ABBOTSLEIGH

August 2002
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks **(84)**
- Attempt Questions 1-7.
- All questions are of equal value.

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Attempt Questions 1-7
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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Differentiate

(i) $\log_e\left(3x^2+2\right)$ **(1)**

(ii) $\left(1+x^2\right)\tan^{-1}x$. **(2)**

(b) Solve the inequality $\frac{2x}{x-2} \le 3$ **(3)**

(c) Evaluate exactly $\int_1^{\sqrt{3}} \frac{dt}{\sqrt{4-t^2}}$ **(2)**

(d) Using the substitution $u = 4-x$ evaluate $\int_3^4 x\sqrt{4-x}\,dx$. **(4)**

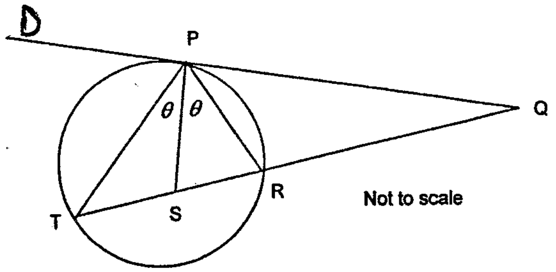
QUESTION 2 (12 Marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\pi} \cos^2 x \, dx$ **(3)**

(b) Show that $x+1$ is a factor of x^3-4x^2+x+6.
Hence or otherwise, factorise x^3-4x^2+x+6 fully. **(3)**

(c) The equation $x^3+2x-8=0$ has a root close to $x=1.6$. Use one application of Newton's method to find a better approximation to the root. (Give your answer to 2 decimal places). **(3)**

(d)



In the diagram the vertices of triangle PTR lie on a circle. The tangent at P meets the secant TR produced at Q. The bisector of $\angle TPR$ meets TR at S.

Copy the diagram into your booklet.
Prove that $PQ=SQ$. **(3)**

QUESTION 3 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) State the domain and range of $y=3\cos^{-1}2x$ **(2)**

(ii) Find the value of y if $x=\frac{1}{4}$ **(1)**

(iii) Sketch the graph of $y=3\cos^{-1}2x$. **(1)**

(b) Let α, β, γ be the roots of the polynomial $3x^3-12x^2-8=0$.
Evaluate $\alpha\beta\gamma$. **(2)**

(c) If $\sin A=\frac{2}{3}$ and $\frac{\pi}{2}<A<\pi$, find the exact value of $\sin 2A$ **(2)**

(d) The acceleration of a particle x metres from 0 at time t seconds is given by

$$\frac{d^2x}{dt^2}=-e^{-2x}$$

If the velocity is 1 metre per second when $x=0$, find the exact velocity when $x=4$ metres. **(4)**

QUESTION 4 (12 Marks) Use a SEPARATE writing booklet.

Marks

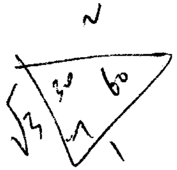
(a) Solve $\sqrt{3}\cos x+\sin x=1$ *for* $0\le x\le 2\pi$. **(4)**

(b) (i) Explain why the function $f(x)=\sqrt{x-2}$ has an inverse function $f^{-1}(x)$. **(1)**

(ii) Write down the equation of the inverse function $f^{-1}(x)$ and sketch both $y=f(x)$ and $y=f^{-1}(x)$ on the same set of axes. **(3)**

(c) (i) Express $\sin A$ and $\cos A$ in terms of t where $t=\tan\frac{A}{2}$. **(1)**

(ii) Hence or otherwise prove that $\frac{\sin 2A}{1+\cos 2A}=\tan A$. **(3)**



QUESTION 5 (12 Marks) Use a SEPARATE writing booklet.

Marks

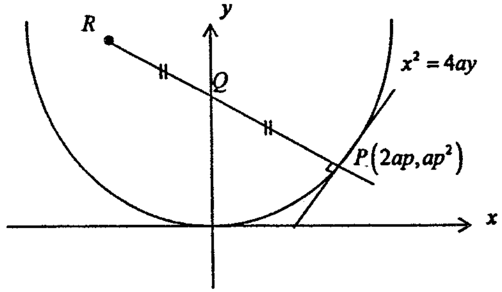
(a) Given that $f(x)=\frac{x}{4-x^2}$

(i) Determine whether $f(x)$ is odd, even or neither. **(1)**

(ii) Show that $f(x)$ has no stationary points. **(3)**

(iii) Find any horizontal or vertical asymptotes. **(2)**

(b)



The normal at $P(2ap,ap^2)$ on the parabola $x^2=4ay$ cuts the y-axis at Q and is produced to a point R such that $PQ=QR$.

