Student Number___________________

ABBOTSLEIGH

AUGUST 2009

YEAR 12

ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided with this paper.
- All necessary working should be shown in every question.

Total marks – 84

- Attempt Questions 1-7.
- All questions are of equal value.
- Answer each question in a separate writing booklet.

Outcomes assessed

Preliminary course

PE2 uses multi-step deductive reasoning in a variety of contexts

PE3 solves problems involving inequalities, polynomials, circle geometry and parametric representations

PE4 uses the parametric representation together with differentiation to identify geometric properties of parabolas

PE5 determines derivatives which require the application of more than one rule of differentiation

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

HE2 uses inductive reasoning in the construction of proofs

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomials, projectiles, simple harmonic motion,or exponential growth and decay

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement

HE6 determines integrals by reduction to a standard form through a given substitution

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

Harder applications of the Mathematics course are included in this course. Thus the Outcomes from the Mathematics course are included.

Outcomes from the Mathematics course

Preliminary course

P2 provides reasoning to support conclusions that are appropriate to the context

P3 performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities

P4 chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques

P5 understands the concept of a function and the relationship between a function and its graph

P6 relates the derivative of a function to the slope of its graph

P7 determines the derivative of a function through routine application of the rules of differentiation

P8 understands and uses the language and notation of calculus

HSC course

- **H2** constructs arguments to prove and justify results
- **H3** manipulates algebraic expressions involving logarithmic and exponential functions
- **H4** expresses practical problems in mathematical terms based on simple given models
- **H5** applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
- **H6** uses the derivative to determine the features of the graph of a function
- **H7** uses the features of a graph to deduce information about the derivative
- **H8** uses techniques of integration to calculate areas and volumes
- **H9** communicates using mathematical language, notation, diagrams and graphs

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Marks

Question 1 (12 marks)

(a) Find: $\int \frac{dx}{64 + x^2}$ **1**

(b) Evaluate: $\lim_{x \to 0} \frac{\sin 3x}{4x}$ **2**

(c) Using the difference of two cubes, simplify:

$\frac{\sin^3 \theta - \cos^3 \theta}{\sin \theta - \cos \theta} - 1$, for $0 \le \theta < \frac{\pi}{4}$ **2**

(d) Find the acute angle between the lines $2y = x + 7$ and $y = -\frac{1}{4}x + 2$ (answer to the nearest degree). **3**

(e) Find the remainder when the polynomial $P(x) = x^3 + 5x$ is divided by $x - 4$. **1**

(f) Differentiate: $e^{2x} \cos^2 x$ **3**

Marks

Question 2 (12 marks)

Start a new booklet

(a) The interval AB has endpoints $A(-3,6)$ and $B(1,3)$. Find the coordinates of the point P which divides the interval AB **externally** in the ratio of 2:3. **2**

(b) Find: $\dfrac{d}{dx}(6x\sin^{-1}x)$ **2**

(c) What is the range for the curve $y = \dfrac{2}{x^2+3}$. **2**

(d) Solve: $\dfrac{4}{x-3} \leq 2$ **3**

(e) Use the substitution, $u = x+8$, to find $\displaystyle\int \frac{x}{\sqrt{x+8}}\,dx$ **3**

Marks

Question 3 (12 marks)

Start a new booklet

(a) Use the definition $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ to find the derivative of x^2 where $x = a$. **2**

(b) Consider the function $y = 2\sin^{-1}\dfrac{x}{2}$.

(i) State the domain and range of the function $y = f(x)$. **2**

(ii) Sketch the graph of $y = f(x)$. **1**

(c) Find the coefficient of x^4 in the expansion of $(x^2 - 3)^6$. **3**

(d) (i) Express $(\cos x + \sin x)$ in the form of $R\cos(x - \alpha)$ where α is in radians. **2**

(ii) Hence, or otherwise, sketch the graph of $y = \cos x + \sin x$ for $0 \le x \le 2\pi$. **2**

Marks

Question 4 (12 marks)

Start a new booklet

(a) The function $f(x) = \cos x - \frac{4}{5}x$ has a zero near $x = 0.8$. Taking $x = 0.8$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. Give your answer correct to three decimal places. **2**

