Student Number___________________

ABBOTSLEIGH

AUGUST 2010
YEAR 12
ASSESSMENT 4

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided with this paper.
- All necessary working should be shown in every question.

Total marks – 84

- Attempt Questions 1-7.
- All questions are of equal value.
- Answer each question in a separate writing booklet.

Outcomes assessed

Preliminary course

- **PE2** uses multi-step deductive reasoning in a variety of contexts
- **PE3** solves problems involving inequalities, polynomials, circle geometry and parametric representations
- **PE4** uses the parametric representation together with differentiation to identify geometric properties of parabolas
- **PE5** determines derivatives which require the application of more than one rule of differentiation
- **PE6** makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

HSC course

- **HE2** uses inductive reasoning in the construction of proofs
- **HE3** uses a variety of strategies to investigate mathematical models of situations involving binomials, projectiles, simple harmonic motion,or exponential growth and decay
- **HE4** uses the relationship between functions, inverse functions and their derivatives
- **HE5** applies the chain rule to problems including those involving velocity and acceleration as functions of displacement
- **HE6** determines integrals by reduction to a standard form through a given substitution
- **HE7** evaluates mathematical solutions to problems and communicates them in an appropriate form

Harder applications of the Mathematics course are included in this course. Thus the Outcomes from the Mathematics course are included.

Outcomes from the Mathematics course

Preliminary course

- **P2** provides reasoning to support conclusions that are appropriate to the context
- **P3** performs routine arithmetic and algebraic manipulation involving surds, simple rational expressions and trigonometric identities
- **P4** chooses and applies appropriate arithmetic, algebraic, graphical, trigonometric and geometric techniques
- **P5** understands the concept of a function and the relationship between a function and its graph
- **P6** relates the derivative of a function to the slope of its graph
- **P7** determines the derivative of a function through routine application of the rules of differentiation
- **P8** understands and uses the language and notation of calculus

HSC course

- **H2** constructs arguments to prove and justify results
- **H3** manipulates algebraic expressions involving logarithmic and exponential functions
- **H4** expresses practical problems in mathematical terms based on simple given models
- **H5** applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
- **H6** uses the derivative to determine the features of the graph of a function
- **H7** uses the features of a graph to deduce information about the derivative
- **H8** uses techniques of integration to calculate areas and volumes
- **H9** communicates using mathematical language, notation, diagrams and graphs

Question 1 (12 marks) **Marks**

(a) The interval AB, where A is $(-3,-4)$ and B is $(5,-1)$, is divided internally in the ratio $3:2$ by the point $P(x,y)$. Find the values of x and y. 2

(b) Differentiate $\log_e(x^3+1)$ with respect to x. 2

(c) Simplify $\dfrac{12^{2n}\times\left(3^n\right)^{-2}}{2^{4n}}$ 2

(d) Evaluate $\displaystyle\int_0^3 \frac{dx}{9+x^2}$ 3

(e) Solve the inequality $\dfrac{x+4}{x-1} \le 6$ 3

Question 2 (12 marks) **Marks**
Start a new booklet.

(a) State the domain and range of $y = 2\cos^{-1}\frac{x}{3}$ 2

(b) Find the coefficient of x^5 in the expansion of $(3+2x)^7$ 2

(c) Find $\int \frac{x}{\sqrt{1+x}}\,dx$ using the substitution $x = u^2 - 1$ 2

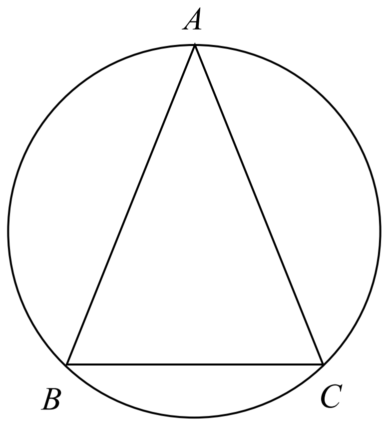
(d) Evaluate $\int_0^{\frac{\pi}{6}} \cos^2 3x\,dx$ 3

