Name: ______________________ Maths Class: _____________

# SYDNEY TECHNICAL HIGH SCHOOL

# TRIAL HIGHER SCHOOL CERTIFICATE

## 2009

# MATHEMATICS

*Time Allowed: 3 hours plus 5 mins reading time*

***Instructions:***

- Write your name at the top of this page, and at the top of each answer sheet
- At the end of the examination this examination paper must be attached to the front of your answers.
- All questions are of equal value and may be attempted.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- Start each question on a new answer page.

**(For Markers Use Only)**

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | Q9 | Q10 | Total |
|---|---|---|---|---|---|---|---|---|---|---|
| /12 | /12 | /12 | /12 | /12 | /12 | /12 | /12 | /12 | /12 | /120 |

**Question 1**

a) Calculate $(9.6 \times 10^4) \div (6.3 \times 10^{-2})$. Write your answer using 3 significant figures. 1

b) Solve i) $x^2 - 3x - 4 = 0$ 2

ii) $x^3 + x^2 + x + 1 = 0$ 2

c) i) Sketch $y = |x + 1|$. Show intercepts on the coordinate axes. 1

ii) Hence or otherwise solve $|x + 1| = x + 1$ 1

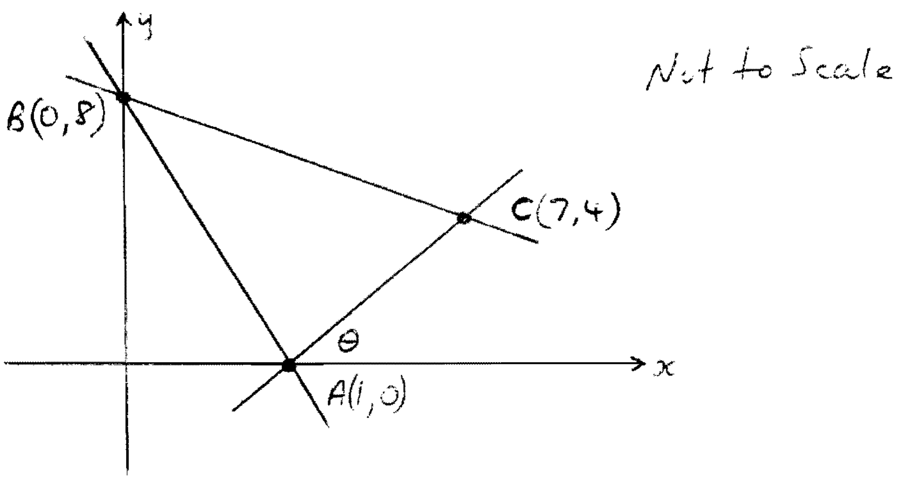
d) Find the gradient of the curve $y = \frac{1}{x^2}$ at the point $(-1,1)$ 2

e) Simplify $\frac{x+4}{x} + \frac{3}{x^2-x}$ 2

f) Write the exact value of $tan\,\frac{5\pi}{6}$ 1

**Question 2 (start a new page)**

a)




Points A, B, C have coordinates as shown above.

The angle between $AC$ and the $x$ $axis$ is $\theta$.

i) Copy this diagram into your answer booklet.

ii) Find the gradient of $AC$. 1

iii) Calculate the size of $\theta$ to the nearest degree. 1

iv) Find the equation of $AC$. Give your answer in general form. 2

v) Find the equation of the perpendicular bisector of the interval $AC$. 3

vi) Find the perpendicular distance from $B$ to $AC$. 2

vii) Write the coordinates of $D$ such that $ABCD$ is a parallelogram. 1

b) Solve $cos\theta = -\frac{1}{2}$ for $0° \leq \theta \leq 360°$ 2

**Question 3 (start a new page)**

a) Differentiate: i) $y = \frac{2x+1}{2x-1}$ 2

ii) $y = (x^2 - 1)^4$ 1

iii) $y = 3xlogx$ 2

iv) $y = sin^2 x$ 1

v) $y = tan\ 2x$ 1

b) Find indefinite integrals of:

i) $cos5x$ 1

ii) $\frac{2x}{x^2+1}$ 1

iii) $(2x + 3)^5$ 1

c) Given $f(x) = x^2$, show how to find $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ 2

