CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2015 Trial Examination

FORM VI

MATHEMATICS 2 UNIT

Friday 31st July 2015

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 92 boys

Examiner

DS

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

1

Which expression represents the product of the roots of the equation $ax^2 + bx + c = 0$?

(A) $-\frac{c}{a}$

(B) $\frac{c}{a}$

(C) $-\frac{b}{a}$

(D) $\frac{b}{a}$

QUESTION TWO

1

What is the range of the function $y = x^2 - 4$?

(A) $y \geq -4$

(B) $y \geq 0$

(C) $-2 \leq y \leq 2$

(D) all real values of y

QUESTION THREE

1

Which of the following is the derivative of x^{n+1} with respect to x?

(A) nx^{n+1}

(B) $\dfrac{x^{n+1}}{n+1}$

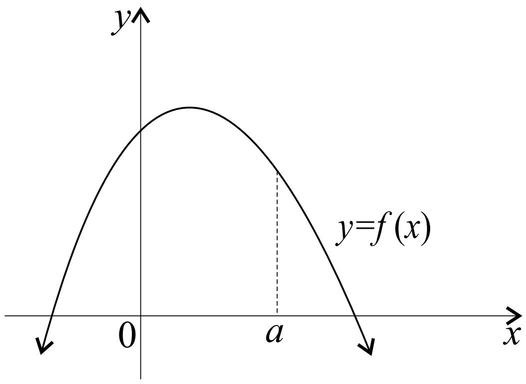
(C) $(n+1)x^n$

(D) $(n+1)x^{n-1}$

Exam continues next page ...

QUESTION FOUR

1



The diagram above shows the graph of the function $y = f(x)$.
Which of the following statements is TRUE?

(A) $f(a) < 0$ and $f'(a) > 0$

(B) $f(a) > 0$ and $f'(a) < 0$

(C) $f(a) > 0$ and $f'(a) > 0$

(D) $f(a) < 0$ and $f'(a) < 0$

QUESTION FIVE

1

What are the period and amplitude, respectively, of the wave $y = \frac{1}{4}\sin 4\pi x$?

(A) $\frac{\pi}{2}$ and $\frac{1}{4}$

(B) $\frac{1}{4}$ and $\frac{\pi}{2}$

(C) $\frac{1}{2}$ and $\frac{1}{4}$

(D) $\frac{1}{4}$ and $\frac{1}{2}$

QUESTION SIX

1

Which expression is equivalent to $\log 2a - \log a$?

(A) $\log 2$

(B) $\dfrac{\log 2a}{\log a}$

(C) $\log a$

(D) $2\log a$

QUESTION SEVEN

1

Simpson's rule is used with three function values to approximate $\int_0^1 9^x \, dx$.
What value is obtained?

(A) $\frac{25}{12}$

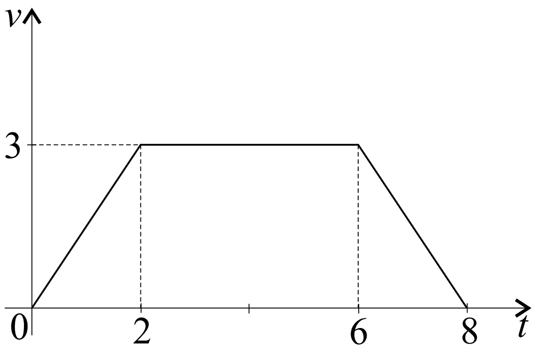
(B) $\frac{22}{3}$

(C) $\frac{5}{2}$

(D) $\frac{11}{3}$

QUESTION EIGHT

1



The diagram above shows the velocity–time graph of a particle moving in a straight line.
Which of the following statements is TRUE?

(A) The velocity is constant throughout the motion.