(e) (i) Show there is a root of $e^x - 3\cos x = 0$ between $x = 0$ and $x = 1$. 1

(ii) Take $x = 1$ as the first approximation and use 1 application of Newton's method to find the root correct to 2 decimal places. 2

Question 3 (12 marks) **Marks**
Start a new booklet

(a) ABC is an isosceles triangle with $AB = AC$. A line parallel to BC is drawn to meet AB and AC in D and E respectively. Copy this diagram into your answer booklet and mark on this information. Prove that $BCED$ is a cyclic quadrilateral. 3



Not to scale

(b) (i) Express $12\cos\theta + 16\sin\theta$ in the form $R\cos(\theta - \alpha)$ where $R>0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. 2

(ii) Hence or otherwise solve $12\cos\theta + 16\sin\theta = 10$ for $0 \le \theta \le 2\pi$. Give your answer correct to 2 decimal places. 2

(c) If $3x^3 - 4x^2 + ax + b$ is divided by $x^2 - 1$ there is no remainder. Find the values of a and b. 2

(d) Prove that $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$ 3

Question 4 (12 marks) **Marks**
Start a new booklet

(a) Two of the roots of $3x^3 - 4x^2 - 35x + 12 = 0$ are $x = -3$ and $x=4$. Find the third root. 1

(b) Let $S(n) = \frac{1}{1\times4} + \frac{1}{4\times7} + \frac{1}{7\times10} + ... + \frac{1}{(3n-2)(3n+1)}$

(i) Prove by induction that $S(n) = \frac{n}{3n+1}$ for all integers $n \geq 1$. 3

(ii) Find $\lim_{n\to\infty} S(n)$. 1

(c) Let $y = \frac{2-\sin\theta}{\cos\theta}$ for $0 \leq \theta \leq \frac{\pi}{4}$

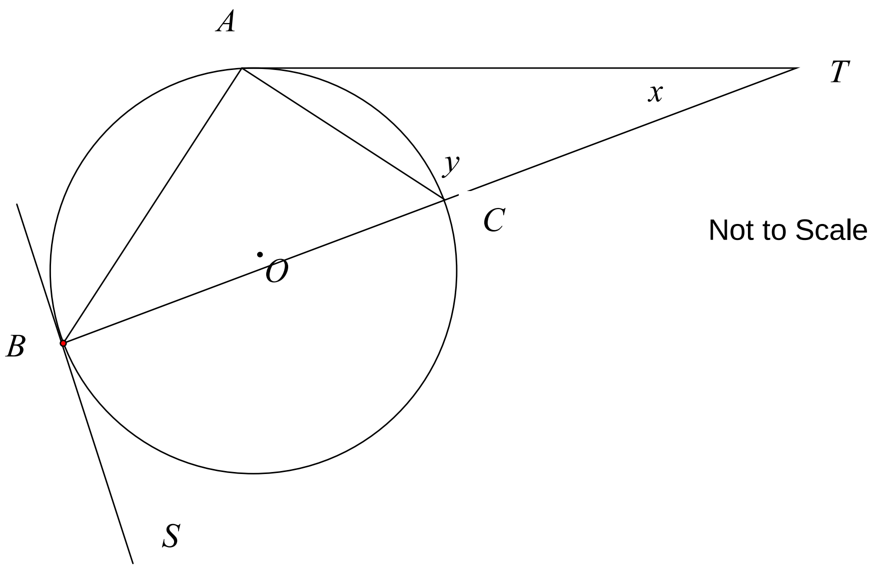
(i) Show that $\frac{dy}{d\theta} = \sec^2\theta(2\sin\theta - 1)$ 2

(ii) Hence or otherwise find the minimum value of $\frac{2-\sin\theta}{\cos\theta}$ for $0 \leq \theta \leq \frac{\pi}{4}$ 3

(iii) Find the maximum value of $\frac{2-\sin\theta}{\cos\theta}$ for $0 \leq \theta \leq \frac{\pi}{4}$ 2

Question 5 (12 marks) **Marks**
Start a new booklet

(a) In the circle below, AT and BS are tangents. The diameter BC produced meets AT at T.