**Question 4 (start a new page)**

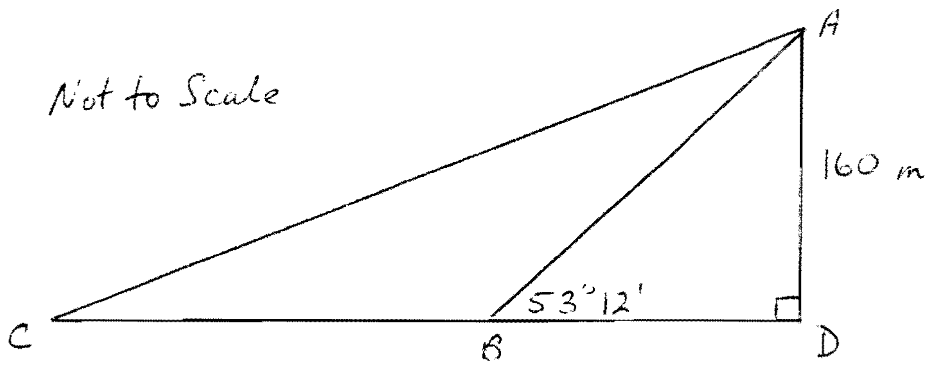
a) A parabola has equation $2y = x^2 - 6x + 7$

i) Write this equation in the form $(x-h)^2 = 4a(y-k)$ and hence, or otherwise, give the coordinates of the vertex. 2

ii) Find the coordinates of the focus. 1

iii) Give the equation of the directrix. 1

b)

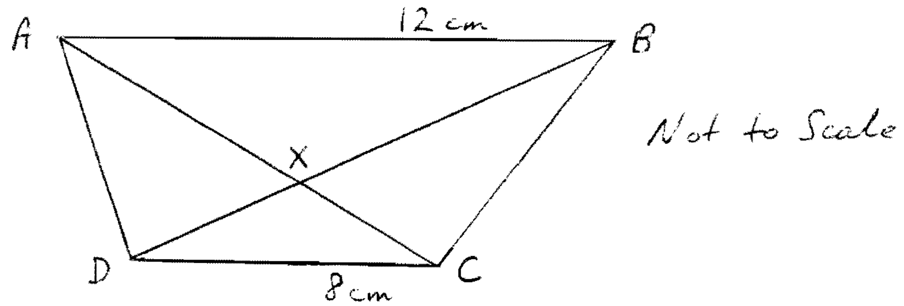



A man in a boat at B observes that the angle of elevation of a cliff top 160 metres high is $53°12'$.

i) Find the direct distance BA. Give your answer correct to 1 decimal place. 2

ii) The man rows 50 metres from B to C. Use the cosine rule in triangle ABC to find AC to the nearest metre. 2

c) ABCD is a trapezium in which $AB \| DC$, $AB = 12cm, DC = 8cm$ and $AC = 9cm$. The diagonals intersect at $X$.




i) Copy the diagram onto your answer page and clearly mark the above information.

ii) Prove that $\Delta AXB$ is similar to $\Delta CXD$. 2

iii) Hence find the length of $AX$. 2

**Question 5 (start a new page)**

a) Find the value(s) of k so that $x^2 - (k + 2)x + 4k - 4 = 0$ has:

i) equal roots. 2

ii) one root the reciprocal of the other. 2

b) Simplify $\frac{\sin(\frac{\pi}{2}-\theta)}{\sin(\pi-\theta)}$ 2

c) Solve $sin^2x + \sin x = 0$ for $0 \leq x \leq 2\pi$ 2

d) i) Sketch the graphs of $y = \frac{1}{x}$, $y = 1$ and $x = 2$ on the same axes. 2

Shade the common area between the three graphs.