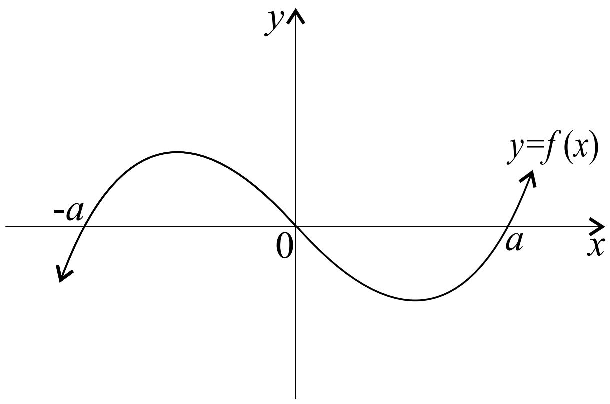
(B) The particle does not return to its starting point.

(C) The particle cannot move because its initial velocity is zero.

(D) The distance travelled by the particle cannot be determined.

QUESTION NINE

1



The diagram above shows the graph of the ODD function $y = f(x)$.
Which of the following statements is FALSE?

(A) $\displaystyle\int_{-a}^{0} f(x)\,dx > 0$

(B) $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

(C) $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(D) $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

QUESTION TEN

1

If x is a negative number, which of the following statements is FALSE?

(A) $|x| = -x$

(B) $2x < x$

(C) $-x > x$

(D) $\sqrt{x^2} = x$

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Write $\dfrac{\sqrt{7}}{\sqrt{7}+2}$ with a rational denominator. **2**

(b) Simplify:

(i) $\dfrac{\sin\theta}{\cos\theta}$ **1**

(ii) $1-\sin^2\theta$ **1**

(c) Find the equation of the line with x-intercept 4 and y-intercept 2. **2**

(d) Calculate the perpendicular distance from the point $(-5,-7)$ to the line $4x-3y-6=0$. **2**

(e) Differentiate:

(i) e^{5+10x} **1**

(ii) $(3x-2)^3$ **1**

(iii) $x\cos x$ **2**

(f) Find:

(i) $\displaystyle\int \frac{1}{x^2}\,dx$ **1**

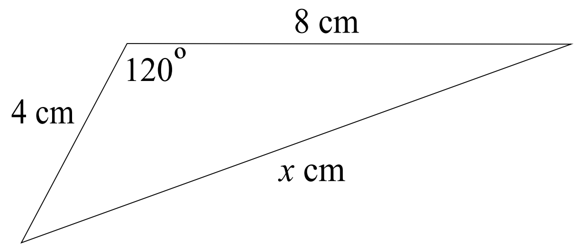
(ii) $\displaystyle\int \cos 7x\,dx$ **1**

(iii) $\displaystyle\int \frac{3}{3x+1}\,dx$ **1**

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) **2**



Use the cosine rule in the diagram above to find x as a surd in simplest form.

(b) (i) Factorise $2x^2 - 3x - 2$. **1**

(ii) Sketch the parabola $y = 2x^2 - 3x - 2$ showing its x intercepts. **1**

(iii) Solve $2x^2 - 3x - 2 < 0$. **1**

(c) Consider the finite sequence 24, 37, 50, …, 7070.

(i) Find the 79th term. **1**

(ii) Determine the number of terms in the sequence. **2**

(d) Find the gradient of the normal to the curve $y = 4\sqrt{x}$ at the point $(9, 12)$. **3**

(e) The parabola $\mathcal{P}$ has equation $y^2 = 8(x + 2)$.

(i) Write down the coordinates of the vertex of $\mathcal{P}$. **1**

(ii) Find the coordinates of the focus of $\mathcal{P}$. **1**

(iii) Write down the equation of the directrix of $\mathcal{P}$. **1**

(iv) Sketch the parabola $\mathcal{P}$ showing the features found above, as well as any x or y intercepts. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) A fuel tank ruptured and as a consequence the fuel began to leak out. The volume V litres of fuel remaining in the tank t minutes after the rupture was given by

$$V = 2000 - 100t - 10t^2.$$

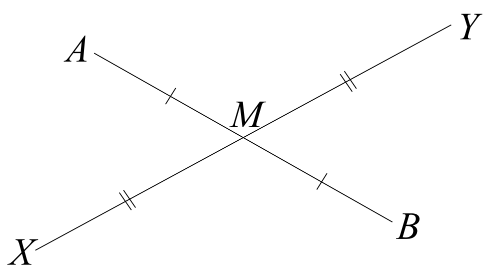
(i) How much fuel was in the tank when the rupture occurred? **1**

(ii) What percentage of the fuel remained in the tank after 4 minutes? **1**

(iii) At what rate was the fuel leaking out after 4 minutes? **1**

(iv) How long did it take for the tank to empty? **2**

(b)



The diagram above shows two intervals AB and XY that bisect each other at M.