(i) Given that the equation of the normal at P is $x+py=2ap+ap^3$, find the coordinates of Q. **(1)**

(ii) Show that R has coordinates $(-2ap,ap^2+4a)$. **(2)**

(iii) Show that the locus of R is a parabola and state its vertex. **(3)**

QUESTION 6 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) A point moves along the curve $y=\frac{1}{x}$ such that the x coordinate is changing at the rate of 2 units per second. At what rate is the y coordinate decreasing when $x=5$? **(3)**

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(b) Molten metal at a temperature of 1400 °C is poured into moulds to form machine parts. After 15 minutes the metal has cooled to 995°C. If the temperature after t minutes is T °C, and if the temperature of the surroundings is 35°C, then the rate of cooling is approximately given by

$$\frac{dT}{dt}=-k(T-35)$$

where k is a positive constant.

(i) Show that a solution of this equation is $T=35+Ae^{-kt}$ where A is a constant. **(1)**

(ii) Find the values of A and k. **(3)**

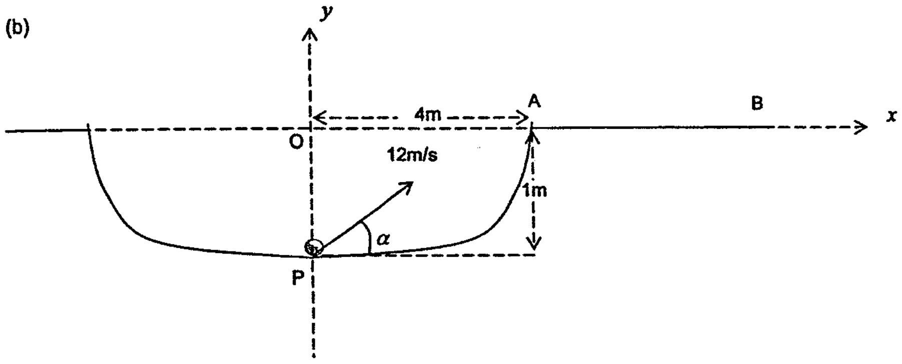
(iii) The metal can be taken out of the moulds when its temperature has dropped to 200°C. How long after the metal has been poured will this temperature be reached? **(2)**

(c) Prove by mathematical induction that $2^{3n}-3^n$ is divisible by 5 for all positive integers n. **(3)**

QUESTION 7 (12 Marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\lim_{x\to 0}\frac{3x}{\tan 4x}$ **(1)**

(b)



A golf ball is lying at point P, at the middle of the bottom of a sand bunker which is surrounded by level ground. The point A is at the edge of the bunker $4m$ from O and AB lies on level ground. The initial velocity is $12m/s$ and P is $1m$ below O.

(i) Using $g=-10m/s^2$, show that the golf ball's trajectory at time t seconds after being hit may be defined by the equations:

$$x=(12\cos\alpha)t \quad \text{and} \quad y=-5t^2+(12\sin\alpha)t-1$$

where x and y are the horizontal and vertical displacements, in metres, of the ball from the origin O shown in the diagram, and α is the angle of projection. **(3)**

(ii) Given $\alpha=30^\circ$, how far from A will the ball land? **(3)**

(iii) Find the range of values of α, to the nearest degree, at which the ball must be hit so that it will land to the right of A. **(4)**

END OF PAPER

Year 12 Mathematics Extension1 Trial Solutions 2002

1) a) (i) $\underline{\underline{\dfrac{6x}{3x^2+2}}}$

(ii) $(1+x^2)\times\dfrac{1}{1+x^2} + \tan^{-1}x\times 2x$

$= \underline{\underline{1 + 2x\tan^{-1}x}}$

b) $\dfrac{2x}{x-2} \leq 3 \quad x\neq 2$

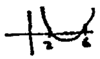
$(x-2)^{\cancel{2}}\times\dfrac{2x}{\cancel{x-2}} \leq 3(x-2)^2$

