Student Number___________________

ABBOTSLEIGH

AUGUST 2007
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided with this paper.
- All necessary working should be shown in every question.

Total marks – 84

- Attempt Questions 1-7.
- All questions are of equal value.
- Answer each question in a separate writing booklet.

Outcomes assessed

Preliminary course

- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives which require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving binomials, projectiles, simple harmonic motion,or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Harder applications of the Mathematics course are included in this course. Thus the Outcomes from the Mathematics course are included.

Outcomes from the Mathematics course

Preliminary course

- **P2** provides reasoning to support conclusions that are appropriate to the context
- **P3** performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities
- **P4** chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques
- **P5** understands the concept of a function and the relationship between a function and its graph
- **P6** relates the derivative of a function to the slope of its graph
- **P7** determines the derivative of a function through routine application of the rules of differentiation
- **P8** understands and uses the language and notation of calculus

HSC course

- **H2** constructs arguments to prove and justify results
- **H3** manipulates algebraic expressions involving logarithmic and exponential functions
- **H4** expresses practical problems in mathematical terms based on simple given models
- **H5** applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
- **H6** uses the derivative to determine the features of the graph of a function
- **H7** uses the features of a graph to deduce information about the derivative
- **H8** uses techniques of integration to calculate areas and volumes
- **H9** communicates using mathematical language, notation, diagrams and graphs

Marks

Question 1 (12 marks)

(a) Evaluate $\lim_{x \to 0} \dfrac{2\sin 2x}{x}$ **1**

(b) Divide the interval AB externally in the ratio $4:3$ where A is the point $(2,-1)$ and B is the point $(1,-3)$. **2**

(c) The polynomial $P(x) = 2x^3 + ax^2 + x + 2$ has a factor $(2x+1)$. Find the value of a. **2**

(d) Find:

(i) $\int \sin^2 \dfrac{x}{2}\, dx$ **2**

(ii) $\int \dfrac{2}{\sqrt{4-9x^2}}\, dx$ **2**

(e) Using the substitution $u = x+2$, evaluate $\int_{-1}^{2} \dfrac{x}{3}\sqrt{x+2}\, dx$. **3**

Marks

Question 2 (12 marks)
Start a new booklet

(a) (i) Differentiate $x\sin^{-1}x+\sqrt{1-x^2}$. **2**

(ii) Hence, evaluate $\int_0^1 \sin^{-1}x\;dx$. **2**

(b) Without using calculus, draw a neat sketch of the graph of $y=\dfrac{1}{x^2+2}$. **1**

(c) Consider the function $f\left(x\right)=\dfrac{1}{2}\sin^{-1}\left(1-3x\right)$.

(i) State the domain and range of $f\left(x\right)$. **2**

(ii) Hence, sketch the graph of $y=f\left(x\right)$. **2**

(d) Show that $\dfrac{d}{dx}\cos^{-1}\left(\sin x\right)$ has 2 possible answers, both of which are constants. **3**

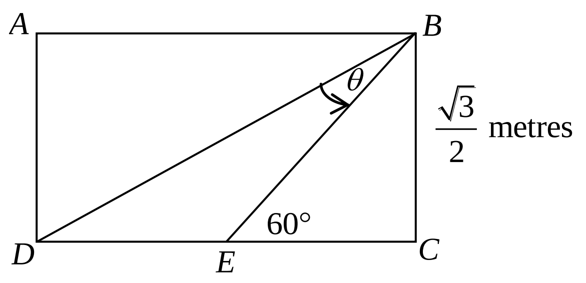
Marks

Question 3 (12 marks)
Start a new booklet

(a) (i) The polynomial equation $P(x)=0$ has a double root at $x=a$. by writing $P(x)=(x-a)^2 Q(x)$, where $Q(x)$ is a polynomial, show that $P'(a)=0$. **2**

(ii) Hence or otherwise, find the values of a and b if $x=1$ is a double root of $x^4-ax^3+bx^2+5x-1=0$. **3**

(b)



Not to scale

In the diagram above, $ABCD$ is a rectangle and E is the mid point of DC.
The length of BC is $\frac{\sqrt{3}}{2}$ metres.

(i) Show that $DB=\frac{\sqrt{7}}{2}$ metres. **2**

(ii) Show that $\cos\theta=\frac{5\sqrt{7}}{14}$. **3**

(c) (i) Show that $f(x)=e^x-x^3+1$ has a zero between 4.4 and 4.6. **1**

(ii) Find an approximation, correct to 1 decimal place, for this zero using one application of the method of halving the interval. **1**

Marks

Question 4 (12 marks)
Start a new booklet

(a) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. **2**

(ii) Hence, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le \frac{\pi}{2}$. **1**

(b) Prove $\tan^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) = \tan^{-1}\left(\frac{7}{4}\right)$. **3**

(c) Sketch the curve $y = x + \frac{4}{x}$ showing clearly all the stationary points and asymptotes.