(i) Prove that the triangles AMX and BMY are congruent. **2**

(ii) Hence prove that AX and BY are parallel. **2**

(c) At the start of the year 2000 there were 40 foxes on McLean's cattle station. Since then the number of foxes F has increased exponentially according to the equation $F = Ae^{kt}$, where A and k are constants and t is the number of years since the start of 2000. The fox population had reached 120 at the start of 2008.

(i) What is the value of A? **1**

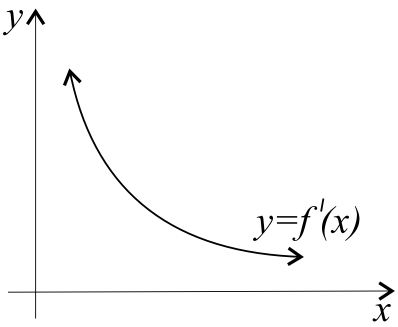
(ii) Find the exact value of k. **2**

(iii) What was the fox population at the start of 2015? **1**

(iv) During which year will the fox population reach 1000? (Assume that the population continues to grow exponentially according to the given equation.) **2**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)

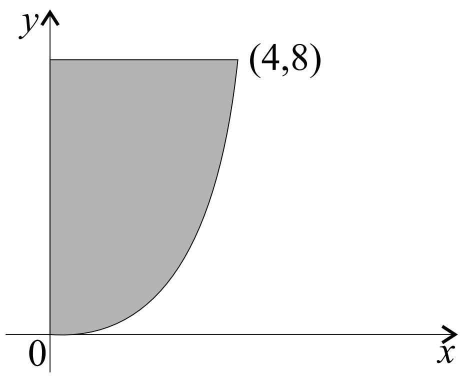


The diagram above shows the graph of the gradient function $y = f'(x)$ of the curve $y = f(x)$.

(i) State whether the curve $y = f(x)$ is increasing or decreasing throughout its domain. **1**

(ii) State whether the curve $y = f(x)$ is concave up or concave down throughout its domain. **1**

(b)



The shaded region in the diagram above is bounded by the curve $y = \frac{1}{8}x^3$, the y-axis and the line $y = 8$.

(i) Make x the subject of the equation $y = \frac{1}{8}x^3$. **1**

(ii) Hence, or otherwise, find the area of the region. **2**

(iii) Find the volume of the solid formed when the region is rotated about the y-axis. **2**

(c) A special commemorative tree in Canberra was 2 metres tall when it was planted many years ago. It grew 90 cm in the first year, and since then each year's growth has been 70% of the previous year's growth.

(i) By how much, correct to the nearest centimetre, did the tree grow in its fifth year? **2**

(ii) What is the limiting height of the tree? **2**

(d) The line $y = mx - 2$ is a tangent to the circle $x^2 + y^2 = 1$.

(i) By solving simultaneously, show that $(1 + m^2)x^2 - 4mx + 3 = 0$. **2**

(ii) Use the discriminant of the quadratic equation in part (i) to find the two possible values of m. **2**

Exam continues overleaf ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the curve with equation $y = \frac{1}{3}x^3 - 3x^2 + 11x - 9$.

(i) Show that the curve has no stationary points. **2**

(ii) Show that there is a point of inflexion at $(3, 6)$. **1**

(iii) How many x intercepts does the curve have? Justify your answer. **1**

(b) A car is travelling along a straight horizontal road in outback Australia, with velocity v km/h given by

$$v = 100 + 20e^{-0{\cdot}2t}.$$

Suppose that the car has displacement $x = 0$ when $t = 0$.

