Name:______________________ Maths Class:__________________

SYDNEY TECHNICAL HIGH SCHOOL

2U MATHS

TRIAL HIGHER SCHOOL CERTIFICATE

2005

Time allowed: 3 hours plus 5 mins reading time

Instructions:

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination this examination paper must be attached to the front of your answers.
- All questions are of equal value and may be attempted.
- All necessary working must be shown. Marks will be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary
- Non programmable calculators may be used.

(For markers use only)

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Total
										120

Question 1 (12 marks)

a) Evaluate $\sin^2 1.7 + \cos^2 1.7$ (1)

b) Factorise fully $x^4 - 16$ (2)

c) Find rational numbers a and b such that $4a + b\sqrt{7} = 36 + \sqrt{63}$ (2)

d) For $3^x = 7$, find x correct to two decimal places (2)

e) Solve $\frac{5}{x} = \frac{4}{x-3}$ (2)

f) i) Sketch $y = x^2 - x - 2$

ii) Hence solve $x^2 - x - 2 < 0$ (3)

Question 2 (12 marks) (Start a new page)

A, B and C are the points (0,4) , (1,-5) and (6,-2) respectively.

a) Sketch the triangle ABC on a number plane (1)

b) Show that the length of AC is $6\sqrt{2}$ units. (1)

c) Show that the line AC has equation $x + y - 4 = 0$ (2)

d) Show that the perpendicular distance from B to AC is $4\sqrt{2}$ units (2)

e) Hence find the area of triangle ABC (1)

f) Find the co-ordinates of D if C is the midpoint of the interval BD (2)

g) Find the length of the interval AB (1)

h) Hence or otherwise find to the nearest degree the size of $\angle BAC$ (2)

Question 3 (12 marks) Start a new page

a) Consider the function $y = x^3 + x$

i) Find $\frac{dy}{dx}$ (1)

ii) Hence explain why the function is increasing for all values of x (1)

b) Differentiate with respect to x

i) $\dfrac{3}{x^2}$ (1)

ii) $\dfrac{\cos 3x}{x}$ (2)

iii) $\dfrac{3}{x^2+1}$ (2)

c) Find $\int \frac{1}{\sqrt{2x+1}}\,dx$ (3)

d) Solve $\log_5 3 = 2\log_5 6 - \log_5 x$ (2)

Question 4 (12 marks) Start a new page

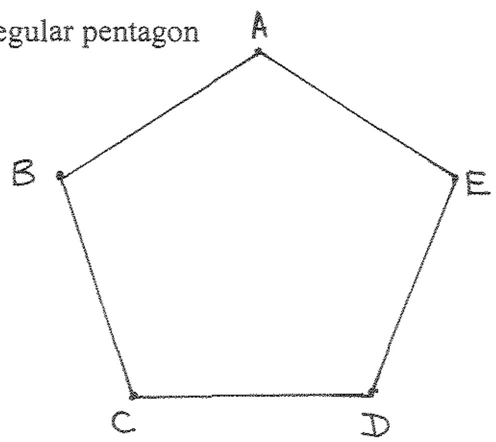
a) An aircraft flies 300 nautical miles from its base B on a bearing of $050°$ to point Q. It then flies 225 nautical miles due north to point P.

i) Draw a diagram to show this information and find angle $\angle BQP$. (2)

ii) Calculate the distance from the base B to point P to the nearest nautical mile. (2)

iii) Find the direction the plane must now fly to return to its base (2)

b) ABCDE is a regular pentagon



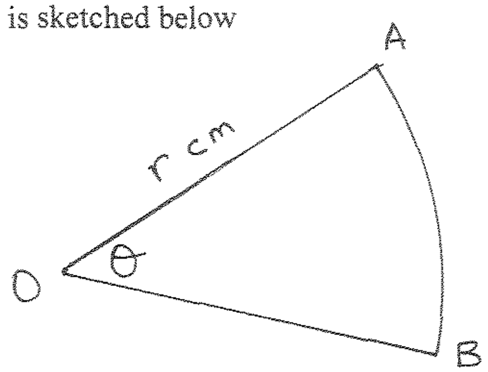
i) Find the size of $\angle ABC$ (1)

ii) Prove $\Delta ABC \equiv \Delta AED$ (3)

iii) Find $\angle CAD$ giving reasons for your answer (2)

Question 5 (12 marks) (Start a new page)

a) The sector OAB is sketched below



i) Write a formula for the arc length AB in terms of r and θ (1)

ii) Write a formula for the area of the sector OAB in terms of r and θ (1)

iii) If the sector has area π cm^2 and the arc length AB is $\frac{\pi}{4}$ cm find r and θ (2)

b) Solve $2\sin 2\theta - 1 = 0$ for $0° \le \theta \le 360°$ (4)

c) i) Sketch $y = \log_{10} x$ (1)

ii) Complete the table below for $y = \log_{10} x$

(leave answers to 3 decimal places) (1)

x	1	1.5	2	2.5	3
y					

iii) Use Simpsons Rule with five function values from part ii) to estimate

$$\int_1^3 \log_{10} x \, dx$$

Give your answer to 2 decimal places (2)

