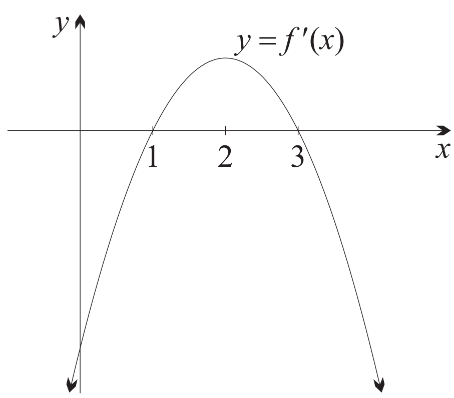


The diagram above shows the curve $y = \dfrac{1}{x+2}$ for $x > -2$.

(i) Show that $x^2 = \dfrac{1}{y^2} - \dfrac{4}{y} + 4$. **2**

(ii) Calculate the exact volume of the solid of revolution formed when the shaded region bounded by the y-axis, the line $y = 2$ and the curve is rotated about the y-axis. **4**

(c)



The diagram above shows the graph of the gradient function $y = f'(x)$ of the function $y = f(x)$.

(i) For which values of x is the curve $y = f(x)$ increasing? **1**

(ii) For which values of x is the curve $y = f(x)$ concave up? **1**

(iii) Given $f(0) = f(2) = f(4) = 0$, sketch the curve $y = f(x)$ for $0 \le x \le 4$. **2**

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

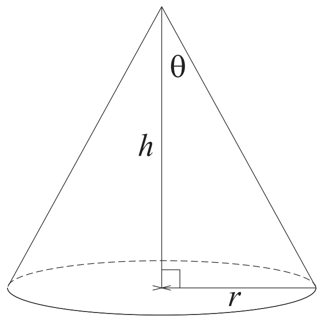
(a) Katherine borrows \$200 000 from the bank. The loan plus the interest is to be repaid in equal monthly instalments of \$$M$ over 25 years. Reducible interest is charged at 6% p.a. and is calculated monthly. Let A_n be the amount owing after n months.

(i) Write down expressions for A_1 and A_2, and show that the amount owing after 3 months is given by $A_3 = 200\,000(1{\cdot}005)^3 - M(1 + 1{\cdot}005 + 1{\cdot}005^2)$. **3**

(ii) Hence write an expression for A_n. **1**

(iii) Calculate the monthly instalment \$$M$ correct to the nearest cent. **2**

(b)



A right circular cone of radius r and height h has a total surface area S and volume V. Note that $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$ and $V = \frac{1}{3}\pi r^2 h$.

(i) Show that $9V^2 = r^2(S^2 - 2\pi r^2 S)$. **2**

(ii) For a fixed surface area S, find $\dfrac{d}{dr}(9V^2)$. **1**

(iii) Hence find the semi-vertical angle θ that gives the maximum volume of the cone for a fixed surface area S. Write your answer correct to the nearest minute. **3**

END OF EXAMINATION

B L A N K P A G E

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\;\; n \neq -1;\;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\;\; a > 0,\;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \;\ln x = \log_e x,\;\; x > 0$$

2 UNIT TRIAL 2009

QUESTION 1

(a) $2^x = \frac{1}{16}$

$x = -4$ ✓

(b) $x^3 + 27 = (x+3)(x^2 - 3x + 9)$ ✓

(c) $\frac{x}{2} - \left(\frac{x-1}{3}\right) = \frac{3x - 2(x-1)}{6}$ ✓

$= \frac{3x - 2x + 2}{6}$

$= \frac{x+2}{6}$ ✓

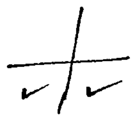
(d) $f(x) = x^{1/2}$

$F(x) = \frac{2}{3}x^{3/2} + c$ ✓

(e) $\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$

$= 14$ ✓

(f) $\sin\alpha = -\frac{1}{2}$



$\alpha = \pi + \frac{\pi}{6}$ or $\alpha = 2\pi - \frac{\pi}{6}$

$\alpha = \frac{7\pi}{6}$ ✓ or $\alpha = \frac{11\pi}{6}$ ✓

(g) $|x-2| < 5$

$x - 2 < 5$ or $x - 2 > -5$

$x < 7$ ✓ or $x > -3$ ✓

$-3 < x < 7$

(h)

$\frac{2}{\sqrt{5}+2} = p\sqrt{5} - q$

$\frac{2}{\sqrt{5}+2} \times \frac{\sqrt{5}-2}{\sqrt{5}-2} = p\sqrt{5} - q$

$\frac{2\sqrt{5} - 4}{5 - 4}$ ✓ $= p\sqrt{5} - q$

$2\sqrt{5} - 4 = p\sqrt{5} - q$

Hence $p = 2$ and $q = 4$ ✓

12

QUESTION 2

(a) $32 + 36 + 40 + \cdots + 92$.