ii) Find the value of the shaded area in exact form. 2

**Question 6 (start a new page)**

a) Solve $3^x = 7$. Give your answer correct to 2 decimal places. 2

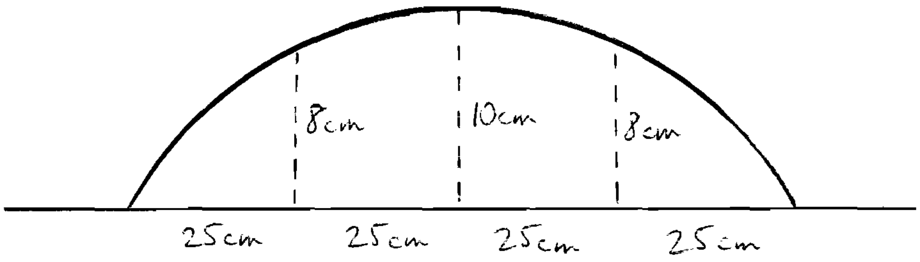
b) The first three terms of a certain geometric series are $x + 2 + 1\frac{1}{2} + \ldots$

i) Find the value of $x$. 1

ii) Find the limiting sum of the series. 1

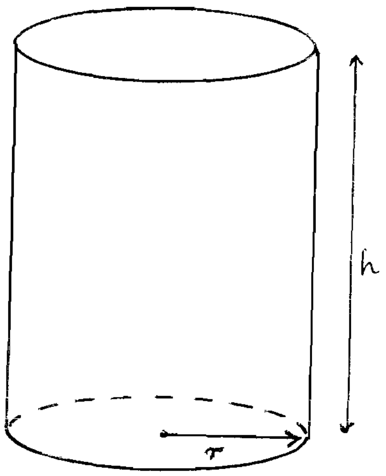
c) Evaluate $\sum_{n=10}^{50}(100 - 4n)$ 2

d) A speed hump has a cross-section as shown:




Use Simpson's rule to find the approximate area of the cross-section. 2

e)




The cylinder varies its volume but the sum of its radius and height is constant at 12cm.

i) Given $V = \pi r^2 h$, find $\dfrac{dv}{dr}$. 2

ii) Find the radius that proves the volume is maximised. 2

**Question 7 (start a new page)**

a) The curve $y = e^{x+1}$ is rotated about the $x$ axis between $x = 0$ and $x = 1$.
Find the volume generated, leaving your answer in exact form. 3

b) Given the function $y = -2sin3x$.

i) State the period of the function. 1

ii) Sketch its graph for $0 \leq x \leq \pi$ 2

c) For a function $y = f(x)$, you are given that $\frac{dy}{dx} = 3x^2 - 2x - 1$.

There is a stationary point at (1,2).

i) Determine the nature of the stationary point. 2

ii) Prove that there is a point of inflexion. 2

iii) Find the equation of the function $y = f(x)$. 2

**Question 8 (start a new page)**

a) i) Solve $4x - x^2 \geq 0$ 2

ii) Hence, state the range of the function $y = \frac{-1}{\sqrt{4x-x^2}}$ 1

b) i) Prove that $1 + tan^2x = sec^2x$ 2

ii) Hence, find $\int tan^2 3x\, dx$ 2

c) Solve $log_2 x + log_2(x^3) = -8$ 2

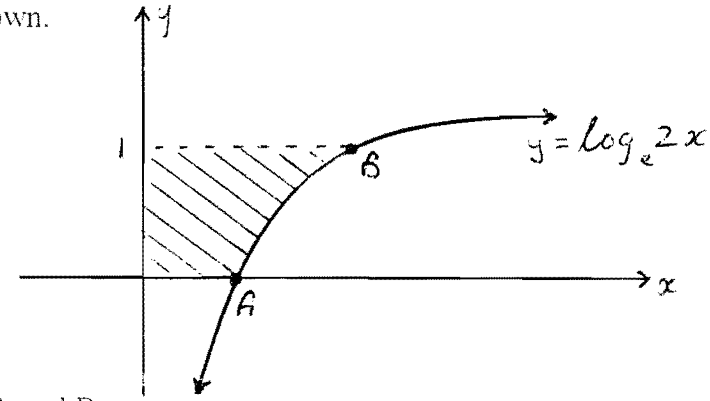
d) The sum of the first two terms of a geometric sequence is 6 and the sum of the second and third terms is -5. 3

Find the first three terms of the sequence.

