ABBOTSLEIGH

2001

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks (**84**)

- Attempt Questions 1-7.
- All questions are of equal value.

Total marks (84)
Attempt Questions 1 – 7
All questions are of equal value

Answer all questions in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the size of the acute angle between the lines $y = 2x + 3$ and $4x - y + 1 = 0$. **2**

(b) Find all solutions to $\frac{1}{x-2} \leq 4$. **3**

(c) The point $P(8, -5)$ divides the interval AB externally in the ratio 3 : 2. If A is the point (-1, 4), find the coordinates of B(x, y). **2**

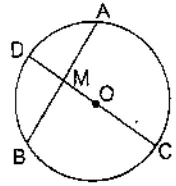
(d) Evaluate exactly $\int_0^3 \frac{1}{\sqrt{9 - x^2}}\, dx$. **2**

(e) Find all values of θ, $0 \leq \theta \leq 2\pi$, which satisfy the equation $\sin\theta = \cos 2\theta$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Use the substitution $u = \ln x$ to evaluate $\int_e^{e^2} \frac{1}{x \ln x}\, dx$. **4**

(b) In the diagram AB and CD are intersecting chords in a circle. AB and CD intersect at M and CD passes through the centre of the circle O.
AM = BM = 5 cm and DM = 2 cm.



Copy or trace the diagram into your writing booklet.

(i) Give a reason why AB is perpendicular to CD. **1**

(ii) Find the radius OC. **2**

(iii) Find the area of the quadrilateral ACBD. **1**

(c) (i) Show that the equation $e^x + x - 5 = 0$ has a root in the interval $1 < x < 2$. **2**

(ii) Taking the first approximation of the root to be $x = 1.5$, use Newton's Method once to find a closer approximation to the root. Answer correct to 1 decimal place. **2**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For the function $f(x) = 2\cos^{-1}\left(\frac{x}{4}\right)$

(i) Write down the domain and range of the function. 2

(ii) Sketch the function showing all main features. 1

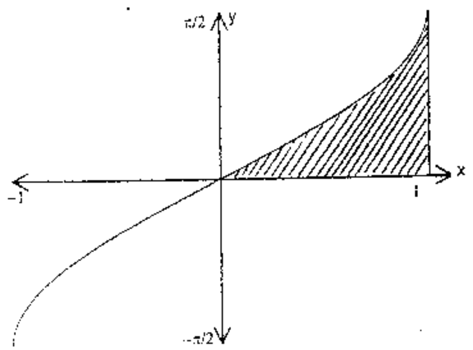
(b) A particle moves in a straight line and its position x cm at time t seconds is given by $x = 5\cos 6t$.

(i) Show that the particle is undergoing Simple Harmonic Motion. 2

(ii) State the period and end points for this motion. 2

(iii) Initially the particle is 5 cm to the right of the centre of motion, O. Find the time taken for the particle to move halfway towards the centre of motion for the first time. 2

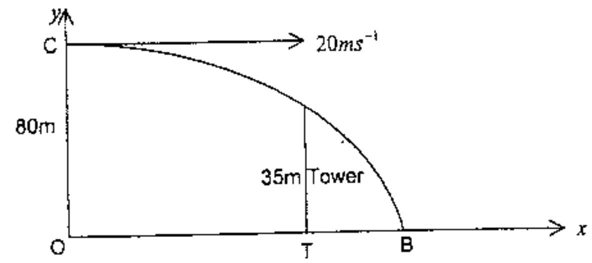
(c) Find the exact area of the region bounded by the curve $y = \sin^{-1} x$ and the x axis from $x = 0$ to $x = 1$. 3



Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Using $t = \tan\frac{x}{2}$ prove that $\frac{1+\cos x}{1-\cos x} = \cot^2\frac{x}{2}$. 2

(b) A particle is projected horizontally from the top of a cliff, C, with a velocity of $20ms^{-1}$. The cliff is 80 metres above the ground and the particle strikes the ground at point B.