Question 6 (12 marks) (Start a new page)

a) Let α *and* β be the roots of the equation $2x^2 - 3x - 4 = 0$. find the values of

i) $\alpha + \beta$ (1)

ii) $\alpha\beta$ (1)

iii) $(\alpha + 1)(\beta + 1)$ (1)

iv) $\alpha^2 + \beta^2$ (1)

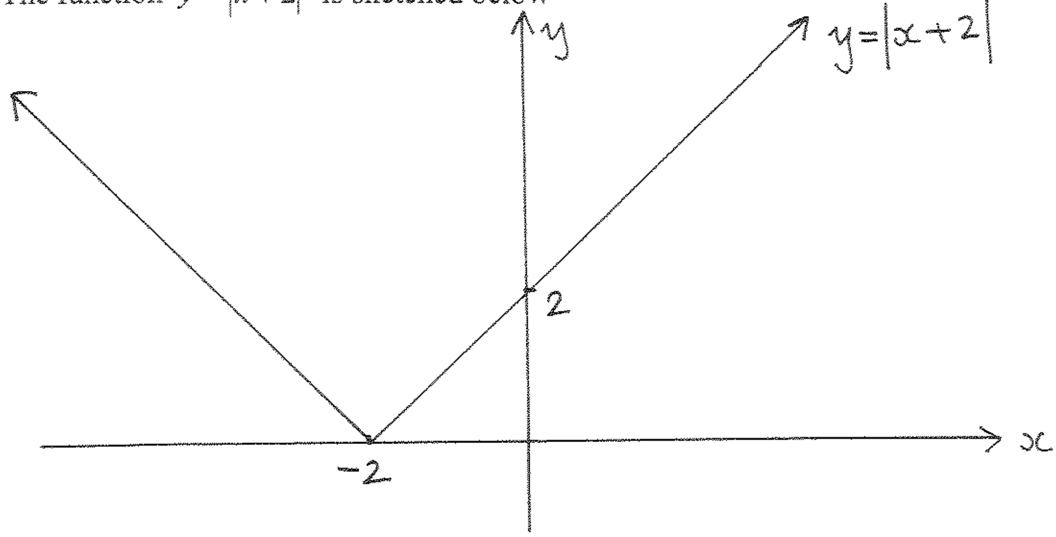
b) A parabola has equation $y^2 - 6y - 3 = 12x$

i) Write the parabola in the form $(y-k)^2 = 4a(x-h)$ (1)

ii) Find the co-ordinates of the vertex. (1)

iii) Sketch the parabola showing the co-ordinates of the focus. (2)

c) The function $y = |x+2|$ is sketched below



i) Copy the graph onto your answer paper and sketch $y = 2x + 6$ on the same number plane (1)

ii) Show the co-ordinates of the point of intersection are $(-2\frac{2}{3}, \frac{2}{3})$ (2)

iii) Hence or otherwise solve $|x+2| > 2x + 6$ (1)

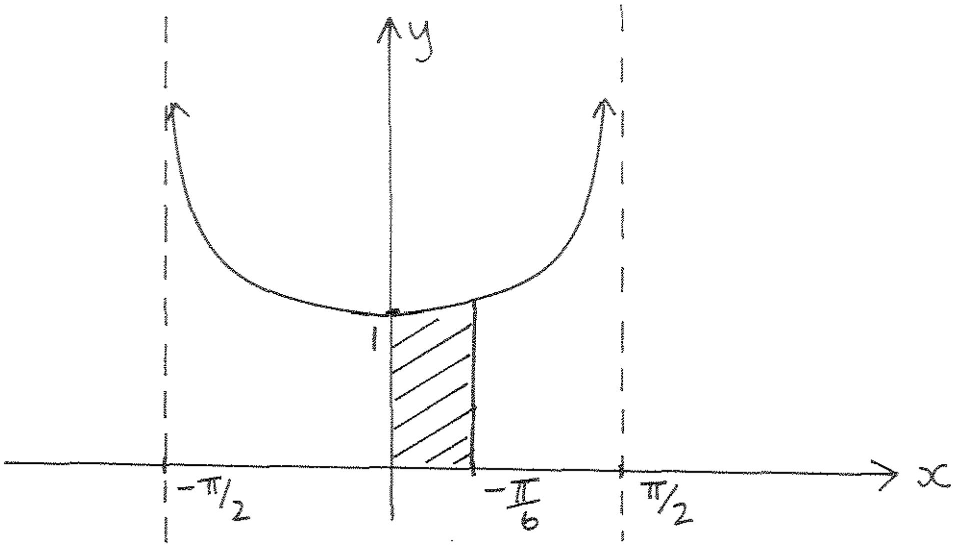
Question 7 **(12 marks)** **(Start a new page)**

a) i) Sketch y=3cos2x for $0 \leq x \leq 2\pi$ (2)