Not to Scale

(i) Copy or trace the diagram into your answer booklet. If $\angle ATC = x$ and $\angle TCA = y$, show that $x + 2y = 270^{\circ}$. 3

(ii) Show $\angle ABS = y$ 2

(b) After t minutes the number of bacteria N in a culture is given by $N = \dfrac{900}{1 + be^{-ct}}$ for some constants $b>0$ and $c>0$. Initially there are 300 bacteria in the culture and the number of bacteria is initially increasing at a rate of 20 per minute.

(i) Show that $\dfrac{dN}{dt} = \dfrac{cN}{900}(900 - N)$. 3

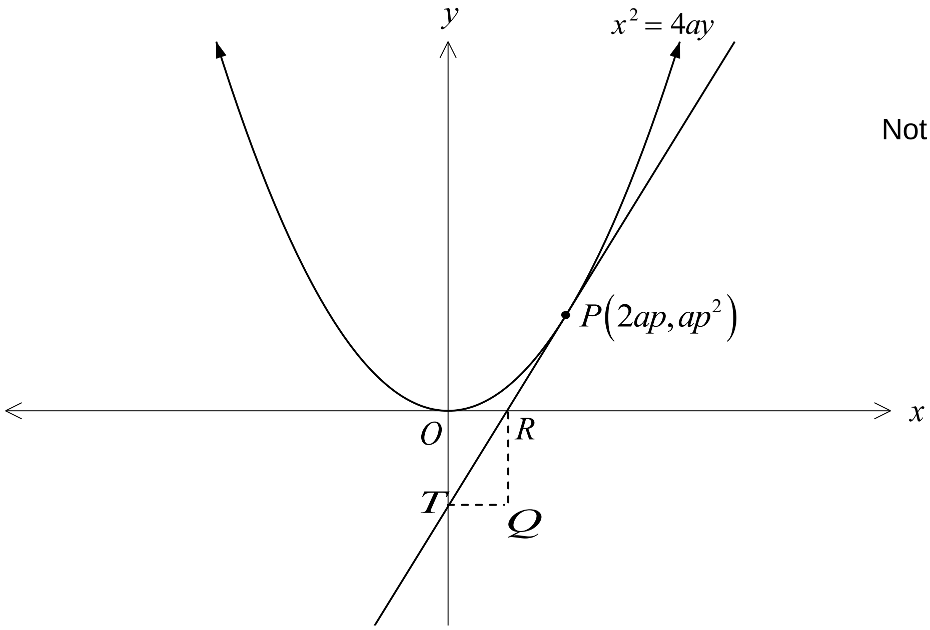
(ii) Show that $b = 2$ and $c = 0.1$. 2

(iii) Show that the maximum rate of increase in the number of bacteria occurs when $N = 450$. 2

Question 6 (12 marks) **Marks**

Start a new booklet

(a) The diagram shows the point $P\left(2ap, ap^2\right)$ which moves along the parabola $x^2 = 4ay$. The tangent at P meets the x axis at R and the y axis at T.