$3(x-2)^2 - 2x(x-2) \geq 0$

$(x-2)(3(x-2)-2x)\geq 0$

$(x-2)(x-6)\geq 0$

$\underline{\underline{x<2 \text{ or } x\geq 6}}$



c) $\displaystyle\int_1^{\sqrt3}\frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1}\left(\frac{t}{2}\right)\right]_1^{\sqrt3}$

$= \sin^{-1}\frac{\sqrt3}{2} - \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{3}-\frac{\pi}{6}$

$= \underline{\underline{\frac{\pi}{6}}}$

d) $\displaystyle\int_3^4 x\sqrt{4-x}\,dx$

$u = 4-x$
$du = -dx$
$x=3 \ u=1$
$x=4 \ u=0$

$= \displaystyle\int_1^0 -(4-u)\sqrt{u}\,du$

$= \displaystyle\int_0^1 4u^{\frac12} - u^{3/2}\,du$

$= \left[2\times\frac{4u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right]_0^1$

$= \left(\frac{8\times1^{3/2}}{3} - \frac{2\times1^{5/2}}{5}\right) - (0-0)$

$= \frac{8}{3} - \frac{2}{5} = \underline{\frac{34}{15} \text{ or } 2\frac{4}{15}}$

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② a) $\displaystyle\int_0^{\pi}\cos^2x\,dx = \int_0^{\pi}\frac12(1+\cos 2x)\,dx$

$= \frac12\left[x+\frac{\sin 2x}{2}\right]_0^{\pi}$

$= \frac12\left[\left(\pi+\frac{\sin 2\pi}{2}\right) - \left(0+\frac{\sin 0}{2}\right)\right]$

$= \frac12\left[\pi+0\right] - 0$

$= \underline{\underline{\frac{\pi}{2}}}$

b) $f(x) = x^3-4x^2+x+6$

$f(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$

$= -1-4-1+6$

$= 0$

$\therefore x+1$ is a factor of $f(x)$

$f(2) = 8-16+2+6$

$= 0$

$\therefore (x-2)$ is a factor

$\therefore$ Third factor must be $(x-3)$ as $1\times -2\times -3 = 6$

$\underline{\underline{f(x) = (x+1)(x-2)(x-3)}}$

or

$$\begin{array}{r} x^2-5x+6 \\ x+1\,\overline{)\,x^3-4x^2+x+6} \\ \underline{x^3+x^2} \\ -5x^2+x \\ \underline{-5x^2-5x} \\ 6x+6 \\ \underline{6x+6} \\ \cdot\cdot \end{array}$$

c) $f(x) = x^3+2x-8$

$f'(x) = 3x^2+2$

$f(1.6) = (1.6)^3+2(1.6)-8 = -0.704$

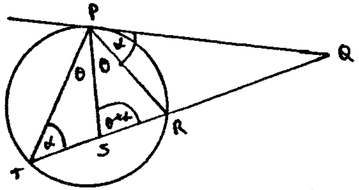
$f'(1.6) = 3(1.6)^2+2 = 9.68$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1.6 - \dfrac{-0.704}{9.68} = 1.6727273$

$\underline{\underline{= 1.67}}$

d)



$\alpha = \angle QPR = \angle PTR$ (angle betw tangent & chord equals angle in alternate segment)

$\angle PSR = \alpha+\theta$ (exterior angle of Δ = sum of 2 interior opp angles)

$\angle QPS = \alpha+\theta$ (by addition)

$\therefore \underline{PQ = SQ}$ (sides opp equal angles)

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③ a) (i) Domain of $\cos^{-1}x \quad -1\leq x\leq 1$

Range of $\cos^{-1}x \quad 0\leq y\leq \pi$

$\frac{y}{3} = \cos^{-1}2x$

Domain $\quad -1\leq 2x\leq 1$

$\therefore \underline{-\frac12\leq x\leq\frac12}$

Range $\quad 0\leq\frac{y}{3}\leq\pi$

$\underline{0\leq y\leq 3\pi}$

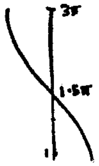
(ii) $y = 3\cos^{-1}\left(\frac{2}{4}\right)$