(i) $36 - 32 = 40 - 36 = 4$

$\therefore$ an AP. ✓

$(a = 32$ and $d = 4)$

(ii) $a + (n-1)d = 92$

$32 + (n-1)4 = 92$

$4n + 28 = 92$

$4n = 64$.

$n = 16$ ✓

There are 16 terms.

(iii) $S_n = \frac{n}{2}(a + l)$

$S_{16} = 8(32 + 92)$

$S_{16} = 992$ ✓

(b) $R = 1.005$

$P = 12\,000$

$n = 60$

$A = PR^n$

$= 12\,000(1.005)^{60}$ ✓

$= \$16,186.20$. ✓

(c) $A(0,4)$, $B(3,0)$

(i) gradient $AB = -\frac{4}{3}$ ✓

(ii) $(0,4)$, $m = -\frac{4}{3}$

$y - y_1 = m(x - x_1)$

$y - 4 = -\frac{4}{3}(x - 0)$ ✓

$3y - 12 = -4x$

$4x + 3y - 12 = 0$

(as required).

(iii) $4x + 3y - 12 = 0$

$C(-2,0)$

$d = \frac{|-8 + 0 - 12|}{\sqrt{16 + 9}}$ ✓

$d = \frac{20}{5}$

$d = 4$ units. ✓

(iv) gradient $CD = \frac{3}{4}$ ✓ $C(-2,0)$

$y - y_1 = m(x - x_1)$

$y = \frac{3}{4}(x + 2)$

$4y = 3x + 6$

$3x - 4y + 6 = 0$ ✓

(v) put $x = 0$.

$3x - 4y + 6 = 0$

$-4y + 6 = 0$

$4y = 6$

$y = \frac{3}{2}$

$H(0, \frac{3}{2})$ // ✓

12

QUESTION 3

(a) (i) $l = r\theta$

$16 = 0.8 \times r$

$r = 20$ cm. ✓

(ii) $A = \frac{1}{2} r^2 \theta$ ✓

$= \frac{1}{2} \times 400 \times 0.8$

$= 160 \text{ cm}^2$ ✓

(b)

(i) $BC^2 = 8.2^2 + 4.5^2 - 2 \times 8.2 \times 4.5 \cos 118^\circ$ ✓

$BC \doteqdot 11.1$ cm ✓

(ii) Area $= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 8.2 \times 4.5 \sin 118^\circ$

$\doteqdot 16.3 \text{ cm}^2$ ✓

(c) (i) $y = x^{-1}$

$\frac{dy}{dx} = -\frac{1}{x^2}$ ✓

(ii) $y = \tan 2x$

$\frac{dy}{dx} = 2 \sec^2 2x$ ✓

(iii) $y = xe^x$

$\frac{dy}{dx} = e^x + xe^x$ ✓✓

$= e^x(1 + x).$

(iv) $y = \frac{\ln x}{x}$

$\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2}$ ✓✓

$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

12

QUESTION 4

(a) $2\log_3 6 - \log_3 4$

$= \log_3 36 - \log_3 4$ ✓

$= \log_3 9$

$= 2$ ✓

(b)

(i) $\int_0^{\ln 3} e^x \, dx = \left[e^x \right]_0^{\ln 3}$ ✓

$= e^{\ln 3} - e^0$

$= 3 - 1$

$= 2$ ✓

(ii) $\int_0^1 \frac{x}{x^2+1} \, dx = \frac{1}{2}\left[\ln(x^2+1) \right]_0^1$ ✓

$= \frac{1}{2}\left\{ \ln 2 - \ln 1 \right\}$

$= \frac{1}{2} \ln 2$ ✓

$= \ln\sqrt{2}$.