(i) Write down a formula for the acceleration of the car after t hours, and hence state whether the car is speeding up or slowing down. **2**

(ii) Find a formula for the displacement of the car after t hours. **2**

(iii) How far, to the nearest kilometre, does the car travel in the third hour? **1**

(iv) What is the limiting velocity of the car? **1**

(v) Sketch the velocity-time graph. **1**

(c) Michael is about to retire and has arranged for his superannuation lump sum of $\$P$ to be deposited into an account which earns 3% per annum interest, with interest paid monthly on the balance. He intends to withdraw \$6400 at the end of each month to live off.

Suppose that $\$A_n$ is the balance immediately after the nth withdrawal.

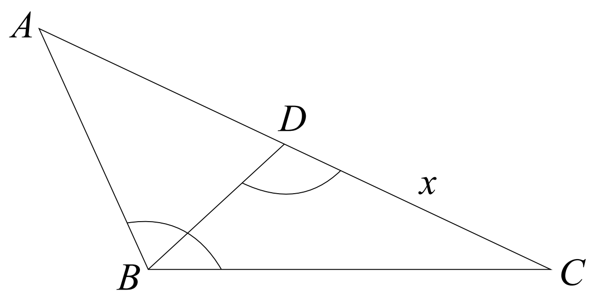
(i) Use the fact that $A_1 = P(1{\cdot}0025) - 6400$ to determine an expression for A_2. **1**

(ii) Hence show that $A_3 = P(1{\cdot}0025)^3 - 6400(1{\cdot}0025)^2 - 6400(1{\cdot}0025) - 6400$. **1**

(iii) Find the value of P to the nearest integer if the balance will be zero after the 180th withdrawal. **2**

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows a triangle ABC. The point D is chosen on AC so that $\angle BDC = \angle ABC$.

(i) Prove that the triangles BDC and ABC are similar. **1**

(ii) Given that $DC = x$ and $3DC = 2BC$, find AD in terms of x. **3**

(b) An open cylindrical water tank has base radius x metres and height h metres. Each square metre of the base costs a dollars to manufacture and each square metre of the curved surface costs b dollars, where a and b are constants. The combined cost of the base and curved surface is c dollars.

(i) Find c in terms of a, b, x and h. (Note that the curved surface has area $2\pi xh$.) **1**

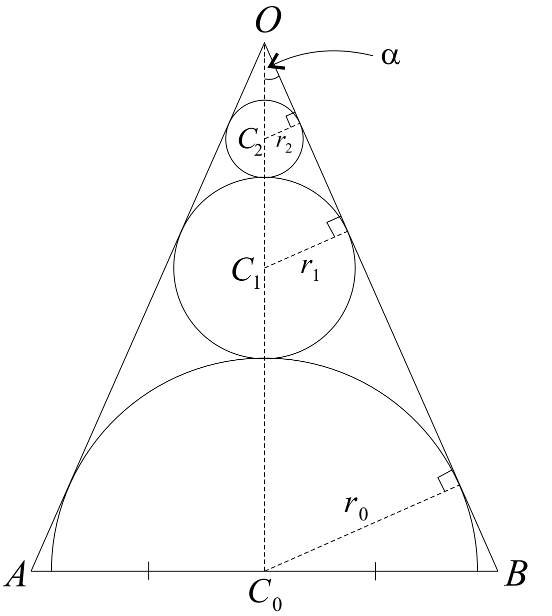
(ii) Show that the volume V of the tank in cubic metres is given by **2**

$$V = \tfrac{x}{2b}\left(c - \pi a x^2\right).$$

(iii) If x can vary, prove that V is maximised when the cost of the base is $\frac{c}{3}$ dollars. **4**

QUESTION SIXTEEN (Continued)

(c)



In the diagram above $\triangle OAB$ is isosceles, with $OA = OB$ and $\angle AOB = 2\alpha$. The semicircle of radius r_0 has centre C_0, the midpoint of AB, and the sides OA and OB are tangents. Infinitely many circles are stacked on top of the semicircle so that each circle is tangent to the circles immediately above and below it, and also tangent to OA and OB. The circles have radii r_1, r_2, r_3, ... and centres C_1, C_2, C_3, ... The semicircle and the first two circles are shown in the diagram.