Hence, or otherwise, find the values of k such that $x + \frac{4}{x} = k$ where there are no real roots. **3**

(d) Use the method of mathematical induction to prove that

$$(1+1)+(2+3)+(3+5)+.....+(n+(2n-1)) = \frac{1}{2}n(3n+1)$$

where n is a positive integer. **3**

Marks

Question 5 (12 marks)
Start a new booklet

(a) The rate at which a body cools in air is given by the differential equation $\frac{dT}{dt} = k(T - S)$.

Where the temperature of the air is S degrees, T is the temperature in degrees of the body after t hours and k is a constant.

(i) Show that $T = S + Be^{kt}$, where B is a constant, is a solution of the differential equation. **1**

(ii) A heater body cools from $80°C$ to $40°C$ in 2 hours. The air temperature S is $20°C$. Find the temperature of the body after one further hour has elapsed. Give your answer correct to the nearest degree. **2**

(b) A particle with displacement x and velocity V, is moving in simple harmonic motion such that $\ddot{x} = -12x$. Initially it is stationary at the point where $x = -4$.

(i) Show that $V^2 = 12(16 - x^2)$. **2**

(ii) By assuming the general form for x or otherwise, find x as a function of t. **2**

(c) An insurance company has calculated that the probability of a woman being alive in 40 years' time is 0.8 and that the probability of her husband being alive in 40 years' time is 0.7. What is the probability that in 40 years' time

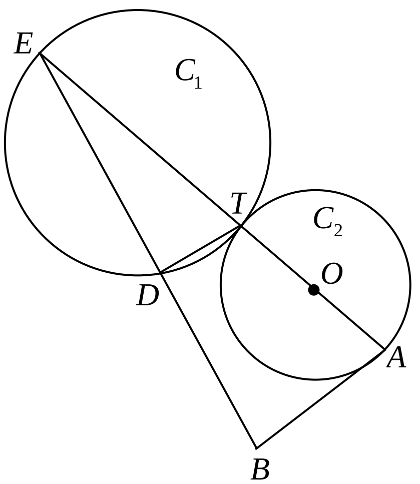
(i) both will be alive? **1**

(ii) only one of them will be alive? **2**

(d) Find the coefficient of x^6 in the expansion of $(x+2)^2(2x-1)^7$. **2**

Question 6 (12 marks)
Start a new booklet

(a)



Not to scale

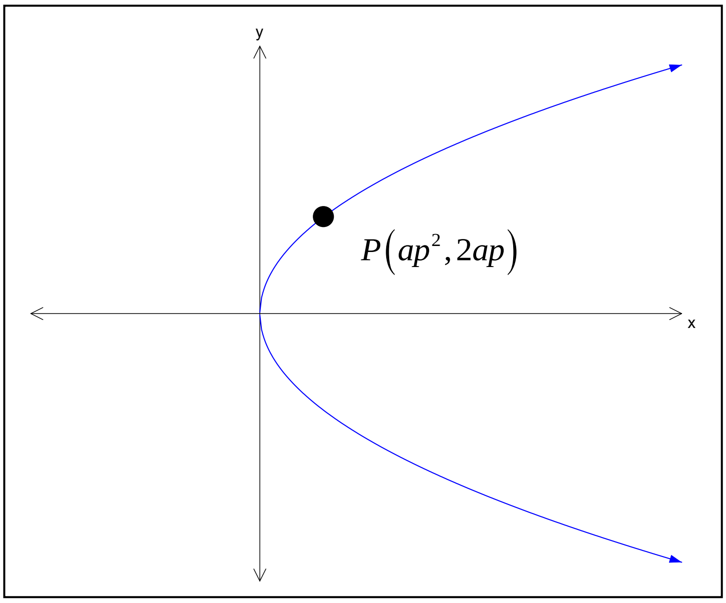
Two circles C_1 and C_2 touch at T. O is the centre of C_2. AE and BD are straight lines. BA is a tangent to C_2. The radius of C_1 is R and the radius of C_2 is r.

(i) Find the size of $\angle EDT$, giving reasons **2**

(ii) If $DE = 2r$, find an expression for the length of EB in terms of R and r. **2**

Question 6 continued on next page

Question 6 (continued) **Marks**



(b) The point $P(ap^2, 2ap)$ lies on the parabola $y^2 = 4ax$ as shown in the diagram above.