Not to scale

(i) Show that the equation of the tangent at P is $y = px - ap^2$. 2

(ii) Find the coordinates of R and T in terms of p. 2

(iii) Q is the vertex of the rectangle $ORQT$. Use the coordinates of Q to find the equation of the locus of Q. 2

Question 6 continues on the next page

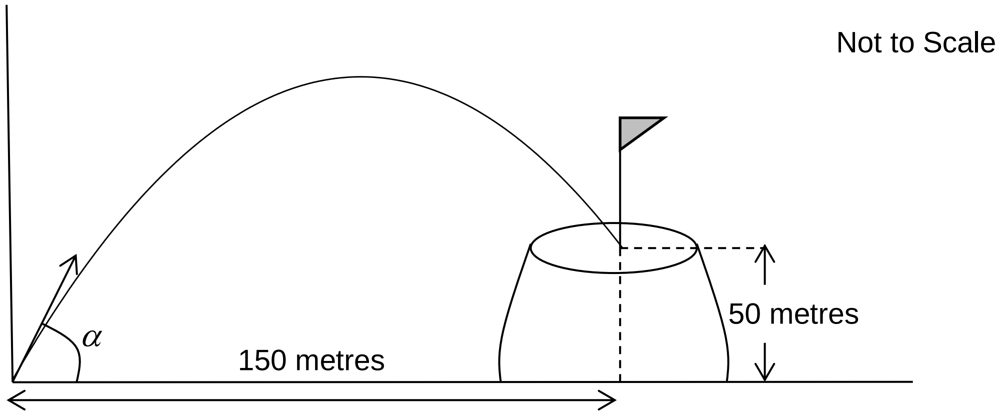
Question 6 (continued) **Marks**

(b) The diagram shows a golf ball being hit from a tee in the ground with velocity $V\ ms^{-1}$ at an angle of α to the horizontal. The position of the ball at time t seconds is given by the parametric equations

$$x = Vt\cos\alpha \qquad (1)$$

$$y = Vt\sin\alpha - \frac{1}{2}gt^2 \qquad (2)$$

where $g\,ms^{-2}$ is the acceleration due to gravity (You are NOT required to derive these.)



(i) A golfer hits the ball with initial velocity of $75ms^{-1}$ at an angle of 30^{o} to the horizontal. Given that $g = 10ms^{-1}$, find the greatest height reached by the ball. 3

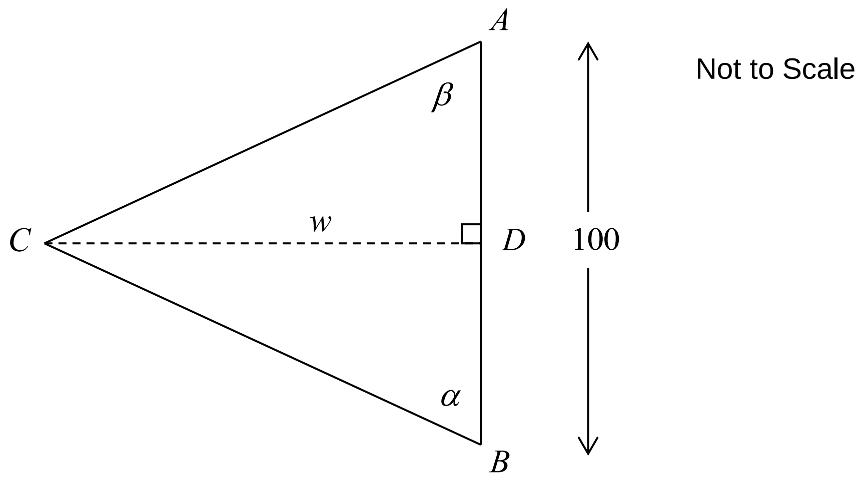
(ii) The Cartesian equation of the path of the ball is given by $y = x\tan\alpha - \dfrac{gx^2\sec^2\alpha}{2V^2}$.

(You are NOT required to derive this.)

A second golfer hits with the same initial velocity of $75ms^{-1}$ and is aiming for a hole in one. The hole is located on a hill 50 metres above the tee. The hole is 150 metres horizontally from the tee. Using the Cartesian equation, or otherwise, calculate at which angle/s she needs to hit the ball so it will land directly in the hole. (Use $g = 10ms^{-1}$ and answer to the nearest degree). 3

Question 7 (12 marks) **Marks**
Start a new booklet

(a) A and B are points on one bank of a straight river. C is a tree on the opposite bank. $\angle ABC = \alpha$, $\angle CAB = \beta$ and $AB = 100$ metres. The width of the river is w metres.