(c) $y = \cos(\pi - x)$

$\left(\frac{\pi}{3}, -\frac{1}{2}\right)$ ✓

$\frac{dy}{dx} = \sin(\pi - x)$ ✓

at $x = \frac{\pi}{3}$, gradient $= \sin \frac{2\pi}{3}$

$= \frac{\sqrt{3}}{2}$

$y - y_1 = m(x - x_1)$

$y + \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$ ✓

$2y + 1 = \sqrt{3}x - \frac{\sqrt{3}\pi}{3}$

$\sqrt{3}x - 2y - 1 - \frac{\sqrt{3}\pi}{3} = 0$

(d) $x^2 - 2x + 4y + 9 = 0$

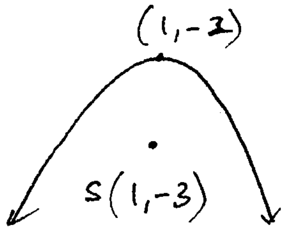
(i) $(x-1)^2 - 1 + 4y + 9 = 0$

$(x-1)^2 = -4y - 8$

$(x-1)^2 = -4(y+2)$ ✓

vertex $(1, -2)$, $a = 1$.

(ii)



focus $(1, -3)$ ✓

(iii) $y = -1$. ✓

12

QUESTION 5

(a)

$$A = \int_{\frac{\pi}{3}}^{\pi} \sin x - \sin 2x \, dx$$

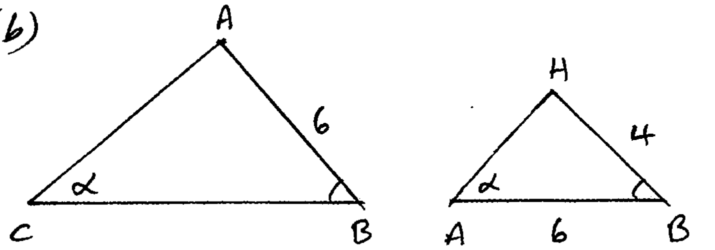
$$= \left[-\cos x + \tfrac{1}{2}\cos 2x \right]_{\frac{\pi}{3}}^{\pi}$$

$$= \left(1 + \tfrac{1}{2}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{2} \times -\tfrac{1}{2}\right)$$

$$= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4}$$

$A = 2\tfrac{1}{4}$ sq. units.

(b)



(i) $\angle ACB = \angle HAB = \alpha$ (given)

$\angle ABC = \angle HBA$ (common)

$\therefore \triangle ABC \,|||\, \triangle HBA$ (A.A)

(ii) $\frac{BC}{6} = \frac{6}{4}$ (matching sides in the same ratio.)

$BC = \frac{36}{4}$

$BC = 9$

$HC = BC - 4$

$= 9 - 4$

$HC = 5$ units.

(c) $x^2 - 2kx + (8k - 15) = 0$

(i) $\Delta = b^2 - 4ac$

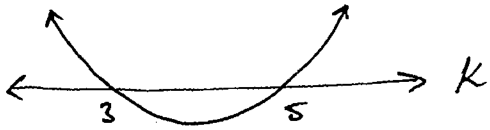
$\Delta = 4k^2 - 4(8k - 15)$

$\Delta = 4k^2 - 32k + 60$

(ii) $4k^2 - 32k + 60 \geq 0$

$k^2 - 8k + 15 \geq 0$

$(k - 3)(k - 5) \geq 0$



$k \leq 3, \quad k \geq 5$

(iii) $\alpha + \beta = -\frac{b}{a}$

$= 2k$

$\alpha\beta = \frac{c}{a}$

$= 8k - 15$

$6k = 2(8k - 15)$

$6k = 16k - 30$

$-10k = -30$

$k = 3.$

12

QUESTION 6

(a)

(i) $h(x) = \sqrt{x^2-1}$

$h(-x) = \sqrt{(-x)^2-1}$

$= \sqrt{x^2-1}$

$= h(x)$ ✓

$\therefore$ $h(x)$ is even.

(ii) Domain $x^2-1 \geq 0$

$(x-1)(x+1) \geq 0$

$x \leq -1$, $x \geq 1$ ✓

(b) $y = (x+1)^3(x-3)$

i) $y' = 3(x+1)^2(x-3) + (x+1)^3$ ✓

$= (x+1)^2\{3x-9+x+1\}$ ✓

$= (x+1)^2(4x-8)$

$= 4(x+1)^2(x-2)$

(as required.)

(ii) $4(x+1)^2(x-2) = 0$

$x=-1$ or $x=2$

stationary points.