**Question 9 (start a new page)**

a) Find $\int \frac{x^3+1}{2x}\, dx$ 2

b) The graph of $y = log_e 2x$ is shown.




i) Find the coordinates of A and B. 2

ii) Find the shaded area. 3

c) \$1000 is deposited into a savings account.

i) If the account earns 6% pa compounded annually, find the account's value at the end of 15 years. 1

ii) If the account is to have a value of \$5000 after 15 years, find the annual compound interest rate needed to achieve this.
Give your answer to the nearest whole percent. 2

iii) \$1000 is deposited at the beginning of each year into the account paying 6% pa. What will be the total value of the account at the end of 15 years, to the nearest dollar. 2

**Question 10 (start a new page)**

a) A block of ice, with mass $M$ grams after $t$ minutes is melting according to the equation $M = Moe^{-kt}$.
Initially, there is 100 grams of ice and, after 35 minutes, the block has reduced to 60 grams.

i) Find the constants $Mo$ and $k$ (exact form). 3

ii) Find the rate of melting when the ice weighs 50 grams. 1

iii) Find how long it takes for the ice to reduce to 5 grams. 2

b) A particle moves so that its position $x$ $cm$ from a fixed point 0 after $t$ seconds, is given by $x = 2te^{-t}$.

i) Find the particle's initial position. 1

ii) Show that $v = 2e^{-t}(1 - t)$. 1

iii) Find when the particle is stationary. 1

iv) Describe the particle's position as $t \to \infty$ 1

v) Use the above information to sketch, on coordinate axes, a graph of the particle's position over time.
Show on your curve the time where you expect the acceleration to be zero.
Justify the location. 2

END OF PAPER.

## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

# SOLUTIONS

1) a) 1520000

b) i) $(x-4)(x+1)=0$

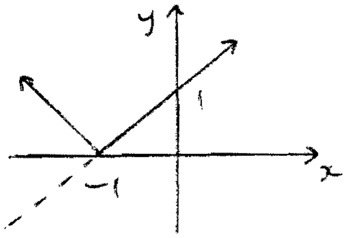
$x = 4, -1$

ii) $x^2(x+1)+1(x+1)=0$

$(x+1)(x^2+1)=0$

$x=-1$

c) i)




ii) $x \geq -1$

d) $y = x^{-2}$, $\frac{dy}{dx} = -2x^{-3}$

$m_T = \frac{-2}{(-1)^3}$

$= 2$

e) $\frac{(x+4)(x-1)}{x(x-1)} + \frac{3}{x(x-1)}$

$= \frac{x^2+3x-4+3}{x(x-1)}$

$= \frac{x^2+3x-1}{x(x-1)}$

f) $-\frac{1}{\sqrt{3}}$

2) i) diagram

ii) $M_{AC} = \frac{4-0}{7-1}$

$= \frac{2}{3}$

iii) $\tan\theta = \frac{2}{3}$

$\therefore \theta \doteqdot 34^\circ$

iv) $y - 0 = \frac{2}{3}(x-1)$

$3y = 2x - 2$

$2x - 3y - 2 = 0$

v) m.p. $=(4,2)$, $m_2 = -\frac{3}{2}$

$\therefore y - 2 = -\frac{3}{2}(x-4)$

$2y - 4 = -3x + 12$

$3x + 2y - 16 = 0$ $\left(y = -\frac{3x}{2} + 8\right)$

vi) p.d. $= \left|\frac{2\times 0 - 3\times 8 - 2}{\sqrt{2^2+(-3)^2}}\right|$

$= \left|\frac{-26}{\sqrt{13}}\right|$

$= \frac{26}{\sqrt{13}}$ (or $2\sqrt{13}$)

vii) $D(8,-4)$

b) $\theta = 60^\circ$ (2nd, 3rd quads.)