Assuming there is no horizontal acceleration and vertical acceleration is due to gravity the equations of motion for the particle are $\ddot{x} = 0$ and $\ddot{y} = -g$. The origin is the base of the cliff, O, and gravity, $g = -10ms^{-2}$.

(i) Using calculus, show that the position of the particle is given by

$$x = 20t \text{ and } y = 80 - 5t^2$$ 2

(ii) If the particle just clears a tower of height 35 metres find the distance from the base of the cliff O to the tower T. 2

(c) The points $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ lie on the parabola $x^2 = 8y$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - 2p^2$. 2

(ii) The tangent at P and the line passing through Q parallel to the y axis intersect at T.
Show that the coordinates of T are $(4q, 4pq - 2p^2)$. 2

(iii) Find the coordinates of M, the midpoint of PT. 1

(iv) Find the Cartesian equation of the locus of M when $pq = -1$. 1

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Express $x^2+8x+20$ in the form $(x+k)^2+C$. 1

(ii) Hence find $\int \frac{dx}{x^2+8x+20}$. 2

(b) The rate of increase of a population P of sandflies at Sand Fly Point on the Milford Track is proportional to the excess of the population over 2000. This can be expressed as $\frac{dP}{dt}=k(P-2000)$ where k is a constant and t represents time in weeks.

(i) Show that $P=2000+Ae^{kt}$, where A is a constant, satisfies the differential equation $\frac{dP}{dt}=k(P-2000)$. 1

(ii) Initially the population is 2500 and 2 weeks later it is 5000. Find the value of A and k. 2

(iii) Find the population after 8 weeks. 1

(c) $S_n=\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\ldots+\frac{1}{n(n+1)(n+2)}$ for all positive integers.

(i) Use mathematical induction to prove that $S_n=\frac{n(n+3)}{4(n+1)(n+2)}$ 4

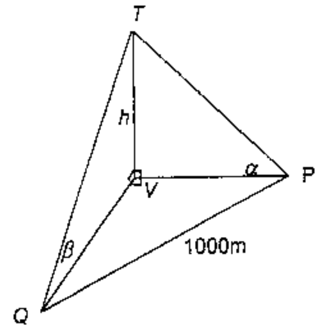
(ii) What is $\lim_{n\to\infty} S_n$? 1

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) Show that the area A of an equilateral triangle of side x cm is given by $A=\frac{\sqrt{3}x^2}{4}$. 1

(ii) The length of the sides of an equilateral triangle are increasing at the rate of $\frac{1}{6}cms^{-1}$. At what rate is the area increasing at the instant when the sides are 12 cm? 2

(b) In the diagram below, the angle of elevation of the top of a Telstra tower, T, from a point P, due east of the tower is α. From a point Q, due south of the tower the angle of elevation to the top of the tower is β. The distance from P to Q is 1000 m and the tower's height is h metres.



(i) Show that $PV=\frac{h}{\tan\alpha}$ 1

(ii) Show that the height of the tower is given by $h=\frac{1000\tan\alpha\ \tan\beta}{\sqrt{\tan^2\alpha+\tan^2\beta}}$ 3

(iii) The angles of elevation of the top of the tower from the points P and Q are 36° and 45° respectively. Calculate the height of the tower. 1

(c) A particle is moving in a straight line. The acceleration of the particle when it is x metres from a fixed point O is given by $\ddot{x}=6x^2$. Initially $x=1m$ and velocity $v=-2ms^{-1}$.

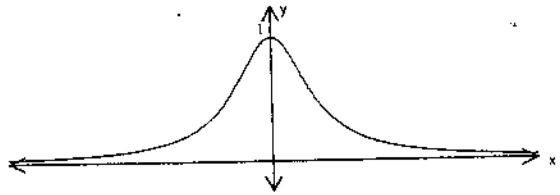
(i) Find v^2 as a function of displacement, x. 2

(ii) Explain why velocity can never be positive. 2

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $(x-2)$ is a factor of x^3+2x^2-kx-6 find the value of k. **2**

(b) Consider the function $f(x)=\dfrac{1}{1+x^2}$.