$(-1,0)$ ✓ and $(2,-27)$ ✓

- nature.

x	-2	-1	0	0	2	3
y'	-16	0	-8	-8	0	64
	\	—	\	\	—	/

$(-1,0)$ is a horizontal pt. of inflex^n ✓

$(2,-27)$ is a minimum stationary point. ✓

(iii) $y'' = 8(x+1)(x-2) + 4(x+1)^2$

$= 4(x+1)\{2(x-2)+(x+1)\}$ ✓

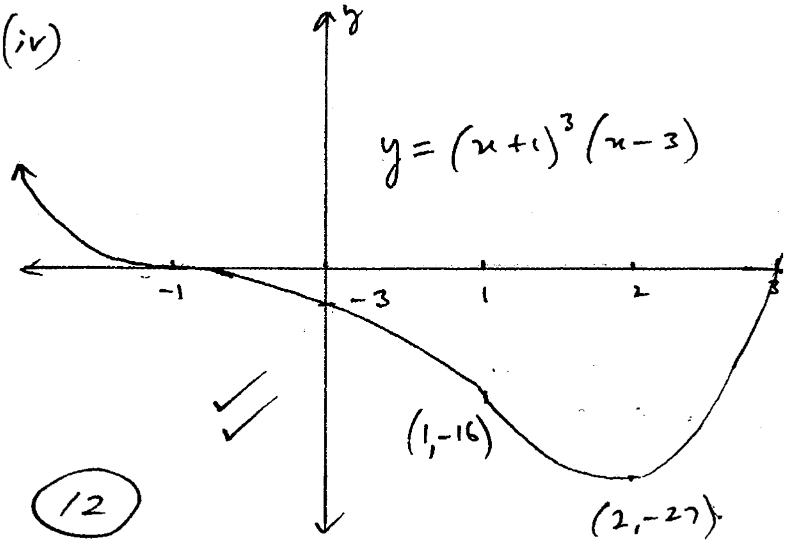
$= 4(x+1)(3x-3)$

$12(x+1)(x-1) = 0$

$x=-1$ or $x=1$

$(-1,0)$ $(1,-16)$ ✓ is the other pt. of inflexion.

(iv)



✓✓

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QUESTION 7

(a) $P = 50e^{kt}$

(i) when $t=4$, $P=100$.

$100 = 50e^{4k}$ ✓

$2 = e^{4k}$

$4k = \ln 2$

$k = \frac{1}{4}\ln 2$ ✓

(ii) put $t=10$.

$P = 50e^{\frac{t}{4}\ln 2}$

$P = 50e^{\frac{10}{4}\ln 2}$ ✓

$P = 50\left(e^{\ln 2}\right)^{\frac{5}{2}}$

$P = 50 \times 2^{5/2}$

$P = 282.8 \quad (200\sqrt{2})$

approximately 283 ✓ mosquitoes.

(iii) $\frac{dP}{dt} = kP$

$= \frac{1}{4}\ln 2 \times 282.8$

$\frac{dP}{dt} \doteqdot 49$ mosquitoes/day ✓ $(50\sqrt{2}\ln 2)$

(b) (i)

x	0	0.5	1	1.5	2
y	0	0.4055	0.6931	0.9163	1.0986

✓

$f(x) = \ln(x+1)$

(ii) $h = \frac{1}{2}$.

$\int_0^2 \ln(x+1)\,dx \doteqdot$

$\frac{1}{6}\left\{0 + 4\times 0.4055 + 0.6931\right\}$ ✓

$+ \frac{1}{6}\left\{0.6931 + 4\times 0.9163 + 1.0986\right\}$

$= 1.295$ ✓ (3 dec. place)

(iii) $\frac{d}{dx}\left((x+1)\ln(x+1) - x\right)$

$= 1\times\ln(x+1) + (x+1)\times\frac{1}{(x+1)} - 1$ ✓✓

$= \ln(x+1) + 1 - 1$

$= \ln(x+1)$ (as required).

(iv) $\int_0^2 \ln(x+1)\,dx = \left[(x+1)\ln(x+1) - x\right]_0^2$

$= (3\ln 3 - 2) - (\ln 1 - 0)$

$= 3\ln 3 - 2$ ✓

$\doteqdot 1.296$ (3 decimal places)

The approximation is not ✓ accurate to 3 decimal places.