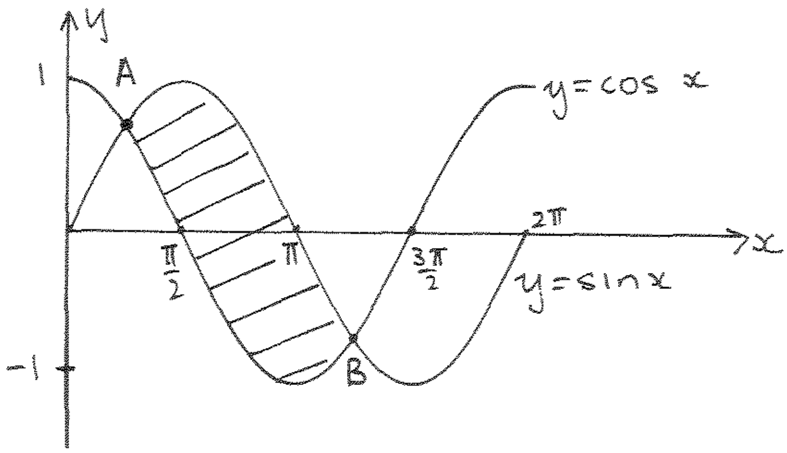
ii) From the sketch, state how many points of inflexion the curve has in the domain $0 \leq x \leq 2\pi$ (1)

b) The diagram below shows the region between the curve $y = \sec x$ and the x axis from $x = 0$ to $x = \dfrac{\pi}{6}$

Find the volume generated when this region is rotated around the x axis (3)



c) The curves $y = \sin x$ and $y = \cos x$ are sketched below for $0 \leq x \leq 2\pi$



i) Show the x co-ordinates of the points of intersection A and B are $\dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$ respectively. (2)

ii) Hence find the shaded area. (4)

Question 8 (12 marks) (Start a new page)

a) The function $y = x^3 - 3x^2 - 9x + 1$ is defined in the domain $-4 \leq x \leq 5$.

i) Find the co-ordinates of any turning points and determine their nature. (3)

ii) Find the co-ordinates of any points of inflexion. (1)

iii) Sketch the function in the domain and label stationary points, points of inflexion and end points. (3)

iv) Determine the minimum value of the function y in the given domain. (1)

b) For the function $y = kx^2 - 4\sqrt{3}x + k - 1$

i) Find an expression for the discriminant. (1)

ii) For what values of k is the function positive definite? (3)

Question 9 (12 marks) (Start a new page)

a) i) Find $\frac{d}{dx}\left(\sin^3 x\right)$ (2)

ii) Hence find $\int \cos x . \sin^2 x \ dx$ (1)

b) Alan borrows \$130,000 to start a signwriting business. He is charged interest on the balance owing at the rate of 9.75%pa compounded monthly and agrees to repay the loan including interest by making equal monthly instalments of \$M.

i) How much does Alan owe at the end of the first month just before he makes an instalment? (1)

ii) Write an expression involving M for the total amount owed by Alan just after the second instalment is paid. (2)

iii) Calculate the value of M (to the nearest cent) which will repay the loan after 13 yrs. (4)

iv) In how many months (to the nearest whole month) will the loan be repaid if Alan made instalments of \$1700 per month (2)

Question 10 (12 marks) (start a new page)

a) In a new Quiz show of "The Sky is the Limit", you win \$6 000 for answering the first question correctly, \$14 000 for answering the second question correctly, \$22 000 for answering the third question correctly and so on for the following questions. The prizes form an arithmetic sequence.

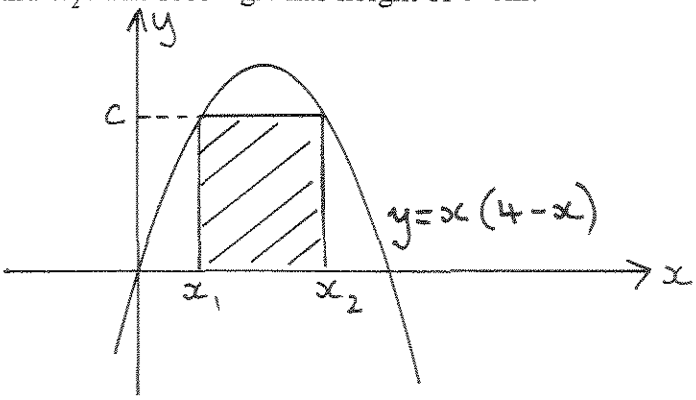
You finish when you answer a question incorrectly. Your total winnings for the contest is the sum of money you win on each question.

i) What is the prize money for the 10^{th} question only? (2)

ii) How many questions must you correctly answer to exceed \$1 000 000 in total winnings? (4)

b) A rectangle has two vertices on the curve $y = x(4 - x)$.