(i) Write down the largest domain that contains $x=1$ for which $f(x)$ has an inverse function. **1**

(ii) Find the inverse function $f^{-1}(x)$ for this domain and state the domain of $f^{-1}(x)$. **3**

(c) A particle P is projected from a point on horizontal ground with velocity V at an angle of projection α.

You may assume that the equations of motion are:

$$\ddot{x}=0 \qquad \ddot{y}=-g$$
$$\dot{x}=V\cos\alpha \qquad \dot{y}=V\sin\alpha-gt$$
$$x=Vt\cos\alpha \qquad y=Vt\sin\alpha-\frac{1}{2}gt^2$$

(i) Show that the particle's maximum height is $\dfrac{V^2\sin^2\alpha}{2g}$. **2**

(ii) A second particle Q is projected from the same point on horizontal ground with velocity $\sqrt{\frac{5}{2}}\,V$ at an angle $\frac{\alpha}{2}$ to the horizontal. Both particles reach the same maximum height.
Show that $\alpha=\cos^{-1}\dfrac{1}{4}$. **4**

End of paper

YEAR 12 TRIAL HSC 2001 - EXTENSION 1 SOLUTIONS

Question 1

(a)

$y = 2x + 3 \qquad m_1 = 2$

$y = 4x + 1 \qquad m_2 = 4$

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\dfrac{2-4}{1+2\times 4}\right|$

$= \dfrac{2}{9}$

$\theta \doteq 13^\circ$

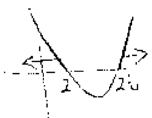
(b) $\dfrac{1}{x-2} \leq 4 \qquad \boxed{x \neq 2}$

$(x-2) \leq 4(x-2)^2$

$4(x-2)^2 - (x-2) \geq 0$

$(x-2)(4x-9) \geq 0$

$x < 2,\ x \geq 2\frac{1}{4}$



(c) $A(-1, 4) \quad B(x, y) \quad 3 : -2$

$8 = \dfrac{-2\times -1 + 3x}{3-2} \qquad -5 = \dfrac{-2\times 4 + 3y}{3-2}$

$8 = 2 + 3x \qquad -5 = -8 + 3y$

$x = 2 \qquad y = 1$

$B(2, 1)$

(d) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1} 1 - \sin^{-1} 0$

$= \dfrac{\pi}{2}$

(e) $\sin\theta = \cos 2\theta$

$\sin\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}, \quad \sin\theta = -1$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}$

Question 2

(a) $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

$u = \ln x$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$x = e \quad u = \ln e = 1$

$x = e^2 \quad u = \ln e^2 = 2\ln e = 2$

$= \displaystyle\int_1^2 \frac{1}{u}\frac{du}{dx}\,dx$

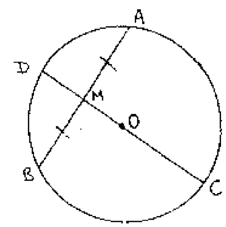
$= \displaystyle\int_1^2 \frac{1}{u}\,du$

$= \left[\ln u\right]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2$

(b)



(i) $AM = BM$ given

Line through the centre that bisects the chord is perpendicular to the chord.

$\therefore AB \perp CD$.

(ii) $AM \times BM = CM \times DM$
product of intercepts of intersecting chords.

$5 \times 5 = CM \times 2$

$CM = 12\frac{1}{2}$

$\therefore DC = 14\frac{1}{2}$ (diameter)

radius $OC = 7\frac{1}{4}$ cm

(iii) Area $= 2 \times$ Area $\triangle ADC$

$= 2 \times \frac{1}{2} \times 14\frac{1}{2} \times 5$

$= 72\frac{1}{2}$ cm^2

(c) $f(x) = e^x + x - 5$

(i) $f(x)$ is a continuous function

$f(1) = e^1 + 1 - 5$

$= -1.28 < 0$

$f(2) = e^2 + 2 - 5$

$= 4.39 > 0$

Since $f(x)$ is continuous and there is a sign change between $x = 1$ and $x = 2$ there is a root in the interval $1 < x < 2$.