(b) Let α, β and γ be the roots of the polynomial $3x^3 - 12x + 4 = 0$. Find $\alpha\beta\gamma$. **1**

(c) (i) By applying the binomial theorem write the expansion for $(1 + x)^n$. **1**

(ii) By differentiating your answer to part (i), write an expression for $n(1+x)^{n-1}$ **1**

(iii) Hence deduce $n(4)^{n-1} = \binom{n}{1} + 6\binom{n}{2} + 27\binom{n}{3} + \ldots + n\binom{n}{n}3^{n-1}$. **1**

(d) Evaluate $\int_1^4 y\, dx$ if $xy = 1$. **2**

(e) By factorising, write $3^{k+1} + 3^k$ as a single term, and hence write $\frac{3^{1001} + 3^{1000}}{12}$ as a power of 3

2

(f) Find: $\int \sin^2 3x\, dx$. **2**

Marks

Question 5 (12 marks)

Start a new booklet

(a) Consider the variable point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$.

Draw a diagram onto your page.

(i) Prove that the equation of the normal at P is $x + py = ap^3 + 2ap$. **2**

(ii) If Q is the point $(2aq, aq^2)$ on the parabola such that the normal at Q is perpendicular to the normal at P, show that $pq = -1$. **1**

(iii) Show that the two normals of part (ii) intersect at the point R, whose coordinates are: **4**

$$x = a\left(p - \frac{1}{p}\right) \text{ and } y = a\left(p^2 + 1 + \frac{1}{p^2}\right).$$

(iv) Find the equation in Cartesian form of the locus of the point R given in part (iii). **2**

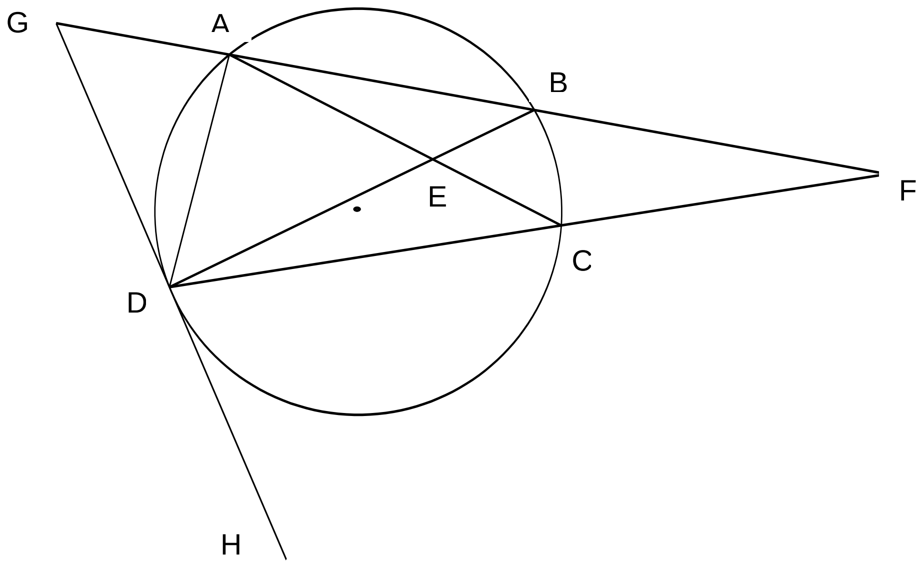
(b) Prove by the method of mathematical induction that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for positive integer $n = 1,\ 2,\ 3,\ \ldots$ **3**

Marks

Question 6 (12 marks)

Start a new booklet

(a)



DG is a tangent to the circle at D.

$GABF$ and DCF are straight lines.

AD, BD, AC and CD are chords of the circle that do NOT pass through the centre of the circle.

(i) Copy the diagram onto your page.