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QUESTION 8

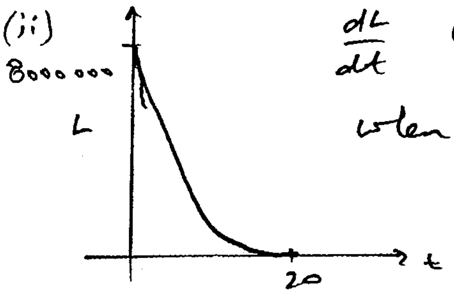
(a) $L = 1000(20-t)^3$

(i) $\frac{dL}{dt} = -3000(20-t)^2$ ✓

at $t = 10$. $\frac{dL}{dt} = -300000$

After 10 minutes the pool is being emptied at 300 000 L/minute. ✓

(ii)



$\frac{dL}{dt}$ is a max. when $t = 0$. ✓

The pool is being emptied at a maximum rate initially.

(b) (i) $(\sqrt{3}u - 1)(u - \sqrt{3})$

$= \sqrt{3}u^2 - 3u - u + \sqrt{3}$ ✓

$= \sqrt{3}u^2 - 4u + \sqrt{3}$

(ii) $\sqrt{3}\tan^2\theta - 4\tan\theta + \sqrt{3} = 0$

$(\sqrt{3}\tan\theta - 1)(\tan\theta - \sqrt{3}) = 0$

$\tan\theta = \frac{1}{\sqrt{3}}$ or ✓ $\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$ or $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$ ✓

(c)

$v = \frac{2}{1+t} - t$ m/s

(i) $\frac{dv}{dt} = \frac{-2}{(1+t)^2} - 1$

$a = \frac{-2}{(1+t)^2} - 1$ ✓

when $t = 0$.

$a = -2 - 1$

$a = -3\ m/s^2$ ✓

(ii) $\frac{dx}{dt} = \frac{2}{1+t} - t$

$x = 2\ln(1+t) - \frac{t^2}{2} + c$ ✓

when $t = 0$, $x = 0$, $\therefore c = 0$

$x = 2\ln(1+t) - \frac{t^2}{2}$ ✓

(iii) $v = 0$. $\frac{2}{1+t} - t = 0$

$2 - t(1+t) = 0$

$2 - t - t^2 = 0$

$t^2 + t - 2 = 0$

$(t-1)(t+2) = 0$ ✓

$t = 1$ or $t = -2$ (omit)

The particle is stationary at $t = 1$ s

(iv)

x	0	0.8863	0.1972
t	0	1	2

$x = 2\ln(1+t) - \frac{t^2}{2}$

distance travelled =

$0.8863 + (0.8863 - 0.1972)$

$= 1.5754$

$= 1.58$ m (3 sig. fig.) ✓

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QUESTION 9

(a)

$$\int_0^3 g(x)\,dx = \tfrac{1}{2}bh - \tfrac{1}{2}\pi r^2$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\pi$$

$$= \tfrac{1}{2}(1-\pi)$$

(b)

(i)

$$y = \frac{1}{x+2}$$

$$x+2 = \frac{1}{y}$$

$$x = \frac{1}{y} - 2$$

$$x^2 = \frac{1}{y^2} - \frac{4}{y} + 4$$

(ii)

$$V = \pi \int_{\frac{1}{2}}^{2} \frac{1}{y^2} - \frac{4}{y} + 4 \; dy$$

$$= \pi \left[-\frac{1}{y} - 4\ln y + 4y \right]_{\frac{1}{2}}^{2}$$

$$= \pi \left\{ \left(-\tfrac{1}{2} - 4\ln 2 + 8\right) - \left(-2 - 4\ln\tfrac{1}{2} + 2\right) \right\}$$

$$= \pi \left\{ -\tfrac{1}{2} - 4\ln 2 + 8 + 2 - 4\ln 2 - 2 \right\}$$

$$= \pi\left(7\tfrac{1}{2} - 8\ln 2\right)$$

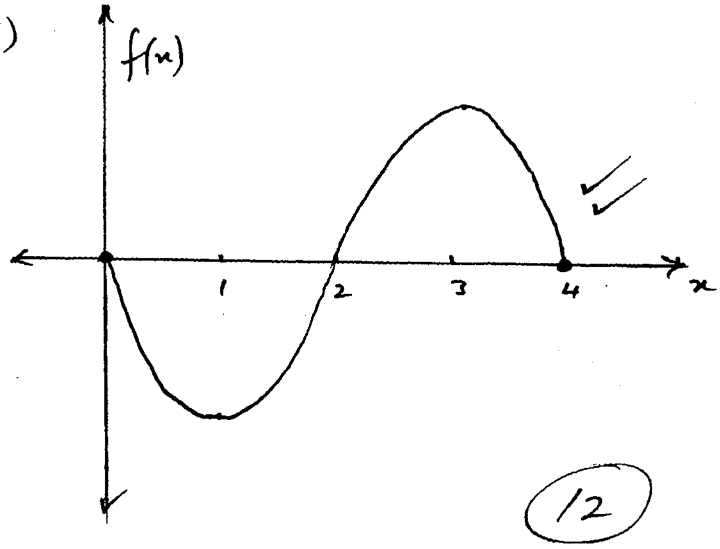
$= \frac{\pi}{2}(15 - 16\ln 2)$ cubic units