(i) Write down an infinite series for OC_0 in terms of r_0, r_1, r_2, ... and write down a similar series for OC_1. **1**

(ii) Prove that $\dfrac{r_1}{r_0} = \dfrac{1 - \sin\alpha}{1 + \sin\alpha}$. **3**

End of Section II

END OF EXAMINATION

2U Trial Solutions 2015 (Total 100)

(1) $\frac{r}{p}$ B

(2) $y \geq -4$ A

(3) $(n+1)x^n$ C

(4) $f(a) > 0$ and $f'(a) < 0$ B

(5) $\frac{1}{2}$ and $\frac{1}{4}$ C

(6) $\log 2$ A

(7) $\frac{11}{3}$ D

(8) B

(9) $\int_{-a}^{a} f(x)\,dx \neq 2\int_0^a f(x)\,dx$ C

(10) $\sqrt{x^2} \neq x$ D

ONE MARK EACH

(11)(a) $\frac{\sqrt{7}}{\sqrt{7}+2} \times \frac{\sqrt{7}-2}{\sqrt{7}-2}$ ✓

$= \frac{7-2\sqrt{7}}{3}$ ✓

(b)(i) $\frac{\sin\theta}{\cos\theta} = \tan\theta$ ✓

(ii) $1 - \sin^2\theta = \cos^2\theta$ ✓

(c) Line through $(4,0)$ and $(0,2)$.

$m = \frac{2-0}{0-4}$

$= -\frac{1}{2}$ ✓

The line has equation

$y = -\frac{1}{2}x + 2$ ✓

(or $x + 2y - 4 = 0$)

(d) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$= \frac{|-20+21-6|}{5}$ ✓

$= 1$ unit ✓

(e)(i) $\frac{d}{dx}\left(e^{5+10x}\right) = 10e^{5+10x}$ ✓

(ii) $\frac{d}{dx}(3x-2)^3 = 9(3x-2)^2$ ✓

(iii) By the product rule

$\frac{d}{dx}(x\cos x) = \cos x - x\sin x$ ✓ ✓

(f)(i) $\int x^{-2}\,dx = \frac{x^{-1}}{-1} + c$ ✓

$= -\frac{1}{x} + c$

(ii) $\int \cos 7x\,dx = \frac{1}{7}\sin 7x + c$ ✓

(iii) $\int \frac{3}{3x+1}\,dx = \log_e(3x+1) + c$ ✓

(No penalty for omission of 'c'.)

(12)(a) $x^2 = 4^2 + 8^2 - 2 \times 4 \times 8 \times \cos 120°$

$= 64 + 16 - 64 \times -\frac{1}{2}$

$= 80 + 32$

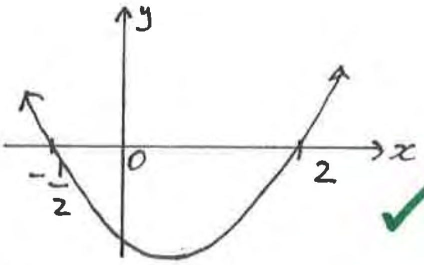
$= 112$

so $x = \sqrt{112}$

$= 4\sqrt{7}$

(b)(i) $2x^2 - 3x - 2 = (2x+1)(x-2)$

(ii)



(iii) $-\frac{1}{2} < x < 2$

(c) The sequence is arithmetic with $a = 24$ and $d = 13$.

(i) $T_{79} = a + 78d$

$= 24 + 78(13)$

$= 1038$

(ii) Let $T_n = 7070$ and solve for n.

$24 + (n-1)(13) = 7070$

$13n + 11 = 7070$

$n = 543$

So there are 543 terms.

(d) $y = 4x^{\frac{1}{2}}$

so $y' = 2x^{-\frac{1}{2}}$

$= \frac{2}{\sqrt{x}}$

when $x = 9$, $y' = \frac{2}{3}$

so the normal has gradient $-\frac{3}{2}$.