The other two vertices are on the x axis in the interval $0 \le x \le 4$ and are called x_1 and x_2. The rectangle has height of c cm.



i) Find the length $(x_2 - x_1)$ in terms of c and hence show the area of the rectangle is given by $A = 2c\sqrt{4-c}\ \ cm^2$ (3)

ii) Show the maximum area of this rectangle is $\frac{32\sqrt{3}}{9} cm^2$. (3)

(showing marks)

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QUESTION 1

a) $\sin^2 1.7 + \cos^2 1.7 = \underline{1}$ (1)

b) $x^4 - 16 = (x^2-4)(x^2+4)$ (2)

$= \underline{(x-2)(x+2)(x^2+4)}$

c) $4a + b\sqrt{7} = 36 + \sqrt{63}$

$= 36 + 3\sqrt{7}$

$\therefore 4a = 36$ (2)

$\underline{a = 9 \qquad b = 3}$

d) $3^x = 7$

$\log_{10} 3^x = \log_{10} 7$

$x \log_{10} 3 = \log_{10} 7$

$x = \dfrac{\log_{10} 7}{\log_{10} 3}$

$\underline{x = 1.77}$ (2)

e) $\dfrac{5}{x} = \dfrac{4}{x-3}$

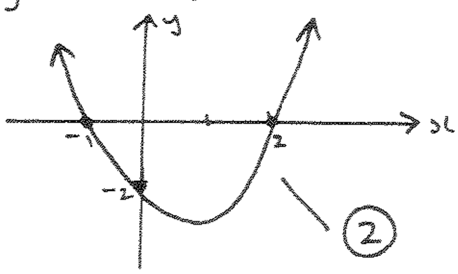
$4x = 5(x-3)$

$4x = 5x - 15$

$\therefore \underline{x = 15}$ (2)

f) i) $y = x^2 - x - 2$

$y = (x-2)(x+1)$



(2)

ii)

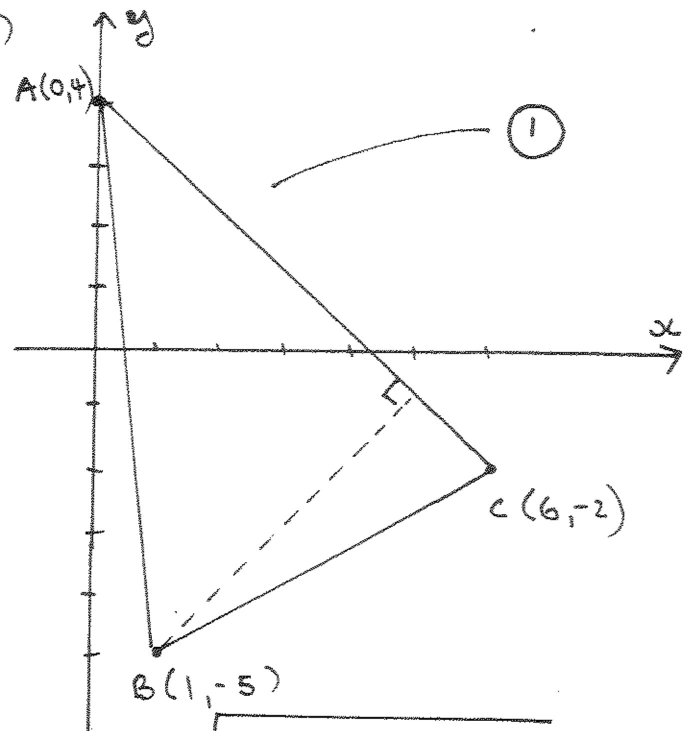
$\therefore x^2 - x - 2 < 0$

for $\underline{-2 < x < 1}$

(1)

QUESTION 2

a)



(1)

b) $AC = \sqrt{(0-6)^2 + (4+2)^2}$

$= \sqrt{36+36}$

$= \sqrt{72}$

$= \underline{6\sqrt{2} \text{ units}}$ (1)

c) AC has gradient $m = \dfrac{4--2}{0-6} = -1$

and y intercept $b = 4$

$\therefore$ eqn AC: $y = -x + 4$

$\therefore \underline{x + y - 4 = 0}$ (2)

d) $p = \left| \dfrac{1\times1 + 1\times-5 - 4}{\sqrt{1+1}} \right|$

$p = \left| \dfrac{-8}{\sqrt{2}} \right|$

$p = \dfrac{8}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$ (2)

$\underline{p = 4\sqrt{2} \text{ units}}$

e) Area $\triangle ABC = \dfrac{6\sqrt{2} \times 4\sqrt{2}}{2}$

$= \underline{24 \text{ unit}^2}$ (1)

f) Let $D(x, y)$

$\therefore \frac{x+1}{2} = 6 \qquad \frac{y-5}{2} = -2$

$x + 1 = 12 \qquad y - 5 = -4$

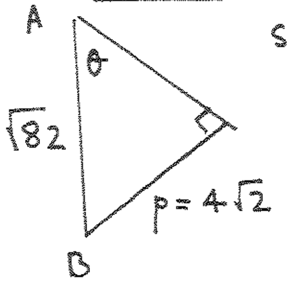
$x = 11 \qquad y = 1$

$\therefore D(11, 1)$ —— ②

g) $AB = \sqrt{(0-1)^2 + (4+5)^2}$

$= \sqrt{82}$ units —— ①

h)