$= 3\cos^{-1}\left(\frac12\right)$

$= 3\times\frac{\pi}{3}$

$= \pi$

(iii)



b) $\alpha\beta\gamma$ = product of roots

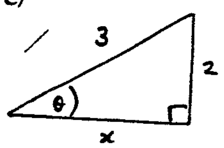
$3x^3 - 12x^2 + 0x - 8 = 0$

$\ - \quad + \quad -$

$\alpha\beta\gamma = -\frac{d}{a}$

$\underline{\alpha\beta\gamma = \frac{8}{3}}$

c)



By Pythagoras

$9 = 2^2+x^2$

$x^2 = 5$

$x = \sqrt5$

$\frac{\pi}{2} < A < \pi$ 2nd quadrant

$\sin A = \frac{2}{3} \quad \cos A = \frac{-\sqrt5}{3}$

$\sin 2A = 2\sin A\cos A$

$= 2\times\frac{2}{3}\times\frac{-\sqrt5}{3}$

$= \underline{\underline{\frac{-4\sqrt5}{9}}}$

d) $\dfrac{d^2x}{dt^2} = -e^{-2x}$

acc $= \dfrac{d}{dx}\left(\frac12 v^2\right)$

$\dfrac{d}{dx}\left(\frac12 v^2\right) = -e^{-2x}$

$\frac12 v^2 = \dfrac{+e^{-2x}}{2} + c$

When $x=0 \quad v=1$

$\frac12 = \frac{e^0}{2} + c$

$\therefore c = 0$

$\frac12 v^2 = \frac{e^{-2x}}{2}$

$v^2 = e^{-2x}$

$v = \pm\sqrt{e^{-2x}}$

$v = e^{-x}$ (take +ve as $v=1$ when $x=0$)

When $x=4 \quad v = e^{-4}$

$v = \frac{1}{e^4}$ metres per second

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④ (a) $\sqrt3\cos x + \sin x = 1 \qquad 0\leq x\leq 2\pi$

Let $\sqrt3\cos x + \sin x = A\cos(x-\alpha)$

$= A\cos x\cos\alpha + A\sin x\sin\alpha$

$A\cos\alpha = \sqrt3 \quad (1)$

$A\sin\alpha = 1 \quad (2)$

$(1)^2+(2)^2 \quad A^2(\sin^2\alpha+\cos^2\alpha) = (\sqrt3)^2+1^2$

$A^2 = 4$

$A = \pm 2$ take positive $A = 2$

$(2)\div(1) \quad \tan\alpha = \frac{1}{\sqrt3}$ First quadrant as $\sin\alpha > 0$

$\alpha = \frac{\pi}{6}$

$2\cos\left(x-\frac{\pi}{6}\right) = 1$

$0\leq x\leq 2\pi$

$0-\frac{\pi}{6}\leq x-\frac{\pi}{6}\leq 2\pi-\frac{\pi}{6}$

$x-\frac{\pi}{6} = \frac{\pi}{3}$ or $2\pi-\frac{\pi}{3}$

Basic angle $= \frac{\pi}{3}$

$x-\frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{3}+\frac{\pi}{6}, \frac{5\pi}{3}+\frac{\pi}{6}$

$= \frac{3\pi}{6}, \frac{11\pi}{6}$

$\underline{\underline{x = \frac{\pi}{2}, \frac{11\pi}{6}}}$

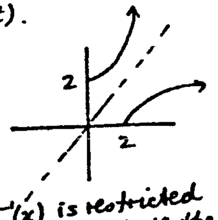
b) (i) $y = \sqrt{x-2}$ has an inverse function because it is a one-to-one function. (horizontal line test).

(ii) $x = \sqrt{y-2}$

$x^2 = y-2$

$y = x^2+2$

$f^{-1}(x) = x^2+2$ for $x>0$



Inverse function $f^{-1}(x)$ is restricted to $x>0$ since it it is only half the para

c) (i) $t = \tan\frac{A}{2} \quad \sin A = \dfrac{2t}{1+t^2} \quad \cos A = \dfrac{1-t^2}{1+t^2}$