(ii) $f(x) = e^x + x - 5$

$f'(x) = e^x + 1$

$x_1 = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$

$= 1.5 - \dfrac{0.981689\ldots}{5.481689\ldots}$

$\doteq 1.3$

Question 3

(a) $f(x) = 2\cos^{-1}\left(\frac{x}{4}\right)$

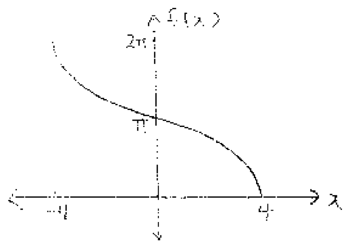
(i) $-1 \leq \frac{x}{4} \leq 1$

Domain $-4 \leq x \leq 4$

$0 \leq \cos^{-1}\left(\frac{x}{4}\right) \leq \pi$

Range $0 \leq 2\cos^{-1}\left(\frac{x}{4}\right) \leq 2\pi$

(ii)



(b) $x = 5\cos 6t$

(i) $\dot{x} = -30\sin 6t$

$\ddot{x} = -180\cos 6t$

$= -36(5\cos 6t)$

$= -6^2 x$

Since $\ddot{x}$ is of the form $-n^2 x$ the motion is simple harmonic.

(ii) amplitude 5, centre of motion 0

∴ endpoints $x = \pm 5$

period $= \frac{2\pi}{6}$

$= \frac{\pi}{3}$ seconds

(iii) $\frac{1}{2}$ way $\Rightarrow x = 2.5$

$2.5 = 5\cos 6t$

$\cos 6t = \frac{1}{2}$

$6t = \frac{\pi}{3}$ (first time)

$t = \frac{\pi}{18}$ seconds

(c)

Area = A(rectangle) − A(to y-axis)

$= \frac{\pi}{2} \times 1 - \int_0^{\frac{\pi}{2}} \sin y \, dy$

$= \frac{\pi}{2} - \left[-\cos y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - \left(-\cos\frac{\pi}{2} + \cos 0\right)$

$= \frac{\pi}{2} - 1$ sq. units

Question 4

(a) $t = \tan\frac{x}{2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$

$\frac{1+\cos x}{1-\cos x} = \frac{1 + \frac{1-t^2}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}}$

$= \frac{\frac{1+t^2+(1-t^2)}{1+t^2}}{\frac{1+t^2-(1-t^2)}{1+t^2}}$

$= \frac{2}{2t^2}$

$= \frac{1}{\tan^2\frac{x}{2}}$

$= \cot^2\frac{x}{2}$

(b) Initially $\dot{x} = 20$, $x = 0$

$\dot{y} = 0$, $y = 80$

(i)

Horizontal	Vertical
$\ddot{x} = 0$	$\ddot{y} = -10$
$\dot{x} = C_1$	$\dot{y} = -10t + C_3$
when $t=0$ $\dot{x} = 20$	when $t=0$ $\dot{y} = 0$
∴ $C_1 = 20$	∴ $C_3 = 0$
$\dot{x} = 20$	$\dot{y} = -10t$
$x = 20t + C_2$	$y = -5t^2 + C_4$
when $t=0$, $x=0$	when $t=0$, $y=80$
∴ $C_2 = 0$	∴ $C_4 = 80$
∴ $x = 20t$	$y = 80 - 5t^2$

(ii) when $y = 35$

$35 = 80 - 5t^2$

$5t^2 = 45$

$t^2 = 9$

$t = 3$ $(t > 0)$

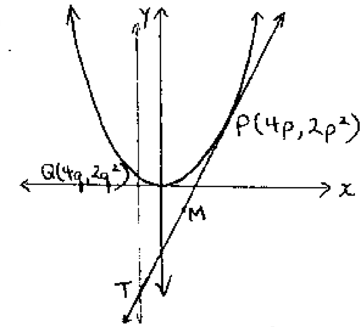
When $t = 3$

$x = 20t$

$= 60$

60 m from cliff to tower

(c)