$\sin\theta = \frac{4\sqrt{2}}{\sqrt{82}}$

$\theta = 38 \cdot 65°$

$\therefore B\hat{A}C = 39°$ —— ②
(nearest degree)

QUESTION 3

a) i) $\frac{dy}{dx} = 3x^2 + 1$ —— ①

ii) for all values of x, $x^2 \geq 0$

$\therefore \frac{dy}{dx} > 0 \quad \therefore$ increasing for all x —— ①

b) i) $\frac{d}{dx}\left(3x^{-2}\right) = -6x^{-3}$

$= \frac{-6}{x^3}$ —— ①

ii) $u = \cos 3x \qquad v = x$

$u' = -3\sin 3x \qquad v' = 1$ —— ②

$\frac{dy}{dx} = \frac{-3x\sin 3x - \cos 3x}{x^2}$

iii) $\frac{d}{dx}\left(3(x^2+1)^{-1}\right)$

$= -3 \times 2x(x^2+1)^{-2}$

$= \frac{-6x}{(x^2+1)^2}$ —— ②

c) $\int (2x+1)^{-1/2}\,dx = \frac{(2x+1)^{1/2}}{2 \times \frac{1}{2}} + c$

take off 1 for no 'c'

$= \sqrt{2x+1} + c$ —— ③

d) $\log_5 3 = 2\log_5 6 - \log_5 x$

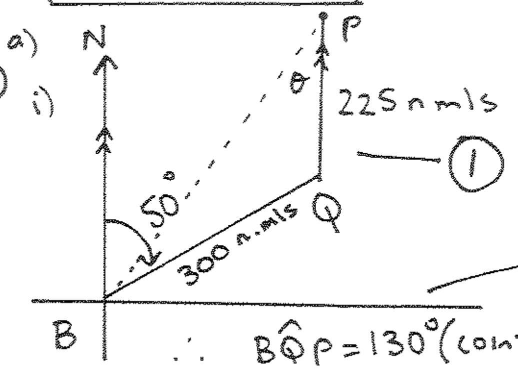
$\log_5 3 = \log_5\left(\frac{36}{x}\right)$

$3 = \frac{36}{x}$

$x = 12$ —— ②

QUESTION 4

a) i)



—— ①

—— ①

$\therefore B\hat{Q}P = 130°$ (cointerior angles parallel lines)

ii) $BP^2 = 300^2 + 225^2 - 2 \times 300 \times 225 \cos 130°$

$BP = 477$ nat mls (to nearest n. ml) —— ②

iii) Let $B\hat{P}Q = \theta$

$\frac{\sin\theta}{300} = \frac{\sin 130°}{477}$

$\sin\theta = \frac{300\sin 130°}{477}$

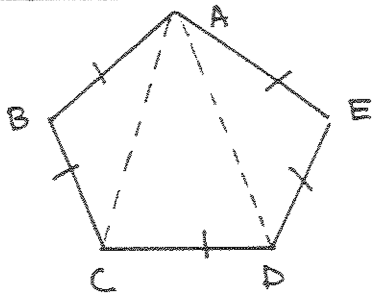
$\theta = 28 \cdot 8°$ —— ①

$\therefore$ Bearing $180° + 28 \cdot 8°$ —— ①

$= 209°$ (to nearest degree)

OR S 29° W

b)



i) angle sum Pentagon = 3×180 (1)

$\therefore \hat{ABC} = \frac{540^\circ}{5} = 108^\circ$ (angle of regular pentagon)

ii) In $\triangle ABC$ and $\triangle AED$

$\hat{ABC} = \hat{AED}$ (angles in regular pentagon)

$AB = AE$, $BC = DE$ (sides of regular pentagon) (3)

$\therefore \triangle ABC \equiv \triangle AED$ (SAS)

iii)

$B\hat{A}C = 36^\circ$ (angle sum isosceles triangle)

$D\hat{A}E = 36^\circ$ (corsp. angles in congruent triangles)

since $B\hat{A}E = 108^\circ$ (angle of regular pentagon)

$\therefore C\hat{A}D = 108^\circ - 2\times 36^\circ$

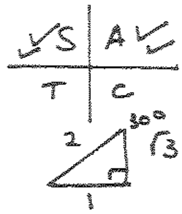
$= 36^\circ$ (2)

b) $2\sin 2\theta - 1 = 0$

$2\sin 2\theta = 1$

$\sin 2\theta = 1/2$

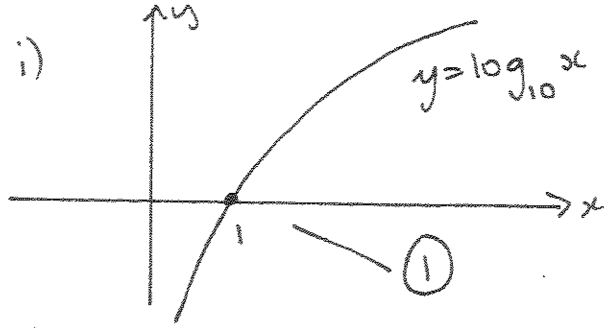
acute 2θ is 30°



$\therefore 2\theta = 30^\circ, 150^\circ, 390^\circ, 510^\circ$ (4)