$= 120^\circ, 240^\circ$

3) a) i) $\frac{dy}{dx} = \frac{2(2x-1) - 2(2x+1)}{(2x-1)^2}$

$= \frac{4x - 2 - 4x - 2}{(2x-1)^2}$

$= \frac{-4}{(2x-1)^2}$

ii) $\frac{dy}{dx} = 4(x^2-1)^3 \times 2x$

$= 8x(x^2-1)^3$

iii) $\frac{dy}{dx} = 3\log x + \frac{1}{x} \cdot 3x$

$= 3\log x + 3$

iv) $\frac{dy}{dx} = 2\sin x \cos x$

v) $\frac{dy}{dx} = 2\sec^2 2x$

b) i) $\frac{\sin 5x}{5} + c$

ii) $\log(x^2+1) + c$

iii) $\frac{(2x+3)^6}{12} + c$

c) $\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= 2x$.

4) a) i) $x^2 - 6x = 2y - 7$

$x^2 - 6x + 9 = 2y + 2$

$(x-3)^2 = 2(y+1)$

$\therefore$ vertex is $(3, -1)$

ii) $4a = 2 \Rightarrow a = \frac{1}{2}$

$\therefore$ Focus is $(3, -\frac{1}{2})$

iii) $y = -1\frac{1}{2}$

b) i) $\sin 53°12' = \frac{160}{BA}$

$\therefore BA = \frac{160}{\sin 53°12'}$

$= 199.8\text{m}$

ii) $AC^2 = 50^2 + 199.8^2 - 2 \times 50 \times 199.8 \times \cos 126°48'$

$= 54388.5315$

$\therefore AC \doteqdot 233$ metres.

c) i) diagram

ii) $\angle A = \angle C$ (alternate angles $AB \parallel DC$)

$\angle B = \angle D$ (same reason)

$\therefore \triangle AXB \sim \triangle CXD$ (equiangular)

iii) $\frac{AX}{CX} = \frac{12}{8} = \frac{3}{2}$ (equal ratio of sides in sim. triangles)

$\therefore AX = \frac{3}{5} \times 9$

$= 5.4$ cm

5) a) i) $b^2 - 4ac = 0$

$[-(k+2)]^2 - 4 \times 1 \times (4k-4) = 0$

$k^2 + 4k + 4 - 16k + 16 = 0$

$k^2 - 12k + 20 = 0$

$(k-10)(k-2) = 0$

$\therefore \underline{\underline{k = 10, 2}}$

ii) $\alpha = \frac{1}{\beta}$

$\therefore \alpha\beta = 1$

$\therefore \frac{c}{a} = 1$

$\therefore \frac{4k-4}{1} = 1$

$\therefore 4k = 5$

$\underline{\underline{k = 5/4}}$

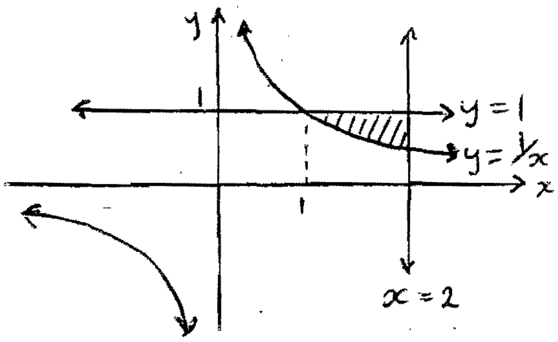
b) $\frac{\cos\theta}{\sin\theta} = \underline{\underline{\cot\theta}}$

c) $\sin x(\sin x + 1) = 0$

$\sin x = 0$ or $-1$

$\therefore x = \underline{\underline{0, \pi, 2\pi, \frac{3\pi}{2}}}$

d) i)