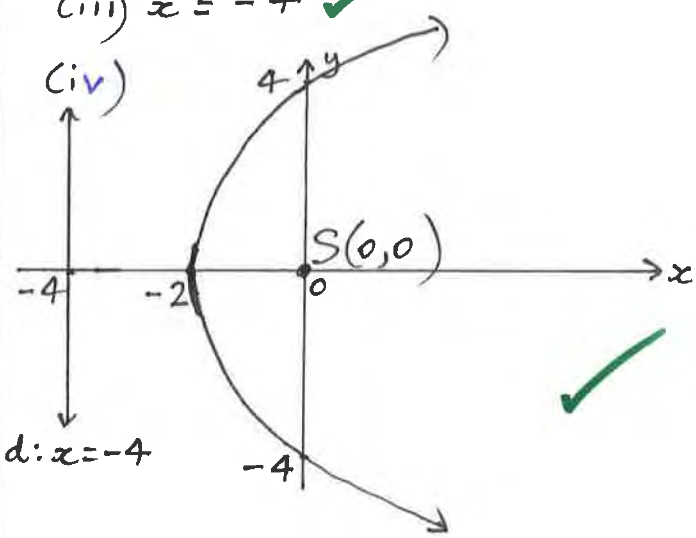
(e)(i) The parabola is concave right with vertex $(-2, 0)$.

(ii) Also $4a = 8$ so $a = 2$.

So the focus is $(0,0)$.

(iii) $x = -4$

(iv)



when $x = 0$,

$y^2 = 16$

$y = \pm 4$ (y-intercepts)

(13)(a)(i) when $t=0$,

$V = 2000\ L$

(ii) when $t=4$,

$V = 2000 - 400 - 160$

$= 1440$

so percentage remaining is

$\frac{1440}{2000} \times \frac{100}{1}\% = 72\%$.

(iii) $\frac{dV}{dt} = -100 - 20t$

when $t=4$,

$\frac{dV}{dt} = -100 - 80$

$= -180$

So the fuel was leaking out at 180 L/min.

(iv) when $V=0$,

$0 = 2000 - 100t - 10t^2$

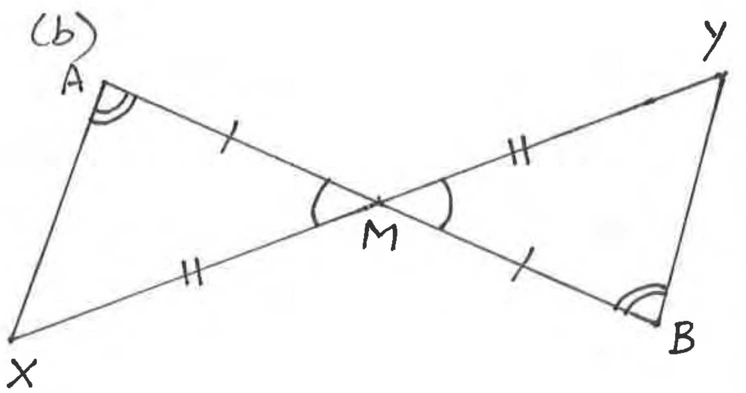
$t^2 + 10t - 200 = 0$

$(t+20)(t-10) = 0$

$t \geq 0$ so $t = 10$

So the tank took 10 min to empty.

(b)



(i) In Δs AMX and BMY:

AM = BM (given)

$\angle AMX = \angle BMY$ (vertically opposite)

XM = YM (given)

so $\Delta AMX \equiv \Delta BMY$ (S.A.S.)

(ii)

$\angle XAM = \angle YBM$ (matching angles of congruent triangles)

so $AX \parallel YB$ (alternate angles are equal)

(c)(i) when $t=0$, $F=40$

so $40 = Ae^0$

$40 = A$

$A = 40$

(ii) when $t=8$, $F=120$

so $120 = 40e^{8k}$

$e^{8k} = 3$

$8k = \log_e 3$

$k = \frac{1}{8}\ln 3$

(iii) when $t=15$,

$F = 40e^{15k}$

$= 313.806\ldots$

$\doteqdot 314$

(iv) when $F=1000$,

$1000 = 40e^{kt}$

$e^{kt} = 25$

$kt = \log_e 25$

$t = \frac{\ln 25}{k}$

$= 23.439\ldots$

so the fox population will reach 1000 during 2023.

(14)(a) (i) $f'(x) > 0$ so $f(x)$ is increasing.

(ii) $f''(x) < 0$ so the curve is concave down.