$\theta = 15^\circ, 75^\circ, 195^\circ, 255^\circ$

c) i)



(1)

ii)

x	1	1·5	2	2·5	3
y	0	·176	·301	·398	·477
	F	y_1	y_2	y_3	L

(1)

iii) $\int_1^3 \log_{10} x \, dx$

$= \frac{\cdot 5}{3}\left[0 + \cdot 477 + 4(\cdot 176 + \cdot 398) + 2\times\cdot 301\right]$ (2)

$= \cdot 56$ (2 dec places)

QUESTION 5

a) i) $AB = r\theta$ (1)

ii) Area Sector $= \frac{1}{2}r^2\theta$ (1)

iii) $\frac{\pi}{4} = r\theta$ —— ①

$\pi = \frac{1}{2}r^2\theta$ —— ②

sub ① into ② (2)

$\pi = \frac{1}{2} r \frac{\pi}{4}$

$r = 8$ cm $\therefore$ sub. into ①

$\theta = \pi/32$

QUESTION 6

a) i) $\alpha + \beta = \frac{3}{2}$ (1)

ii) $\alpha\beta = -\frac{4}{2} = -2$ (1)

iii) $(\alpha+1)(\beta+1) = \alpha\beta + (\alpha+\beta) + 1$

$= -2 + \frac{3}{2} + 1$

$= \frac{1}{2}$ (1)

iv) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{3}{2}\right)^2 + 4$

$= 6\frac{1}{4}$ (1)

b) $y^2 - 6y - 3 = 12x$

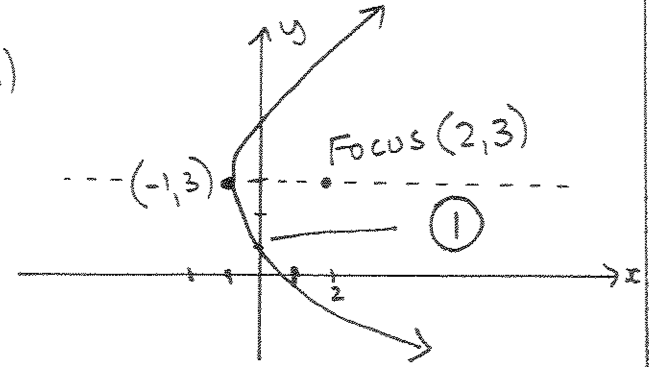
i) $y^2 - 6y + 9 = 12x + 3 + 9$

$(y-3)^2 = 12x + 12$

$(y-3)^2 = 12(x+1)$ — (1)

ii) Vertex $(-1, 3)$ — (1)

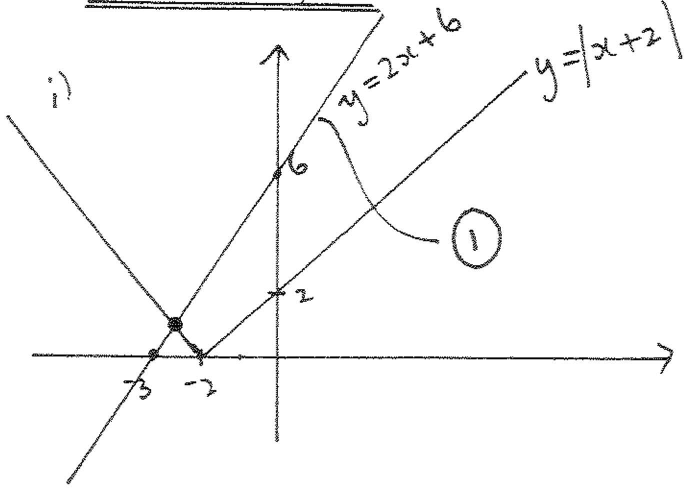
iii)



$4a = 12 \quad \therefore \quad a = 3$

Focus $(2,3)$ — (1)

c) i)



ii) Sim eq. $y = 2x+6 \quad y = |x+2|$

$2x + 6 = x + 2$ OR $2x + 6 = -x - 2$

$x = -4$ | $3x = -8$

no a solution from graph | $x = \frac{-8}{3}$

$x = -2\frac{2}{3}$ only solution

by subst $y = 2x - 2\frac{2}{3} + 6$

$y = 2/3$ — (2)

$\therefore$ pt intersection is $\left(-2\frac{2}{3}, \frac{2}{3}\right)$

(or can be done by subst. into both equations)