(i) Show that the gradient of the tangent to the parabola $y^2 = 4ax$ at the point $P(ap^2, 2ap)$ is $\frac{1}{p}$. **1**

(ii) Hence, find the equation of the normal at $P(ap^2, 2ap)$ on $y^2 = 4ax$. **2**

(iii) This normal intersects the x-axis at Q. Find the coordinates of Q and hence find the coordinates of R where R is the mid point of PQ. **2**

(iv) Hence, find the Cartesian equation of the locus of R. Describe this locus in words and state two of its main features. **3**

Marks

Question 7 (12 marks)
Start a new booklet

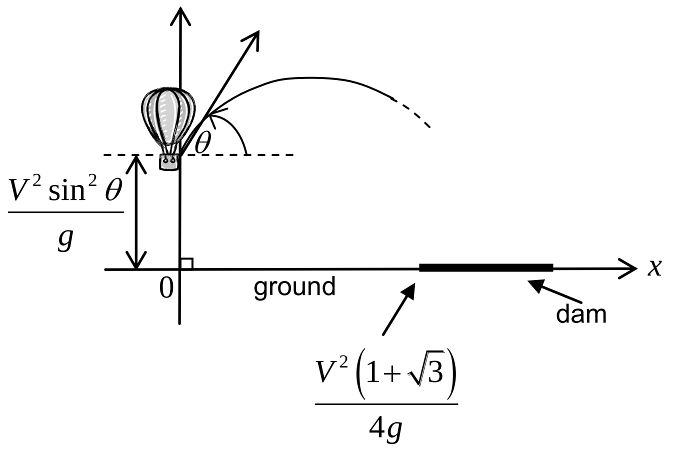
(a) Suppose $(5+3x)^{25} = \sum_{k=0}^{25} t_k x^k$.

(i) Use the binomial theorem to write an expression for t_k, $0 \le k \le 25$. **1**

(ii) Show that $\dfrac{t_{k+1}}{t_k} = \dfrac{3(25-k)}{5(k+1)}$. **2**

(iii) Hence, or otherwise, find the largest coefficient in the expansion of $(5+3x)^{25}$.
You may leave your answer in the form $\binom{25}{b} 5^c 3^d$. **2**

(b)



A man in an ascending hot air balloon throws a set of car keys to his wife who is on the ground. The keys are projected at a constant velocity of $V\text{ms}^{-1}$ at an angle θ to the horizontal and from a point $\dfrac{V^2 \sin^2\theta}{g}$ metres vertically above the ground.

Also, the near edge of a dam located where the balloon took off, lies $\dfrac{V^2(1+\sqrt{3})}{4g}$ m horizontally from the point of projection. The dam is $\dfrac{V^2}{2g}$ metres wide.

The position of the keys at time t seconds after they are projected is given by
$x = Vt\cos\theta$ and $y = \dfrac{-gt^2}{2} + Vt\sin\theta + \dfrac{V^2\sin^2\theta}{g}$.

Marks

Question 7 (cont)

(i) Show the Cartesian equation of the path of the keys is given by **1**

$$y = \frac{-gx^2 \sec^2 \theta}{2V^2} + x \tan \theta + \frac{V^2 \sin^2 \theta}{g}$$

(ii) Show that the horizontal range of the keys on the ground is given by **2**

$$x = \frac{V^2 \left(1+\sqrt{3}\right) \sin 2\theta}{2g}$$

(iii) Find the values of θ for which the keys will <u>NOT</u> land in the dam or on the edge of the dam. **4**

END OF PAPER

Year 12 Extension 1 Mathematics TRIAL 2007

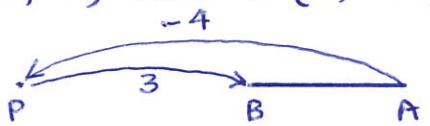
Question 1

a) $\lim\limits_{x \to 0} \dfrac{2 \sin 2x}{x}$

$= \lim\limits_{x \to 0} 4 \dfrac{\sin 2x}{2x}$

$= 4 \quad \left(\text{since } \lim\limits_{x \to 0} \dfrac{\sin X}{X} = 1\right)$

b) $A(2,-1)$ and $B(1,-3)$



$P: \; x = \dfrac{-4(1) + 3(2)}{-1} = -2$

$y = \dfrac{-4(-3) + 3(-1)}{-1} = -9$

$P \; (-2, -9)$

c) $P(x) = 2x^3 + ax^2 + x + 2$

Since $(2x+1)$ is a factor

$P(-\tfrac{1}{2}) = 0$

$\therefore 2(-\tfrac{1}{2})^3 + a(-\tfrac{1}{2})^2 + (-\tfrac{1}{2}) + 2 = 0$

$-\tfrac{1}{4} + \tfrac{a}{4} + \tfrac{3}{2} = 0$

$\tfrac{a}{4} = -\tfrac{5}{4}$

$a = -5$

d) (i) $\int \sin^2\left(\frac{x}{2}\right) dx$

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\sin^2\theta = \tfrac{1}{2}(1 - \cos\theta)$

$\therefore \int \sin^2\left(\frac{x}{2}\right) dx = \tfrac{1}{2} \int (1 - \cos x)\, dx$