(b)(i)
$$y = \frac{1}{8}x^3$$
$$x^3 = 8y$$
$$x = 2y^{\frac{1}{3}}$$

(ii)
$$\text{Area} = \int_0^8 2y^{\frac{1}{3}}\,dy$$
$$= 2\left[\frac{3}{4}y^{\frac{4}{3}}\right]_0^8$$
$$= \frac{3}{2}(16-0)$$
$$= 24 \text{ units}^2$$

OR
$$\text{Area} = 4\times 8 - \int_0^4 \frac{1}{8}x^3\,dx$$
$$= 32 - \frac{1}{8}\left[\frac{x^4}{4}\right]_0^4$$
$$= 32 - \frac{1}{8}(64)$$
$$= 32 - 8$$
$$= 24 \text{ units}^2$$

(iii)
$$\text{Volume} = \pi\int_0^8 x^2\,dy$$
$$= \pi\int_0^8 4y^{\frac{2}{3}}\,dy$$
$$= 4\pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8$$
$$= \frac{12\pi}{5}(32-0)$$
$$= \frac{384\pi}{5} \text{ units}^3$$

(c)(i) The annual growths form a GP with $a = 90$ and $r = 0.7$.
$$T_5 = ar^4$$
$$= 90(0.7)^4$$
$$= 21.609$$
$$\doteqdot 22\text{ cm}$$

(ii)
$$S_\infty = \frac{a}{1-r}$$
$$= \frac{90}{0.3}$$
$$= 300\text{ cm}$$
$$= 3\text{ m}$$
So the limiting height is 5 m.

(d)(i)
$$y = mx - 2 \quad (1)$$
$$x^2 + y^2 = 1 \quad (2)$$

Substitute (1) into (2):
$$x^2 + (mx-2)^2 = 1$$
$$x^2 + m^2x^2 - 4mx + 4 = 1$$
$$(1+m^2)x^2 - 4mx + 3 = 0$$

(ii)
$$\Delta = b^2 - 4ac$$
$$= (-4m)^2 - 4(3)(1+m^2)$$
$$= 16m^2 - 12 - 12m^2$$
$$= 4m^2 - 12$$
$$= 4(m^2-3)$$

Since the line is a tangent there is only one point of intersection.

So $\Delta = 0$.

So
$$m^2 - 3 = 0$$
$$m^2 = 3$$
$$m = \pm\sqrt{3}$$

(15)(a)(i) $y = \frac{1}{3}x^3 - 3x^2 + 11x - 9$

so $y' = x^2 - 6x + 11$

$= (x^2 - 6x + 9) + 2$

$= (x-3)^2 + 2$

≥ 2 for all x

so y' is always positive,

so y' is never zero,

so there are no stationary points.

(ii) $y'' = 2x - 6$

So $y'' = 0$ at $x = 3$.

We must show that the concavity changes at $x = 3$.

x	2	3	4
y''	-2	0	$+2$
CONCAVITY	down		up

When $x = 3$,
$y = 9 - 27 + 33 - 9$
$= 6$.

So $(3, 6)$ is a point of inflexion.

(iii) The curve has ONE x-intercept because it is strictly increasing for all values of x.

(b)(i) $v = \dot{x} = 100 + 20e^{-0.2t}$

so $\ddot{x} = -4e^{-0.2t}$

which is negative for all $t \geq 0$.

But v is positive for all $t \geq 0$

so the car is slowing down.

(ii) $x = \int (100 + 20e^{-0.2t})\,dt$

$= 100t - 100e^{-0.2t} + c$

When $t = 0$, $x = 0$

so $c = 100$

so $x = 100t - 100e^{-0.2t} + 100$

$= 100(t - e^{-0.2t} + 1)$

(iii) When $t = 2$, $x = 200 - 100e^{-0.4} + 100$

$= 232.967\ldots$

When $t = 3$, $x = 300 - 100e^{-0.6} + 100$

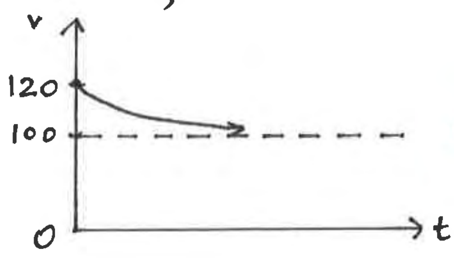
$= 345.118\ldots$

So the car has travelled 112 km (to the nearest km).