(ii) $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

Let $t = \tan A$ from above

$\sin 2A = \dfrac{2t}{1+t^2} \quad \cos 2A = \dfrac{1-t^2}{1+t^2}$

LHS $= \dfrac{2t}{1+t^2} \div \left(1+\dfrac{1-t^2}{1+t^2}\right)$

$= \dfrac{2t}{1+t^2} \div \left(\dfrac{1+t^2+1-t^2}{1+t^2}\right)$

$= \dfrac{\cancel{2}t}{\cancel{1+t^2}} \times \dfrac{\cancel{1+t^2}}{\cancel{2}}$

$= t$

$= \tan A$

$=$ RHS

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⑤ a) $f(x) = \dfrac{x}{4-x^2}$

(i) $f(-x) = \dfrac{-x}{4-(-x)^2} = \dfrac{-x}{4-x^2} = -f(x)$

$\therefore$ odd function

(ii) $f'(x) = \dfrac{(4-x^2)1 - x(-2x)}{(4-x^2)^2}$

$= \dfrac{4-x^2+2x^2}{(4-x^2)^2}$

$= \dfrac{4+x^2}{(4-x^2)^2}$

$\underline{4+x^2}$ $\neq 0$ since $4+x^2>0$ for all values of x (since x^2 is always positive)

$\therefore$ since $f'(x)\neq 0$ there are no stat pts.

) As $x\to\infty$ $\lim_{x\to\infty}\frac{x}{4-x^2}$

$$=\lim_{x\to\infty}\frac{\frac{x}{x^2}}{\frac{4}{x^2}-1}=\frac{\frac{1}{x}}{\frac{4}{x^2}-1}$$

$$=\frac{0}{-1}=0$$

$y=0$ is an asymptote (horizontal)

Denominator $4-x^2\neq 0$

$(2-x)(2+x)=0$

$x=2,\ x=2$ are asymptotes (vertical)

)(i) $x+py=2ap+ap^3$

sub in $x=0$ $\quad py=2ap+ap^3$

$y=2a+ap^2$

$Q\ (0,\ 2a+ap^2)$

(ii) $R(x,y)$ $\quad Q(0,2a+ap^2)$ $\quad P(2ap,ap^2)$

Q is midpt so

$\frac{x+2ap}{2}=0$ and $\frac{y+ap^2}{2}=2a+ap^2$

$x+2ap=0$ $\quad y+ap^2=4a+2ap^2$

$x=-2ap$ $\quad y=4a+ap^2$

$\therefore R$ is $(-2ap,\ 4a+ap^2)$

(iii) $x=-2ap$ $\quad y=4a+ap^2$

$p=\frac{-x}{2a}$ sub into y

$y=4a+a\left(-\frac{x}{2a}\right)^2=4a+\frac{x^2}{4a^2}$

$y=\frac{16a^3+ax^2}{4a^2}$

$y=\frac{16a^2+x^2}{4a}$ $\quad 4ay=x^2+16a^2$

$x^2=4a(y-4a)$

bola, vertex $(0,4a)$.

)(a) $\frac{dx}{dt}=2$

$y=\frac{1}{x}=x^{-1}$

$\frac{dy}{dx}=-x^{-2}$

$=\frac{-1}{x^2}$

$\frac{dy}{dt}=\frac{dx}{dt}\times\frac{dy}{dx}$

$=2\times-\frac{1}{x^2}$

When $x=5$

$\frac{dy}{dx}=2\times-\frac{1}{25}$

$-?$

b)(i) $T=35+Ae^{-kt}$ $\quad$ (1)

$\frac{dT}{dt}=-kAe^{-kt}$

$=-k(T-35)$ from (1)