$= \dfrac{x}{2} - \tfrac{1}{2}\sin x + C$

(ii) $\int \dfrac{2}{\sqrt{4 - 9x^2}}\, dx$

$= 2 \int \dfrac{1}{\sqrt{2^2 - (3x)^2}}$

$= \tfrac{2}{3} \sin^{-1}\left(\dfrac{3x}{2}\right) + C$

e) $\displaystyle\int_{-1}^{2} \frac{x}{3}\sqrt{x+2}\, dx = I$

Let $u = x + 2 \Rightarrow x = u - 2$

$\dfrac{du}{dx} = 1 \Rightarrow du = dx$

$\left.\begin{array}{l} x: -1 \to 2 \\ u: 1 \to 4 \end{array}\right\}$ limits

$I = \displaystyle\int_1^4 \frac{u-2}{3}\sqrt{u}\, du$

$= \dfrac{1}{3}\displaystyle\int_1^4 \left(u^{3/2} - 2u^{1/2}\right) du$

$= \tfrac{1}{3}\left[\tfrac{2}{5}u^{5/2} - \tfrac{4}{3}u^{3/2}\right]_1^4$

$= \dfrac{1}{3}\left(\left(\tfrac{2}{5} \times 32 - \tfrac{4}{3} \times 8\right) - \left(\tfrac{2}{5} - \tfrac{4}{3}\right)\right)$

$= \tfrac{1}{3}\left(\dfrac{62}{5} - \dfrac{28}{3}\right)$

$= \dfrac{46}{45} \quad (\text{or } 1.0\dot{2})$

Question 2

a) (i) $y = x \sin^{-1} x + \sqrt{1 - x^2}$

$\dfrac{dy}{dx} = \sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}} + \tfrac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x)$

$= \sin^{-1} x + \dfrac{x}{\sqrt{1-x^2}} - \dfrac{x}{\sqrt{1-x^2}}$

$= \sin^{-1} x$

(ii) $\displaystyle\int_0^1 \sin^{-1} x\, dx$

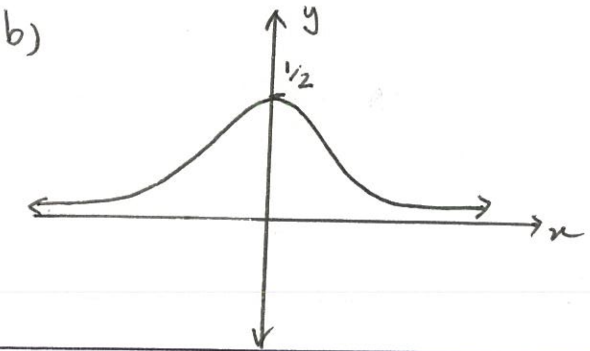
$= \left[x \sin^{-1} x + \sqrt{1 - x^2}\right]_0^1$

$= \left(\tfrac{\pi}{2} + 0\right) - (0 + 1)$

$= \tfrac{\pi}{2} - 1$

Question 2 (cont)

b)



c) $f(x) = \frac{1}{2}\sin^{-1}(1-3x)$

(i) Domain : $-1 \le 1-3x \le 1$

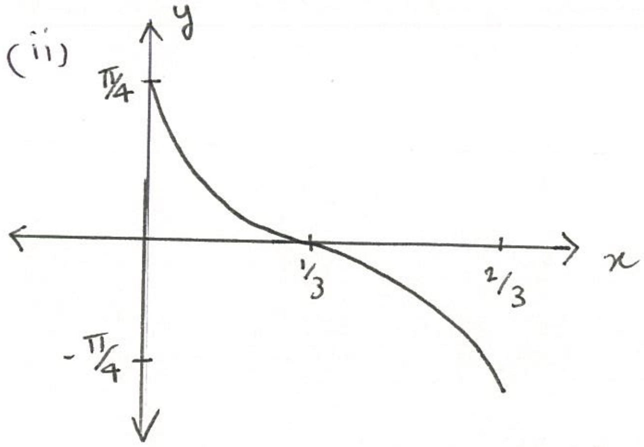
$-2 \le -3x \le 0$

$0 \le x \le 2/3$

Range : $-\pi/2 \le 2y \le \pi/2$

$-\pi/4 \le y \le \pi/4$

(ii)



$f(0) = \pi/4$ (y-intercept)

d) $\frac{d}{dx}\cos^{-1}(\sin x) = \frac{-\cos x}{\sqrt{1-\sin^2 x}}$

$= \frac{-\cos x}{|\cos x|}$

$= \begin{cases} -1 & \text{when } \cos x > 0 \\ 1 & \text{when } \cos x < 0 \end{cases}$

Question 3

a) (i) $P(x) = (x-a)^2 Q(x)$

$P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x)$

$P'(a) = 2(a-a)Q(x) + (a-a)^2 Q'(x)$

$= 0$

(ii) $(x-1)^2 Q(x) = x^4 - ax^3 + bx^2 + 5x - 1$

$P(1) = 1 - a + b + 5 - 1 = 0$

$a - b = 5 \quad \ldots (1)$

$P'(1) = 0$ from (i)

$4x^3 - 3ax^2 + 2bx + 5 = 0$

$3a - 2b = 9 \quad \ldots (2)$

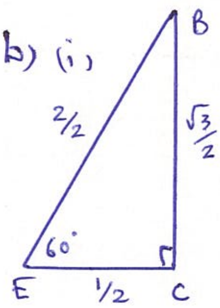
$a - b = 5 \quad \ldots (1)$

(1) × 3 $\quad 3a - 3b = 15$

$3a - 2b = 9$

$\left.\begin{array}{l} b = -6 \\ a = -1 \end{array}\right\}$

b) (i)