ii) $\text{Area} = 1 - \int_1^2 \frac{1}{x}\,dx$

$= 1 - [\log]_1^2$

$= 1 - \log 2 + \log 1$

$= 1 - \log 2$

~~$= (-1 + e^{-2}) + e^2 - 2 + 1$~~

~~$= 1 + e^{-2} + e^2 - 3$~~

~~$= e^{-2} + e^2 - 2$~~

6) a) $\log(3^x) = \log 7$

$x \log 3 = \log 7$

$x = \frac{\log 7}{\log 3}$

$\doteqdot \underline{\underline{1.77}}$

b) i) $\frac{2}{x} = \frac{1\frac{1}{2}}{2}\left(\frac{3}{4}\right)$

$3x = 8$

$\underline{\underline{x = 2\frac{2}{3}}}$

ii) $a = 2\frac{2}{3}$, $r = 3/4$

$S_\infty = \frac{2\frac{2}{3}}{1 - 3/4}$

$= 2\frac{2}{3} \times 4$

$= \underline{\underline{10\frac{2}{3}}}$

c) $a = 60$, $l = -100$, $n = 41$

$S_{41} = \frac{41}{2}(60 - 100)$

$= \underline{\underline{-820}}$

d) $\text{Area} \doteqdot \frac{25}{3}(0 + 4 \times 8 + 10) \times 2$

$= \underline{\underline{700\ cm^2}}$

e) i) $r + h = 12 \Rightarrow h = 12 - r$

$V = \pi r^2 h$

$= \pi r^2(12 - r)$

$= 12\pi r^2 - \pi r^3$

$\frac{dV}{dr} = \underline{\underline{24\pi r - 3\pi r^2}}$

2) ii) Max volume when $\frac{dV}{dr} = 0$

$\therefore 24\pi r - 3\pi r^2 = 0$

$3\pi r(8 - r) = 0$

$\therefore r = 0, 8$

$\frac{d^2V}{dr^2} = 24\pi - 6\pi r$

When $r = 8$, $\frac{d^2V}{dr^2} < 0$

$\therefore$ Volume is maximised when radius = 8 cm.

7) a) $V = \pi \int_0^1 (e^{x+1})^2 \, dx$

$= \pi \int_0^1 e^{2x+2} \, dx$

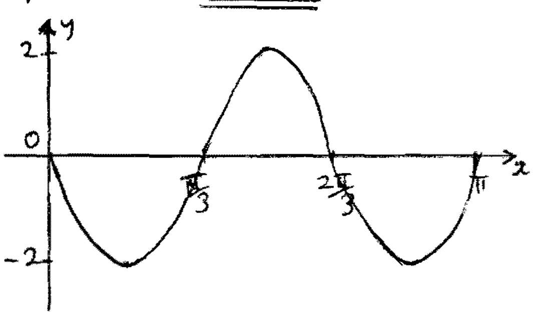
$= \pi \left[\frac{1}{2} e^{2x+2}\right]_0^1$

$= \pi \left(\frac{1}{2}e^4 - \frac{1}{2}e^2\right)$

$= \frac{\pi}{2}(e^4 - e^2)\ u^3$

b) i) period $= \frac{2\pi}{3}$

ii)




c) i) $\frac{d^2y}{dx^2} = 6x - 2$

When $x = 1$, $\frac{d^2y}{dx^2} > 0$

$\therefore$ maximum turning point at (1, 2)

ii) Point of inflexion when $\frac{d^2y}{dx^2} = 0$ and changes concavity.