(i) $x^2 = 8y \Rightarrow y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}$

$P(4p, 2p^2)$ $m_T = \frac{4p}{4} = p$

equation $y - 2p^2 = p(x - 4p)$

$y - 2p^2 = px - 4p^2$

$y = px - 2p^2$

(ii) through Q, ∥ to y axis

∴ $x = 4q$

sub into eq. tangent

$y = 4pq - 2p^2$

$T(4q, 4pq - 2p^2)$

(iii) $M = \left(\frac{4p+4q}{2}, \frac{2p^2 + 4pq - 2p^2}{2}\right)$

$= (2p + 2q, 2pq)$

(iv) when $pq = -1$

$y = 2pq$

$= -2$

∴ locus of M is $y = -2$

tion 5

(i) $x^2 + 8x + 20 = (x+4)^2 + 4$

(ii) $\int \frac{dx}{x^2+8x+20} = \int \frac{dx}{(x+4)^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+4}{2}\right) + C$

(i) $P = 2000 + Ae^{kt}$

$\frac{dP}{dt} = Ake^{kt}$

$= k(Ae^{kt})$

$= k(P - 2000)$

) $t = 0 \quad P = 2500$

$2500 = 2000 + Ae^0$

$A = 500$

$t = 2 \quad P = 5000$

$5000 = 2000 + 500e^{2k}$

$e^{2k} = 6$

$k = \frac{1}{2}\ln 6$

$\doteqdot 0.8958\ldots$

) $P = 2000 + 500e^{4\ln 6}$

$= 650\,000$

(i) Show true for $n = 1$

$S_1 = \frac{1}{1\times 2\times 3} = \frac{1}{6}$

$\frac{1(1+3)}{4(1+1)(1+2)} = \frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{4\times 2\times 3} = \frac{1}{6}$

true for $n = 1$

Assume true for $n = k$

$S_k = \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \ldots + \frac{1}{k(k+1)(k+2)}$

$= \frac{k(k+3)}{4(k+1)(k+2)}$

Show true for $n = k+1$

$S_{k+1} = \frac{1}{1\times 2\times 3} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$

$S_{k+1} = S_k + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{1}{(k+1)(k+2)}\left[\frac{k(k+3)}{4} + \frac{1}{k+3}\right]$

$= \frac{1}{(k+1)(k+2)}\left(\frac{k(k+3)^2+4}{4(k+3)}\right)$

$= \frac{1}{(k+1)(k+2)}\left(\frac{k^3+6k^2+9k+4}{4(k+3)}\right)$

$= \frac{1}{(k+1)(k+2)}\left(\frac{(k+1)(k^2+5k+4)}{4(k+3)}\right)$

$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$ as required

∴ true for $n=k+1$ if true for $n=k$

Since true for $n=1$ also true for $n=1+1=2$, thus true for $n=3$ and for $n=4$ and so on for all positive integers.

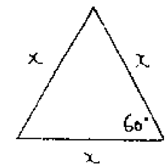
(ii) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n^2+3n}{4n^2+12n+8}$

$= \lim_{n\to\infty} \frac{1+\frac{3}{n}}{4+\frac{12}{n}+\frac{8}{n^2}}$

$= \frac{1}{4}$

Question 6

(a)