$\therefore T=35+Ae^{-kt}$ is a soln to $\frac{dT}{dt}=-k(T-35)$

(ii) $t=0$ $T=1400$

$t=15$ $T=995$

When $t=0$ $\quad 1400=35+Ae^0$

$\underline{A=1365}$

$t=15$ $\quad 995=35+1365e^{-15k}$

$\frac{960}{1365}=e^{-15k}$

$\log_e\left(\frac{960}{1365}\right)=15k$

$k=\frac{-1}{15}\log_e\left(\frac{64}{91}\right)$

$\underline{k=0.023465094}$

(iii) $T=200$ $\quad t=?$

$200=35+1365e^{-0.023465..t}$

$\frac{165}{1365}=e^{-0.0234654t}$

$t=\frac{\ln\left(\frac{165}{1365}\right)}{0.02346..}=9.04712$

It will take $\underline{90 \text{ minutes}}$

c) $2^{3n}-3^n$ is divisible by 5

Prove true for $n=1$

$2^3-3^1=8-3=5$

$\therefore$ True for $n=1$

Let it be true for $n=k$

$2^{3k}-3^k=5m$ where m is a positive integer

$\therefore 2^{3k}=5m+3^k$

Prove true for $n=k+1$

$$2^{3(k+1)}-3^{k+1}=2^{3k}\times2^3-3^{k+1}$$

$$=(5m+3^k)\times8-3^k\times3^1$$

$$=40m+8\times3^k-3\times3^k$$

$$=40m+5\times3^k$$

$=5(8m+3^k)$ which is divisible by 5 if m is a positive integer.

If it is true for $n=k$ we have proven it true for $n=k+1$. Since it is true for $n=1$, then it is true for $n=1+1=2$ and so on for all positive integral n.

) a) $\lim_{x\to0}\frac{3x}{\tan4x}=\frac{3}{4}\lim_{x\to0}\frac{4x}{\tan4x}$

$=\frac{3}{4}\times1$

$=\underline{\underline{\frac{3}{4}}}$

b)(i) $\ddot{x}=0$

$\dot{x}=c_1$

When $t=0$ $\dot{x}=V\cos\alpha$

$=12\cos\alpha$

$\therefore c_1=12\cos\alpha$

$\dot{x}=12\cos\alpha$

$x=12t\cos\alpha+c_2$

When $t=0$ $x=0$ $\therefore c_2=0$

$x=(12\cos\alpha)t$

$\ddot{y}=-10$

$\dot{y}=-10t+c_3$

When $t=0$ $\dot{y}=V\sin\alpha$

$=12\sin\alpha$

$\therefore c_3=12\sin\alpha$

$y=-5t^2+12t\sin\alpha+c_4$

When $t=0$ $y=-1$ (starts at bottom of bunker).

$\therefore c_4=-1$

$y=-5t^2+(12\sin\alpha)t-1$

(ii) $\alpha=30°$. Ball will hit ground when $y=0$

$y=-5t^2+(12\sin\alpha)t-1$

$0=-5t^2+(12\sin30°)t-1$

$0=-5t^2+6t-1$

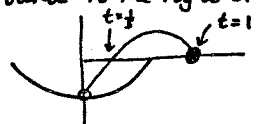
$5t^2-6t+1=0$

$(5t-1)(t-1)=0$

$t=\frac{1}{5}$ or $t=1$

$t=\frac{1}{5}$ gives first time ball crosses x axis which is not on the ground (it is left of A).

$t=1$ gives the time the ball hits ground to the right of A



When $t=1$ $\quad x=(12\cos30°)\times1$

$=12\times\frac{\sqrt3}{2}$

$=6\sqrt3$

But $OA=4$ metres, so ball will land

(iii) For the ball to land to the right of A look at angle necessary to go through A

A is $(4,0)$

$4=(12\cos\alpha)t$

$t=\frac{4}{12\cos\alpha}$

$=\frac{1}{3\cos\alpha}$

sub into $y=-5t^2+(12\sin\alpha)t-1$

when $y=0$

$0=-5\left(\frac{1}{9\cos^2\alpha}\right)+(12\sin\alpha)\left(\frac{1}{3\cos\alpha}\right)-1$

$0=-\frac{5}{9}\sec^2\alpha+\frac{4\sin\alpha}{\cos\alpha}-1$

$0=-5(1+\tan^2\alpha)+36\tan\alpha-9$ $\quad(\times9)$

$=-5-5\tan^2\alpha+36\tan\alpha-9$

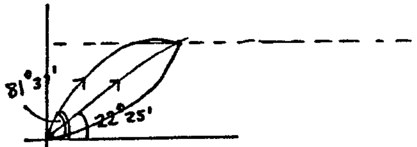
$5\tan^2\alpha-36\tan\alpha+14=0$

Use formula

$\tan\alpha=\frac{36\pm\sqrt{36^2-4\times5\times14}}{10}$

$=0.4125245$ or 6.78

$\alpha=22°25'$ or $81°37'$ (first quadrant only)



Anything less than $22°25'$ or bigger than $81°37'$ will hit the bank of the bunker

So, to land to the right of A

$23°\leqslant\alpha\leqslant81°$ (to nearest degree)