(iv) As $t \to \infty$, $e^{-0.2t} \to 0$,

so $v \to 100$ km/h.

(v) When $t = 0$, $v = 120$.



(c) 3% p.a. $= 0.25\%$ per month

so $r = 0.0025$.

(i) $A_2 = A_1(1.0025) - 6400$

$= P(1.0025)^2 - 6400(1.0025) - 6400$

(ii) $A_3 = A_2(1.0025) - 6400$

$= P(1.0025)^3 - 6400(1.0025)^2 - 6400(1.0025) - 6400$

(iii)

$A_n = P(1.0025)^n - 6400(1 + 1.0025 + \ldots + 1.0025^{n-1})$

$A_{180} = 0$

so

$$P(1.0025)^{180} = \frac{6400(1.0025^{180} - 1)}{0.0025}$$

$$P = \frac{6400(1.0025^{180} - 1)}{(1.0025)^{180}(0.0025)}$$

$= \$926\,755$ (to the nearest dollar)

(16)(a)(i) In Δs BDC and ABC:

$\angle BDC = \angle ABC$ (given)

$\angle BCD = \angle ACB$ (common)

so $\Delta BDC \parallel \Delta ABC$ (A.A.)

(ii) $\frac{BC}{AC} = \frac{DC}{BC}$ (matching sides in the same ratio)

so $BC^2 = AC \times DC$

so $\left(\frac{3x}{2}\right)^2 = AC \times x$

so $AC = \frac{9x}{4}$

so $AD = \frac{9x}{4} - x$

$= \frac{5x}{4}$.

(b)(i) $c = \pi x^2 a + 2\pi x h b$

(ii) $V = \pi x^2 h$

From (i) $h = \frac{c - \pi x^2 a}{2\pi x b}$

so $V = \frac{\pi x^2}{2\pi x b}(c - \pi x^2 a)$

$= \frac{x}{2b}(c - \pi x^2 a)$.

(iii) $V = \frac{cx}{2b} - \frac{\pi a x^3}{2b}$

so $V' = \frac{c}{2b} - \frac{3\pi a x^2}{2b}$

Let $V' = 0$.

Then $x^2 = \frac{c}{2b} \cdot \frac{2b}{3\pi a}$

$= \frac{c}{3\pi a}$.

Since $x > 0$, the only stationary point under consideration is at

$x = \left(\frac{c}{3\pi a}\right)^{\frac{1}{2}}$.

$V'' = -\frac{3\pi a x}{b} < 0$ when $x > 0$.

So V is maximised when $x = \left(\frac{c}{3\pi a}\right)^{\frac{1}{2}}$.

The cost of the base is $\pi a x^2$.

So when $x^2 = \frac{c}{3\pi a}$, the cost is

$\pi a \cdot \frac{c}{3\pi a} = \frac{c}{3}$ dollars.

(c)(i) $OC_0 = r_0 + 2r_1 + 2r_2 + \ldots$

and $OC_1 = r_1 + 2r_2 + 2r_3 + \ldots$

(ii) It follows from (i) that

$OC_0 - OC_1 = (r_0 + 2r_1) - r_1$

$= r_0 + r_1$. (A)

From the diagram,

$\sin\alpha = \frac{r_0}{OC_0} = \frac{r_1}{OC_1}$,

so $OC_0 = \frac{r_0}{\sin\alpha}$

and $OC_1 = \frac{r_1}{\sin\alpha}$ (B)

Substituting from (B) into (A):

$\frac{r_0}{\sin\alpha} - \frac{r_1}{\sin\alpha} = r_0 + r_1$

$r_0 - r_1 = r_0\sin\alpha + r_1\sin\alpha$

$r_0 - r_0\sin\alpha = r_1 + r_1\sin\alpha$

$r_0(1 - \sin\alpha) = r_1(1 + \sin\alpha)$

so $\frac{r_1}{r_0} = \frac{1 - \sin\alpha}{1 + \sin\alpha}$.