ABBOTSLEIGH

TRIAL HIGHER SCHOOL CERTIFICATE

1990

MATHEMATICS

3 UNIT/4 UNIT COMMON PAPER

Time Allowed: 2 hours

All questions may be attempted
Answer each question in a separate booklet
All questions are of equal value
Approved, silent calculators may be used
Show all necessary working

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0.$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0.$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \ln x\ \{x+\sqrt{(x^2-a^2)}\},\ |x| > |a|.$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \ln x\ \{x+\sqrt{(x^2-a^2)}\}.$$

Question 1

(a) Differentiate $e^x \tan x$

(b) Find the primitive function of $\frac{x}{1 - 4x^2}$

(c) Find the primitive function of $\frac{1}{\sqrt{1 - 4x^2}}$

(d) Use the substitution $u = e^x + 2$ to evaluate

$$\int_0^{\log_e 4} \frac{e^{2x}}{e^x + 2} \, dx$$

Question 2

(a) Sketch, showing all essential features,

$$y = 3\sin^{-1}2x$$

(b) Find the exact value of $\tan(\cos^{-1}\frac{\sqrt{3}}{2})$

(c) Write down the inverse function of $y = 3 + e^x$ in the form $y = g(x)$ and state the range and domain of $g(x)$.

(d) (i) Sketch the curve $y = 1 + \sin x$ for $0 \leq x \leq 2\pi$

(ii) Calculate the volume generated when the arc of $y = 1 + \sin x$ between $x = 0$ and $x = \frac{3\pi}{2}$ is rotated about the x-axis.

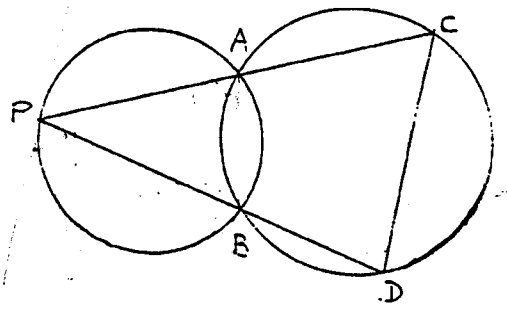
Question 3

(a) Prove by induction that

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

(b) In the expansion $(1 + x)(a - bx)^{12}$ the coefficient of x^8 is zero. $(a \neq 0, b \neq 0)$
Find in its simplest form, the ratio $\frac{a}{b}$

(c)



Copy this diagram into your booklet.
In the diagram PAC and PBD are straight lines. Prove that CD is parallel to the tangent at P.

Question 4

(a) Find $\lim_{x \to 0} \frac{\sin 2x}{3x}$

(b) Find the general solution of the equation
$\sin 2x = \cos x$ (Answer in radians)

(c) It is given that the rate of decrease of temperature of a body hotter than surrounding air is proportional to the temperature difference. If A is the air temperature, and T the temperature of the body after t minutes, then

$$\frac{dT}{dt} = -k(T - A)$$

(i) Show that if I is the initial temperature of the body, then the following function satisfies this condition:

$$T = A + (I - A).e^{-kt}$$

(ii) An ingot of pig-iron, initially at a temperature of 1500°C. is allowed to cool in the open air, where the temperature is 20°C. If it cools to 1200°C in five minutes, find the temperature of the ingot after one hour, to four significant figures.

Question 5

(a) Find the co-ordinates of the point P, dividing the interval AB externally in the ratio 3:2. A is (-2,5) and B is (1,3)

(b) The tangent at $P(2ap,ap^2)$ on the parabola $x^2 = 4ay$ meets the x-axis at C.

(i) Prove that the equation of the tangent at P is $y = px - ap^2$

(ii) Find the co-ordinates of C

(iii) Find the co-ordinates of M, the midpoint of PC

(iv) Show that the locus of M is the parabola $2x^2 = 9ay$ and find the co-ordinates of its focus.

(c) A particle moves with simple harmonic motion and has a speed of 5cm/sec when passing through the centre O of its path. The period is π secs. Find the speed of the particle when it is 2cm from O.

Question 6

(a) Use one application of Newton's method to find a solution, which is more accurate than $x = 1$, to the equation,

$$x - \tan^{-1}2x = 0$$

(give your answer correct to 1 decimal place)

(b) If α, β, γ are the roots of $x^3 - 5x - 4 = 0$

(i) evaluate $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$

(ii) form the equation with roots $2\alpha, 2\beta, 2\gamma$

(c) When $f(x) = x^3 + ax^2 + bx + c$ is divided by $(x - 3)$ the remainder is 30. If $(x^2 - 4)$ is a factor of $f(x)$, find a, b and c.

Question 7

(a) A cat sitting on top of a wall 3.2m high, spots a mouse on the ground, below her, 4m from the foot of the wall. the cat jumps horizontally from the top of the wall with an initial speed of 6m/sec. Taking acceleration due to gravity as $10m/sec^2$ and ignoring air friction,

(i) find how long the cat takes to reach the ground

(ii) show that the cat misses the mouse and determine the extent of the miss

(iii) find the speed with which the cat strikes the ground

(b) Find the maximum and minimum values of

$$\frac{2 - \sin\theta}{\cos\theta} \qquad \text{for } 0 \le \theta \le \frac{\pi}{4}$$