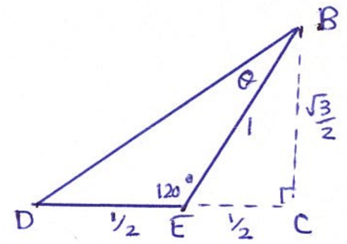


Using proportions

EC = 1/2

Hence DE = 1/2

EB = 1



$DB^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + 1^2$

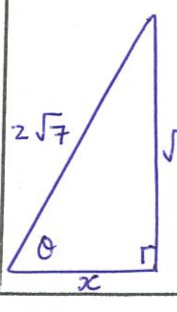
$= \frac{7}{4}$

$DB = \frac{\sqrt{7}}{2}$ m

(ii) $\frac{\sin\theta}{1/2} = \frac{\sin 120°}{\sqrt{7}/2}$

$\sin\theta = \frac{\sin 120°}{\sqrt{7}}$

$= \frac{\sqrt{3}}{2\sqrt{7}}$



$x^2 = (2\sqrt{7})^2 - (\sqrt{3})^2$

$= 25$

$x = 5$

$\cos\theta = \frac{5}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{5\sqrt{7}}{14}$

Question 3 (continued)

c) (i) $f(x) = e^{x} - x^{3} + 1$

$f(4\cdot4) = -2\cdot733\ldots$

$f(4\cdot6) = 3\cdot148\ldots$

$\therefore$ the root lies between $x = 4\cdot4$ and $x = 4\cdot6$

(ii)

x	$4\cdot5$	$4\cdot55$
$f(x)$	$-\cdot108$	$1\cdot44$

$\therefore f(4\cdot5) \doteqdot 0$

Hence $x = 4\cdot5$ is one of the zeros correct to 1 decimal place.

Question 4

a) (i) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$R^2 = (\sqrt{3})^2 + (1)^2$

$R = 2$

Equate like terms

$\sqrt{3}\cos x - \sin x = 2\cos x\cos\alpha - 2\sin x\sin\alpha$

$\sqrt{3} = 2\cos\alpha$

$\cos\alpha = \frac{\sqrt{3}}{2}$

$\alpha = \pi/6$

$\therefore \sqrt{3}\cos x - \sin x = 2\cos(x + \pi/6)$

(ii) $2\cos(x+\pi/6) = 1$

$\cos(x+\pi/6) = 1/2$

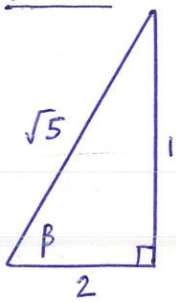
$x + \pi/6 = \pi/3$

$x = \pi/6$

4 b)

Prove $\tan^{-1}\left(\frac{2}{3}\right) + \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) = \tan^{-1}\left(\frac{7}{4}\right)$

Proof



$\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

Let $\alpha = \tan^{-1}(2/3)$, $\beta = \tan^{-1}(1/2)$

Hence $\tan\alpha = 2/3$, $\tan\beta = 1/2$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{2/3 + 1/2}{1 - (2/3)(1/2)}$

$= 7/4$

$\therefore \tan^{-1}\left(\frac{7}{4}\right) = \alpha + \beta$

$= \tan^{-1}(2/3) + \tan^{-1}(1/2)$

$= \tan^{-1}(2/3) + \cos^{-1}(2/\sqrt{5})$

c) $y = x + 4/x$

Vertical asymptote at $x = 0$

$\lim_{x\to\infty}(x + 4/x) = x$

$\therefore$ Oblique asymptote at $y = x$

$f(-x) = -x + 4/-x$

$= -(x + 4/x)$

$= -f(x)$ $\therefore$ ODD

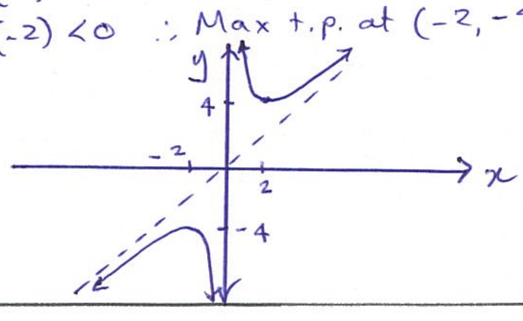
$\frac{dy}{dx} = 1 - \frac{4}{x^2}$

When $\frac{dy}{dx} = 0$: $x^2 = 4$, $x = \pm 2$

$\frac{d^2y}{dx^2} = \frac{8}{x^3}$ $\therefore$ No pts of inflexion

$f''(2) > 0$ $\therefore$ Min t.p. at $(2, 4)$

$f''(-2) < 0$ $\therefore$ Max t.p. at $(-2, -4)$



Question 4 (continued)

c) (ii) $x + \frac{4}{x} = k$

$-4 < k < 4$

d)