(i) $A = \frac{1}{2}\times x \times x \times \sin 60°$

$= \frac{\sqrt{3}\,x^2}{4}$

(ii) $\frac{dx}{dt} = \frac{1}{6} \qquad \frac{dA}{dx} = \frac{\sqrt{3}\,x}{2}$

$\frac{dA}{dt} = \frac{dA}{dx}\times\frac{dx}{dt}$

$= \frac{\sqrt{3}\times 12}{2}\times\frac{1}{6}$

$= \sqrt{3}\ \text{cm}^2\,\text{s}^{-1}$

(b) (i) In ΔPVT

$\tan\alpha = \frac{h}{PV}$

$PV = \frac{h}{\tan\alpha}$

(ii) Similarly for ΔQVT

$QV = \frac{h}{\tan\beta}$

$PV^2 + QV^2 = 1000^2$

$\frac{h^2}{\tan^2\alpha} + \frac{h^2}{\tan^2\beta} = 1000^2$

$h^2\left(\frac{\tan^2\beta+\tan^2\alpha}{\tan^2\alpha\tan^2\beta}\right) = 1000^2$

$h^2 = \frac{1000^2\tan^2\alpha\tan^2\beta}{\tan^2\alpha+\tan^2\beta}$

$h = \frac{1000\tan\alpha\tan\beta}{\sqrt{\tan^2\alpha+\tan^2\beta}}$

(iii) $\alpha = 36° \quad \beta = 45°$

$h = \frac{1000\tan 36\tan 45}{\sqrt{\tan^2 36+\tan^2 45}}$

$= 588$ m (nearest metre)

(c) (i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^2$

$\frac{1}{2}v^2 = 2x^3 + c$

$x = 1, \quad v = -2$

$\frac{1}{2}(-2)^2 = 2(1)^3 + c$

$2 = 2 + c$

$c = 0$

$\frac{1}{2}v^2 = 2x^3$

$v^2 = 4x^3$

(ii) Initially particle moving left, velocity negative.

Particle moves from $x=1$ towards $x=0$

If $x=0 \quad \ddot{x}=0$

$v^2 = 0 \quad (v=0)$

no acceleration ∴ if particle reaches origin it stops and no acceleration for it to start again

∴ velocity won't change from negative to positive.

[$x=0$ is limiting position of the particle

...stion 7

a) If ... factor $P(2)=0$

$P(2) = 2^3 + 2(2^2) - 2k - 6$

$10 - 2k = 0$

$k = 5$

b) (i) $x \geq 0$

(ii) Inverse $x = \frac{1}{1+y^2}$

$1+y^2 = \frac{1}{x}$

$y^2 = \frac{1}{x} - 1$

$y = \pm\sqrt{\frac{1-x}{x}}$

positive case, from (i) $y \geq 0$

$f^{-1}(x) = \sqrt{\frac{1-x}{x}}$

domain: $0 < x \leq 1$

c) (i) for max ht, $\dot{y} = 0$

$V\sin\alpha - gt = 0$

$gt = V\sin\alpha$

$t = \frac{V\sin\alpha}{g}$

$y = V \cdot \frac{V\sin\alpha}{g} \cdot \sin\alpha - \frac{1}{2}g \cdot \frac{V^2\sin^2\alpha}{g^2}$

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$

$= \frac{V^2\sin^2\alpha}{2g}$

(ii) max. ht. of Q is

$y = \frac{\left(\sqrt{\frac{5}{2}}V\right)^2 \sin^2\frac{\alpha}{2}}{2g}$

$y = \frac{5V^2\sin^2\frac{\alpha}{2}}{4g}$

Both reach same max ht.

$\frac{5V^2\sin^2\frac{\alpha}{2}}{4g} = \frac{V^2\sin^2\alpha}{2g}$

$5\sin^2\frac{\alpha}{2} = 2\sin^2\alpha$

$5\left(\frac{1}{2}(1-\cos\alpha)\right) = 2(1-\cos^2\alpha)$

$5 - 5\cos\alpha = 4 - 4\cos^2\alpha$

$4\cos^2\alpha - 5\cos\alpha + 1 = 0$

$(4\cos\alpha - 1)(\cos\alpha - 1) = 0$

$\cos\alpha = \frac{1}{4}$, $\cos\alpha = 1$

$\alpha = \cos^{-1}\frac{1}{4}$ — no soln ($\alpha = 0$ not possible)