$\therefore 6x - 2 = 0 \Longrightarrow$ test $x = \frac{1}{3}$

| $x$ | $\frac{1}{3}^-$ | $\frac{1}{3}$ | $\frac{1}{3}^+$ |
|---|---|---|---|
| $\frac{d^2y}{dx^2}$ | $-$ | 0 | + |

$\therefore$ concavity changes

$\therefore$ a pt. of inflexion occurs when $x = \frac{1}{3}$.

iii) $y = \int (3x^2 - 2x - 1) \, dx$

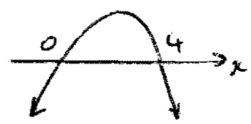
$= x^3 - x^2 - x + c$

Sub $x = 1, y = 2$

$\therefore 2 = 1 - 1 - 1 + c \quad (c = 3)$

$\therefore y = x^3 - x^2 - x + 3$

8) a) i) $x(4 - x) \geq 0$




$\therefore 0 \leq x \leq 4$

ii) $y \leq -\frac{1}{2}$

b)

i) LHS $= 1 + \tan^2 x$

$= 1 + \frac{\sin^2 x}{\cos^2 x}$

$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$

$= \sec^2 x$

$=$ RHS

ii) $\int \tan^2 3x\,dx = \int (\sec^2 3x - 1)\,dx$

$= \frac{\tan 3x}{3} - x + c$

c) $\log_2 x + 3\log_2 x = -8$

$4\log_2 x = -8$

$\log_2 x = -2$

$\therefore x = 2^{-2}$

$= \frac{1}{4}$

d) $a + ar = 6,\ ar + ar^2 = -5$

$a(1+r) = 6$ — ①

$ar(1+r) = -5$ — ②

② ÷ ① : $r = -\frac{5}{6}$

Sub in ① : $a = \frac{6}{\frac{1}{6}}$

$= 36$

$\therefore 36, -30, 25$

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⑨ a) $\int \left(\frac{x^3}{2x} + \frac{1}{2x}\right) dx$

$= \int \left(\frac{x^2}{2} + \frac{1}{2x}\right) dx$

$= \frac{x^3}{6} + \frac{1}{2}\log x + c$

b) i) $A\left(\frac{1}{2}, 0\right),\ B\left(\frac{e}{2}, 1\right)$

ii) $e^y = 2x \Rightarrow x = \frac{1}{2}e^y$

Area $= \int_0^1 \frac{1}{2}e^y\,dy$

$= \left[\frac{1}{2}e^y\right]_0^1 = \left(\frac{1}{2}e - \frac{1}{2}\right)u^2$

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c) i) $A = 1000(1.06)^{15}$

$= \$2396.56$

ii) $5000 = 1000(1+r)^{15}$

$5 = (1+r)^{15}$

$\therefore r = \sqrt[15]{5} - 1$

$\doteqdot 0.11326$

$\doteqdot 11\%$

iii) First \$1000 $\Rightarrow 1000(1.06)^{15}$

Second \$1000 $\Rightarrow 1000(1.06)^{14}$

⋮

last \$1000 $\Rightarrow 1000(1.06)^1$

Total $= \frac{1000(1.06)(1.06^{15} - 1)}{1.06 - 1}$

$= \$24,673$

(10)

a) i) $M_0 = 100$

and $60 = 100 e^{-35k}$

$e^{-35k} = \frac{3}{5}$

$-35k = \log\left(\frac{3}{5}\right)$

$k = -\frac{1}{35}\log\left(\frac{3}{5}\right)$

ii) $\frac{dM}{dt} = -kM$

$= \frac{1}{35}\log\left(\frac{3}{5}\right) \times 50$

$\doteqdot 0.73$ g/m

iii) $5 = 100 e^{-kt}$

$\frac{1}{20} = e^{-kt}$

$\therefore -kt = \log\left(\frac{1}{20}\right)$

or $kt = \log 20$

$\therefore t = \frac{\log 20}{k}$

$\doteqdot 205$ min.

b) i) $t = 0$, $x = 0 \times e^0$

$x = 0$

ii) $v = \frac{dx}{dt}$

$= 2 \times e^{-t} + e^{-t} \times (-1) \times 2t$

$= 2e^{-t} - 2te^{-t}$

$= 2e^{-t}(1-t)$

iii) $v = 0 \Rightarrow t = 1$ second

iv) $x = \frac{2t}{e^t}$

as $t \to \infty$, $x \to 0^+$

v)