$(1+1)+(2+3)+(3+5)+\dots+(n+(2n-1)) = \frac{1}{2}n(3n+1)$

Prove true for $n=1$

LHS $= 1+(2-1) = 2$ RHS $= \frac{1}{2}(3+1) = 2$

$\therefore$ True for $n=1$

Assume true for $n=k$

$(1+1)+(2+3)+\dots+(k+(2k-1)) = \frac{1}{2}k(3k+1)$

Prove true for $n=k+1$

Show that

$(1+1)+(2+3)+\dots+(k+1+(2k+1)) = \frac{1}{2}(k+1)(3k+4)$

Proof

$(1+1)+(2+3)+\dots+(k+1+(2k+1)) = \frac{1}{2}k(3k+1)+(3k+2)$

$= \frac{1}{2}(3k^2+k+6k+4)$

$= \frac{1}{2}(3k+4)(k+1)$

$\therefore$ True for $n=k+1$

Since true for $n=1$ then true for $n=2$, Since true for $n=2$, then true for $n=3$ and so on for all integer n, $n \geq 1$.

Question 5

a) (i) $e^{kt} = \frac{T-S}{B}$

$T = Be^{kt} + S$

$\frac{dT}{dt} = kBe^{kt}$

since $Be^{kt} = T-S$

$\frac{dT}{dt} = k(T-S)$

Alternative solution (long way)

a) (i) $\frac{dT}{dt} = k(T-S)$

$\frac{dt}{dT} = \frac{1}{k} \times \frac{1}{T-S}$

$t = \frac{1}{k}\log_e(T-S) + c$

When $t=0$, $T=T_0$

$c = -\frac{1}{k}\log_e(T_0-S)$

$\therefore t = \frac{1}{k}(\ln(T-S) - \ln(T_0-S))$

$kt = \ln\left[\frac{T-S}{T_0-S}\right]$

$e^{kt} = \frac{T-S}{T_0-S}$

Let $B = T_0 - S$

$e^{kt} = \frac{T-S}{B}$

$\therefore T = S + Be^{kt}$ as required

(ii) $t=0$, $T=80$, $S=20$ } given initial conditions

$80 = 20 + B$

$B = 60$ //

$T = 20 + 60e^{kt}$

when $t=2$, $T=40$

$40 = 20 + 60e^{2k}$

$20 = 60e^{2k}$

$\frac{1}{3} = e^{2k}$

$2k = \ln(\frac{1}{3})$

$k = -0.54931$ //

$T = 20 + 60e^{-0.54931t}$

After 3 hours

$T = 20 + 60e^{-0.54931 \times 3}$

$= 31.547$

$= 32°C$ //

Question 5 (cont)

(b) $\ddot{x} = -12x$

Given $V=0,\ t=0,\ x=-4$

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -12x$$

$$\frac{1}{2}v^2 = -6x^2 + c$$

$$0 = -6(-4)^2 + c$$

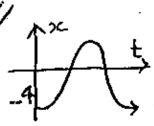
$$c = 96$$

$$\frac{1}{2}v^2 = 96 - 6x^2$$

$$v^2 = 192 - 12x^2$$

$$= 12(16 - x^2) \quad //$$

(ii) $V^2 = 12(16 - x^2)$



$x = \pm a\cos(nt - \alpha)$

Since $(0, -4)$ is the initial condition $a = -4$ and $\alpha = 0$

$\therefore x = -4\cos(\sqrt{12}t) \quad //$

OR

$$V = \frac{dx}{dt} = \sqrt{192 - 12x^2}$$

$$\frac{dt}{dx} = \frac{1}{\sqrt{12}} \times \frac{1}{\sqrt{16 - x^2}}$$

$$t = \frac{1}{\sqrt{12}}\sin^{-1}\left(\frac{x}{4}\right) + c$$

When $x = -4,\ t = 0$

$$0 = \frac{1}{\sqrt{12}}\sin^{-1}(-1) + c$$

$$c = -\frac{1}{\sqrt{12}} \times \frac{-\pi}{2}$$

$$= \frac{\pi}{2\sqrt{12}}$$

$$t = \frac{1}{\sqrt{12}}\left(\sin^{-1}\left(\frac{x}{4}\right) + \frac{\pi}{2}\right)$$

$$\sin^{-1}\left(\frac{x}{4}\right) = \sqrt{12}t - \frac{\pi}{2}$$

$$\frac{x}{4} = \sin(\sqrt{12}t - \pi/2)$$

$$x = 4\sin(\sqrt{12}t - \pi/2)$$

OR $x = -4\cos(\sqrt{12}t)$

c) $P(W) = 0.8 \qquad P(M) = 0.7$

(i) $P(W \cap M) = 0.8 \times 0.7$

$= 0.56 \quad //$

(ii) $P(\bar{W} \cap M) + P(W \cap \bar{M})$

$= (0.8 \times 0.3) + (0.2 \times 0.7)$

$= 0.38 \quad //$

d) $(x+2)^2(2x-1)^7$

$= (x^2 + 4x + 4)(2x-1)^7$

$= x^2(2x-1)^7 + 4x(2x-1)^7 + 4(2x-1)^7$

Coefficient of x^6 term

$$1 \times \binom{7}{3}2^4(-1)^3 + 4 \times \binom{7}{2}2^5(-1)^2 - 4 \times \binom{7}{1} \times 2^6(-1)^1$$