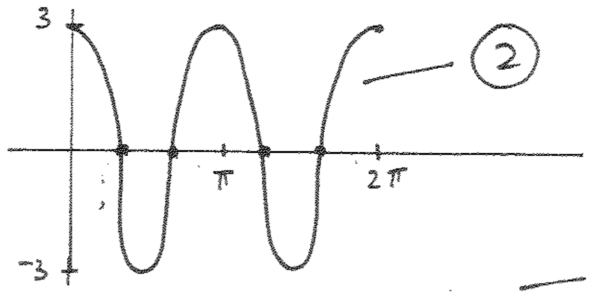
iii) $|x+2| > 2x+6$ — (1)

$x < -2\frac{2}{3}$ from graph

QUESTION 7

a) i) $y = 3\cos 2x$

amp = 3 period = π



ii) 4 points of inflexion — (1)

b) $V = \pi \int_0^{\pi/6} \sec^2 x \, dx$ — (1)

$= \pi \left[\tan x \right]_0^{\pi/6}$

$= \pi \left[\tan \frac{\pi}{6} - \tan 0 \right]$ (2)

$= \frac{\pi}{\sqrt{3}}$ unit^3

c) i) by sim. eq.

$\sin x = \cos x$

$\therefore \tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}$ for A & B respectively (2)

S | A ✓
✓T | C

ii) $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) \, dx$ — (1)

$= \left[-\cos x - \sin x \right]_{\pi/4}^{5\pi/4}$ — (1)

$= \left[\left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \right]$

$= \left[\left(\frac{1}{\sqrt2} + \frac{1}{\sqrt2}\right) - \left(-\frac{1}{\sqrt2} - \frac{1}{\sqrt2}\right)\right]$

$= \frac{2}{\sqrt2} - \left(-\frac{2}{\sqrt2}\right)$ — (2)

$= \frac{4}{\sqrt2}$ or $2\sqrt2 \text{ unit}^2$

QUESTION 8

a) $y = x^3 - 3x^2 - 9x + 1 \quad -4 \leq x \leq 5$

i) $\frac{dy}{dx} = 3x^2 - 6x - 9$

$\frac{d^2y}{dx^2} = 6x - 6$ } (1)

st. pts $\frac{dy}{dx} = 0 \quad x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

(2) $\therefore x = 3 \quad (3, -26) \quad y'' > 0$ min

$x = -1 \quad (-1, 6) \quad y'' < 0$ max

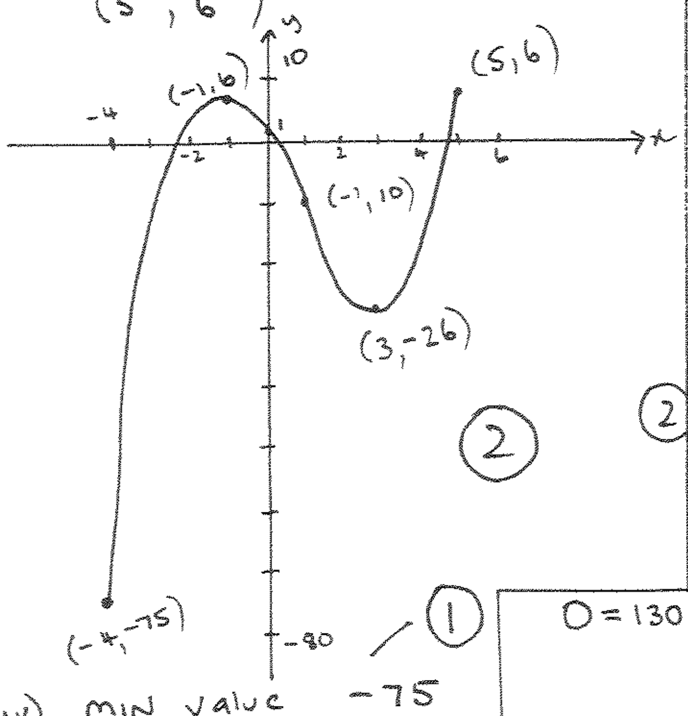
ii) pt inflexion $y'' = 0 \quad x = 1$

(1) $(1, -10)$ (lies on continuous curve between a max and a min)

iii) end pts

$(-4, -75)$ — (1)

$(5, 6)$



(2)

(1)

iv) min value -75

b) $y = kx^2 - 4\sqrt3\, x + k - 1$

i) $\Delta = (-4\sqrt3)^2 - 4k(k-1)$

$= 48 - 4k^2 + 4k$ — (1)

ii) +ve def if

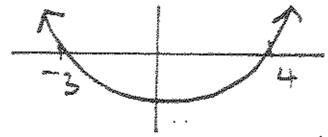
$a > 0$ and $\Delta < 0$ — (1)