(ii) Explain why $\angle ACD = \angle ADG$. **1**

(iii) Prove that $2\angle ADG = \angle BEC + \angle BFC$. **2**

Marks

Question 6 CONTINUED

(b) (i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{dv}{dt}$. **2**

(ii) A particle moves in a straight line with acceleration $\frac{dv}{dt} = 4x + 2$, where x metres is the distance measured from a fixed point O. Initially, the particle is at the origin with velocity equal to -1 ms^{-1}. Show that $x = \frac{1}{2}(e^{-2t} - 1)$. **4**

(iii) By considering the displacement, velocity and acceleration of the particle with respect to time, describe the motion of the particle as t increases indefinitely. **3**

Marks

Question 7 (12 marks)

Start a new booklet

(a) Anne heats a mug of milk up to 85^0C in a microwave oven. She takes it out into a room where the temperature is a constant 22^0C. The milk cools to 60^0C in 5 minutes. Assume that the cooling rate of the milk is such that:

$$\frac{dT}{dt} = -k\left(T - M\right)$$

where T is its temperature at any time t and M is the temperature of the surroundings.

(i) What would be its temperature after 10 minutes (answer to the nearest degree Celsius)? **3**

(ii) Show it will take approximately 20 minutes for the temperature to reach 30^0C. **1**

Question 7 CONTINUED

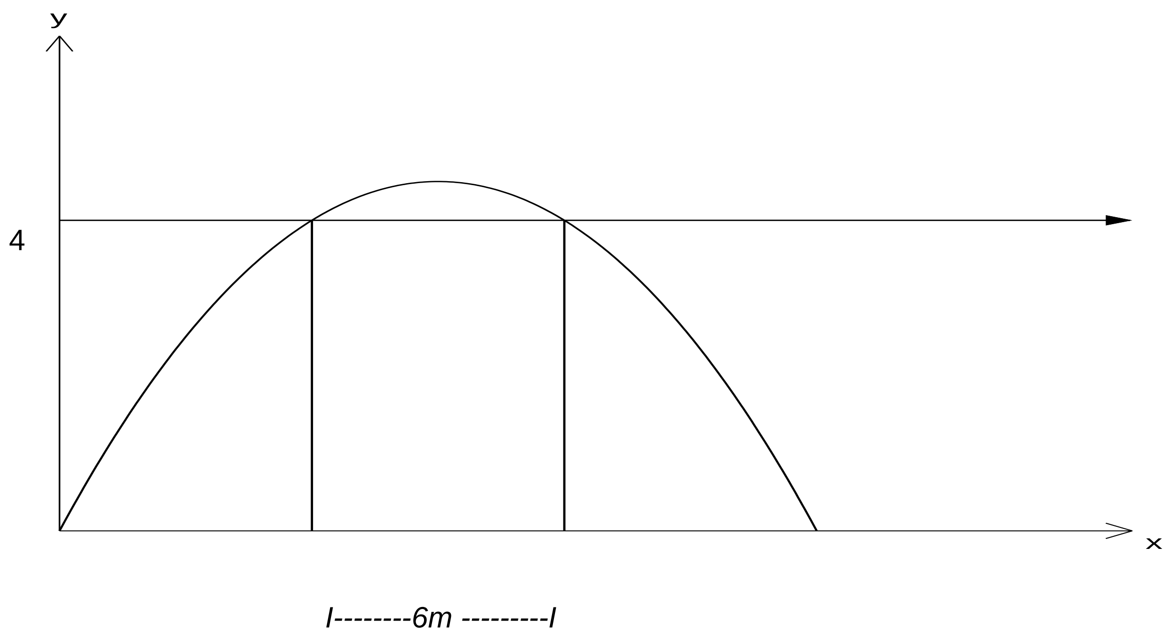
(b) A particle is projected from a point O with a speed $V\ ms^{-1}$ and at an angle α to the horizontal. Air resistance is ignored and $g\ ms^{-2}$ is the acceleration due to gravity.

(i) Derive the equations for the horizontal position x and for the vertical position y of the particle at any time t. **2**

(ii) If R is the range on the horizontal plane of this projectile, show that the Cartesian equation of the path is given by: $y = x(1 - \frac{x}{R})\tan\alpha$ **3**

(you may assume that $R = \dfrac{2V^2 \cos\alpha \sin\alpha}{g}$ without derivation).

(iii)



If $\alpha = 45^0$ and the particle just clears two vertical posts, distant $6m$ apart and each $4m$ above the level of projection, calculate R. **3**

END OF EXAMINATION