$$= 336 \quad //$$

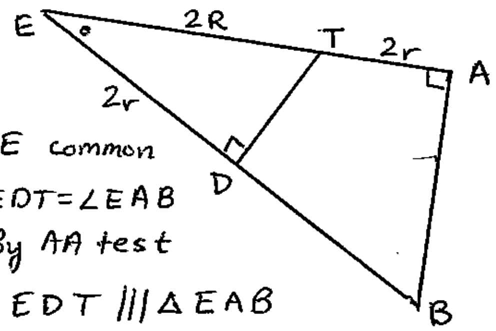
Question 6

a) (i) The circles C_1 and C_2 have a common point of contact, T. Since AT is the diameter of C_2 then ET is the diameter of C_1 (EA is a straight line through the centres and point of contact)

$\angle EDT$ is the angle at the circumference in a semicircle

$\therefore \angle EDT = 90° \quad //$

(ii) $DE = 2r$



$\angle E$ common

$\angle EDT = \angle EAB$

$\therefore$ By AA test

$\Delta EDT \ ||| \ \Delta EAB$

Hence corresponding sides are in the same proportion.

$$\frac{ET}{EB} = \frac{ED}{EA}$$

$$\frac{2R}{EB} = \frac{2r}{(2R + 2r)} = \frac{r}{R + r}$$

$$2R = EB \times \frac{r}{R + r}$$

$$EB = \frac{2R(R + r)}{r} \quad //$$

Question 6 (cont)

b) (i) $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a$$

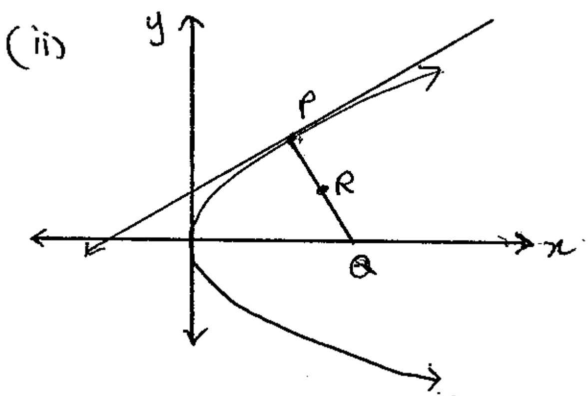
$$\frac{dy}{dx} = \frac{4a}{2y}$$

At P, $y = 2ap$

$$\frac{dy}{dx} = \frac{4a}{2(2ap)}$$

$$= \frac{1}{p} \quad //$$

(ii)



Gradient of normal is $-p$

Equation of normal at P

$$y - 2ap = -p(x - ap^2)$$

$$y - 2ap = -px + ap^3 \quad //$$

When $y = 0$

(iii) $px = 2ap + ap^3$

$$x = 2a + ap^2$$

$$x = a(2 + p^2)$$

$\therefore$ Q $(a(2+p^2), 0)$ //

R is the midpoint PQ

$$x = \frac{ap^2 + 2a + ap^2}{2}$$

$$= a + ap^2$$

$$= a(1+p^2)$$

$$y = ap$$

R $(a(1+p^2), ap)$

(iv) Rearrange $y = ap$

$$p = \frac{y}{a}$$

Substitute

$$x = a + a\left(\frac{y}{a}\right)^2$$

$$x = a + \frac{y^2}{a}$$

$$x - a = y^2/a$$

$$y^2 = a(x - a)$$

This is the equation of a parabola with the same orientation as P.

Vertex at $(a, 0)$.

Directrix is $x = \frac{3a}{4}$

Focus at $\left(0, \frac{5a}{4}\right)$

focal length is $a/4$.

Question 7

a) $(5+3x)^{25} = \sum_{k=0}^{25} t_k x^k$

(i) $t_k = 5^{25-k} \times 3^k \times \binom{25}{k}$

(ii) $t_{k+1} = 5^{24-k} \times 3^{k+1} \times \binom{25}{k+1}$

$$\frac{t_{k+1}}{t_k} = \frac{3^{k+1} \times 5^{24-k} \times \binom{25}{k+1}}{3^k \times 5^{25-k} \times \binom{25}{k}}$$

$$= \frac{3}{5} \times \frac{(25-k)!\,k! \times 25!}{25! \times (k+1)!\,(24-k)!}$$