$\therefore a = k > 0$

and $-4k^2 + 4k + 48 < 0$

$k^2 - k - 12 > 0$

$(k-4)(k+3) > 0$



(2) ie $k < -3$ and $k > 4$

but since $k > 0$ from above

$\therefore k > 4$ only

QUESTION 9

(2)

a) i) $\frac{d}{dx}(\sin x)^3 = 3\cos x \sin^2 x$

(1)

ii) $\therefore \int \cos x \sin^2 x \, dx = \frac{1}{3}\sin^3 x + c$

do not worry about '+c'

b) i) Let A_n be amount owing after n months — (1)

$A_1 = 130{,}000\left(1 + \frac{.8125}{100}\right)^1 = \$131{,}056{\cdot}25$

(before repayment made)

since 9.75% pa ⇒ .8125% p.m

ii) $A_1 = 130{,}000(1{\cdot}008125)^1 - M$

$A_2 = \left(130{,}000(1{\cdot}008125)^1 - M\right)(1{\cdot}008125)^1 - M$

(2) → $= 130{,}000(1{\cdot}008125)^2 - M(1{\cdot}008125)^1 - M$

iii) 13 yrs $n = 156$ Loan repaid $A_{156} = 0$

$A_{156} = 130{,}000(1{\cdot}008125)^{156} - M(1{\cdot}008125)^{155} \ldots\ldots M$

$0 = 130{,}000(1{\cdot}008125)^{156} - M[1 + 1{\cdot}008125^1 + \cdots + 1{\cdot}008125^{155}]$

GP $a = 1 \quad r = 1{\cdot}008125$

$n = 156$

$$M\left[\frac{1.008125^{156}-1}{1.008125-1}\right] = 130{,}000(1.008125)^{156}$$ (4)

$M = \$1473.11$

iv) Let $M = 1700$ and number of months be n

$$1700\frac{(1.008125^n-1)}{.008125} = 130{,}000(1.008125)^n$$

$$1700(1.008125^n-1) = 1056.25(1.008125)^n$$

$$1700(1.008125)^n - 1700 = 1056.25(1.008125)^n$$

$$643.75(1.008125)^n = 1700$$

$$1.008125^n = \frac{1700}{643.75}$$

$$1.008125^n = 2.64077\ldots$$

$$n\log_{10}1.008125 = \ln(2.64072.)$$

$n = 120$ (2)

QUESTION 10

a) i) $6000 + 14000 + 22000 \ldots\ldots$

AP $a = 6000$ $d = 8000$

$$T_{10} = 6000 + 9\times 8000$$
$$= \$78{,}000$$ (2)

ii) $S_n > 1000{,}000$

$$S_n = \frac{n}{2}\left(12000 + (n-1)8000\right)$$

$$\frac{n}{2}\left(12000 + 8000n - 8000\right) > 1000{,}000$$

$$\frac{n}{2}(12 + 8n - 8) > 1000$$

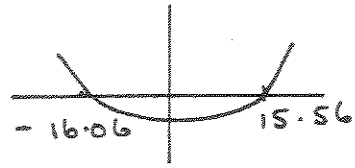
$$n(4+8n) > 2000$$

$$8n^2 + 4n - 2000 > 0$$ (2)

$$2n^2 + n - 500 > 0$$

$$n = \frac{-1 \pm \sqrt{1-4\times 2\times -500}}{4}$$

$$n = \frac{-1 \pm \sqrt{4001}}{4}$$ $n = 15.56$, $n = -16.06$



since $2n^2 + n - 500 > 0$

and $n > 0$

$\therefore n > 15.56$

$n = 16$ (2)

$\therefore$ 16 questions correct

b) i) $c = 4x - x^2$

$$x^2 - 4x + c = 0$$

$$x = \frac{4 \pm \sqrt{16-4c}}{2}$$

$$\therefore x = \frac{4 \pm 2\sqrt{4-c}}{2}$$

$$x = 2 \pm \sqrt{4-c}$$

$x_1 = 2 - \sqrt{4-c}$ $x_2 = 2 + \sqrt{4-c}$

$$\therefore (x_2 - x_1) = 2 + \sqrt{4-c} - (2 - \sqrt{4-c})$$

$$(x_2 - x_1) = 2\sqrt{4-c}$$ (2)

$$\therefore A_{rectangle} = 2c\sqrt{4-c}$$ (1)

ii) $u = 2c$ $v = \sqrt{4-c} = (4-c)^{\frac{1}{2}}$

$u' = 2$ $v' = -\frac{1}{2}(4-c)^{-1/2}$

$$v' = \frac{-1}{2\sqrt{4-c}}$$

$$\frac{dA}{dc} = 2\sqrt{4-c} - \frac{2c}{2\sqrt{4-c}}$$ (2)

$$= \frac{2(4-c) - c}{\sqrt{4-c}}$$

$$\frac{dA}{dc} = \frac{8-3c}{\sqrt{4-c}}$$ $\frac{dA}{dc} = 0$ $\therefore c = \frac{8}{3}$

Test max/min

c	1	8/3	3
A'	+		−

max

$$\therefore \max A = 2\times\frac{8}{3}\sqrt{4-\frac{8}{3}}$$ (1)
$$= \frac{32\sqrt{3}}{9}$$