(c)

(i) $1 < x < 3$

(ii) $x < 2$

(iii)



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QUESTION 10

(a) $R = 1.005$

(i) $A_1 = 200\,000(1.005) - M$ ✓

$A_2 = \big(200\,000(1.005) - M\big)1.005 - M$

$A_2 = 200000(1.005)^2 - M(1.005) - M$ ✓

$A_3 = A_2 \times 1.005 - M$

$= 200000(1.005)^3 - M(1.005)^2 - M(1.005) - M$ ✓

$= 200\,000(1.005)^3 - M\{1 + (1.005) + (1.005)^2\}$

(as required).

(ii) $A_n = 200\,000(1.005)^n - M\{1 + (1.005) + (1.005)^2 + \dots + (1.005)^{n-1}\}$

Using $S_n = \dfrac{a(r^n - 1)}{r-1}$,

$A_n = 200\,000(1.005)^n - \dfrac{M(1.005^n - 1)}{0.005}$ ✓

(iii) Put $A_n = 0$. and $n = 300$.

$M = \dfrac{200\,000(1.005)^{300}(0.005)}{(1.005)^{300} - 1}$ ✓

$M = \$1288.60$. ✓

(b)(i) $V = \frac{1}{3}\pi r^2 h$

$3V = \pi r^2 h$

$9V^2 = \pi^2 r^4 h^2$

now $S = \pi r^2 + \pi r(r^2+h^2)^{1/2}$

$S - \pi r^2 = \pi r(r^2+h^2)^{1/2}$ ✓

square both sides.

$S^2 - 2S\pi r^2 + \pi^2 r^4 = \pi^2 r^2(r^2+h^2)$

$S^2 - 2S\pi r^2 + \pi^2 r^4 = \pi^2 r^4 + \pi^2 r^2 h^2$

$S^2 - 2S\pi r^2 = \pi^2 r^2 h^2$

$S^2 - 2S\pi r^2 = \dfrac{\pi^2 r^4 h^2}{r^2}$ ✓

$S^2 - 2S\pi r^2 = \dfrac{9V^2}{r^2}$

$\therefore\ 9V^2 = r^2(S^2 - 2S\pi r^2)$

as required.

(ii) $9V^2 = r^2S^2 - 2S\pi r^4$

$\dfrac{d}{dr}(9V^2) = 2rS^2 - 8S\pi r^3$ ✓

(iii) The maximum value of $9V^2$ will give the radius required for the maximum value of V.

$2rS^2 - 8S\pi r^3 = 0$

$2rS(S - 4\pi r^2) = 0$

$S = 0$ (omit) or $S = 4\pi r^2$ ✓

Now $S = \pi r^2 + \pi r(r^2+h^2)^{1/2}$

$4\pi r^2 = \pi r^2 + \pi r(r^2+h^2)^{1/2}$

$3\pi r^2 = \pi r(r^2+h^2)^{1/2}$

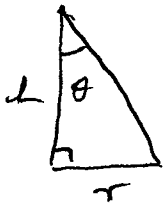
$9\pi^2 r^4 = \pi^2 r^2(r^2+h^2)$

$9\pi^2 r^4 = \pi^2 r^4 + \pi^2 r^2 h^2$

$9r^2 = r^2 + h^2$ ✓

$8r^2 = h^2$

$\dfrac{r^2}{h^2} = \dfrac{1}{8}$



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$\dfrac{r}{h} = \dfrac{1}{2\sqrt{2}}$

$\tan\theta = \dfrac{1}{2\sqrt{2}}$

$\theta = 19°28'$. ✓

$\dfrac{d^2}{dr^2}(9V^2) = 2S^2 - 24\pi r^2 S$

(when $S = 4\pi r^2$)

$= -64\pi r^2 < 0$

$\therefore$ a max. exists at $S = 4\pi r^2$