$$= \frac{3}{5} \times \frac{(25-k)}{(k+1)}$$

(iii) $\frac{t_{k+1}}{t_k} > 1$; $\frac{3}{5}\frac{(25-k)}{(k+1)} > 1$

$$3(25-k) > 5(k+1)$$

$$75 - 3k > 5k + 5$$

$$8k < 70$$

$$k < 8.75$$

$$k = 8$$

t_{k+1} is the greatest coefficient

$$t_9 = \binom{25}{9} \times 5^{16} \times 3^9$$

Question 7 (continued)

b) $x = Vt\cos\theta \Rightarrow t = \dfrac{x}{V\cos\theta}$ ①

(i) $y = -\dfrac{gt^2}{2} + Vt\sin\theta + \dfrac{V^2\sin^2\theta}{g}$ ②

Substitute ① into ②

$$y = -\frac{g}{2}\left(\frac{x^2}{V^2\cos^2\theta}\right) + \frac{Vx\sin\theta}{V\cos\theta} + \frac{V^2\sin^2\theta}{g}$$

$$= -\frac{gx^2\sec^2\theta}{2V^2} + x\tan\theta + \frac{V^2\sin^2\theta}{g}$$

(ii) The keys hit the ground when $y=0$

Method 1

$$0 = \frac{gt^2}{2} - Vt\sin\theta - \frac{V^2\sin^2\theta}{g}$$

$a = g/2$, $b = V\sin\theta$, $c = -\dfrac{V^2\sin^2\theta}{g}$

$$t = \frac{V\sin\theta \pm \sqrt{V^2\sin^2\theta + \frac{4}{2}V^2\sin^2\theta}}{g}$$

$$= \frac{V\sin\theta \pm V\sin\theta \times \sqrt{3}}{g}$$

$$= \frac{V(1+\sqrt{3})\sin\theta}{g}$$

$$x = Vt\cos\theta$$

$$= V\cos\theta \times \frac{V(1+\sqrt{3})\sin\theta}{g}$$

$$= \frac{V^2(1+\sqrt{3})}{g}\sin\theta\cos\theta$$

$\sin 2\theta = = 2\sin\theta\cos\theta$

$\therefore \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$x = \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g}$$

Method 2

$$0 = \frac{gx^2\sec^2\theta}{2V^2} - x\tan\theta - \frac{V^2\sin^2\theta}{g}$$

$a = \dfrac{g\sec^2\theta}{2V^2}$

$b = -\tan\theta$

$c = -\dfrac{V^2\sin^2\theta}{g}$

$$x = \frac{\tan\theta \pm \sqrt{\tan^2\theta + \frac{4g\sec^2\theta}{2V^2} \times \frac{V^2\sin^2\theta}{g}}}{\frac{2g\sec^2\theta}{2V^2}}$$

$$= \frac{\tan\theta \pm \sqrt{\tan^2\theta + 2\tan^2\theta}}{\frac{g\sec^2\theta}{V^2}}$$

$$= \frac{V^2(\tan\theta + \tan\theta \times \sqrt{3})\times\cos^2\theta}{g}$$

$$= \frac{V^2}{g}(1+\sqrt{3})\sin\theta\cos\theta$$

$$= \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g} \quad //$$

(iii) GROUND DAM GROUND

$\dfrac{V^2(1+\sqrt{3})}{4g}$ ← $\dfrac{2V^2}{4g}$ → $\dfrac{V^2(3+\sqrt{3})}{4g}$

$$\frac{2V^2(1+\sqrt{3})\sin 2\theta}{4g} < \frac{V^2(1+\sqrt{3})}{4g}$$

$2\sin 2\theta < 1$

$2\theta < 30^\circ$ or $2\theta > 150^\circ$

$\theta < 15^\circ$ or $\theta > 75^\circ$

(near side of dam)

$$\frac{2V^2(1+\sqrt{3})\sin 2\theta}{4g} > \frac{V^2(3+\sqrt{3})}{4g}$$

$$2\sin 2\theta > \frac{3+\sqrt{3}}{1+\sqrt{3}} \times \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

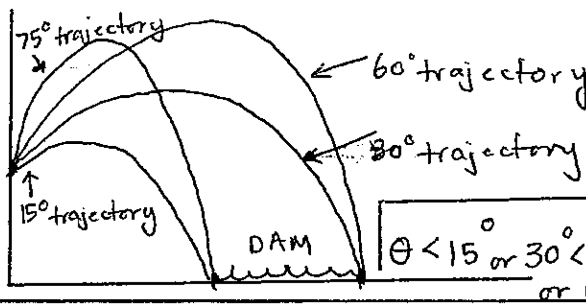
$$2\sin 2\theta > -\frac{2\sqrt{3}}{-2}$$

$$\sin 2\theta > \frac{\sqrt{3}}{2}$$

$2\theta > 60^\circ$ or $2\theta < 120^\circ$

$\theta > 30^\circ$ or $\theta < 60^\circ$

to land on far side of dam.



$\theta < 15^\circ$ or $30^\circ < \theta < 60^\circ$ or $\theta > 75^\circ$