Show $w = \dfrac{100}{\cot\alpha + \cot\beta}$ 2

(b) In the expansion of $(1+ax)^n$ in ascending powers of x, the first three terms are 1, $-45x$ and $900x^2$. Find the values of a and n. 3

(c) Find y if $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{9}{9-x^2}$ and $y = 2\pi$ when $x = 3$. 3

(d) (i) Prove $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ 1

(ii) Hence or otherwise show that $\int_0^t \sin(\theta x)\cos\big(\theta(t-x)\big)dx = \dfrac{1}{2}t\sin(\theta t)$ 3

END OF PAPER

Mathematics Extension 1 Trial 2010 Solutions

1. a) $A(-3,-4)$ $\quad B(5,-1)$

$3:2$

$x = \frac{3\times5+2\times-3}{3+2}\ ;\ y = \frac{3\times-1+2\times-4}{3+2}$

$\therefore x = \frac{9}{5},\quad y = \frac{-11}{5}$

b) $f'(x) = \frac{3x^2}{x^3+1}$

c) $\frac{12^{2n}\times(3^n)^{-2}}{2^{4n}} = \frac{2^{4n}\times3^{2n}\times3^{-2n}}{2^{4n}}$

$= 1$

d) $\int_0^3 \frac{dx}{9+x^2} = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= \frac{1}{3}\left[\tan^{-1}1 - \tan^{-1}0\right]$

$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{12}$

e) $\frac{x+4}{x-1} \leq 6 \quad , \quad x \neq 1$

$(x-1)^2\frac{(x+4)}{\cancel{x-1}} \leq 6(x-1)^2$

$6(x-1)^2 - (x-1)(x+4) \geq 0$

$(x-1)[6(x-1)-(x+4)] \geq 0$

$(x-1)(5x-10) \geq 0$

$\therefore x < 1$ or $x \geq 2$ $\quad$ 1 $\quad$ 2

OR Solve $\frac{x+4}{x-1} = 6, \quad x \neq 1$

$x+4 = 6x-6$

$x = 2$

✓ o × ● ✓

1 $\quad$ 2 $\qquad \therefore x<1$ or $x \geq 2$

2 a) $y = \cos^{-1}x \ :\ -1 \leq x \leq 1$

$0 \leq y \leq \pi$

$\therefore y = 2\cos^{-1}\frac{x}{3}$

$D: -1 \leq \frac{x}{3} \leq 1 \qquad \therefore -3 \leq x \leq$

$R: \quad 0 \leq y \leq 2\pi$

b) Term $= \binom{7}{5}3^2(2x)^5$

$\therefore$ Coeff $= 21\times9\times32$

$= 6048$

c) $x = u^2 - 1$

$\frac{dx}{du} = 2u$

$\therefore dx = 2u\,du$

$\sqrt{1+x} = \sqrt{u^2} = u$

$\therefore \int\frac{x}{\sqrt{1+x}}dx = \int\frac{(u^2-1)2\cancel{u}\,du}{\cancel{u}}$

$= 2\left(\frac{u^3}{3} - u\right)$

$= 2\left[\frac{(\sqrt{x+1})^3}{3} - \sqrt{x+1}\right] + C$

d) $\int_0^{\frac{\pi}{6}}\cos^2 3x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}}(\cos 6x+1)\,dx$

$= \frac{1}{2}\left[\frac{\sin 6x}{6} + x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(0 + \frac{\pi}{6} - (0+0)\right)$

$= \frac{\pi}{12}$

2. e) i) $f(x) = e^x - 3\cos x$

$f(0) = e^0 - 3\cos 0 = -2$

$f(1) = e^1 - 3\cos 1 \doteqdot 1.097$

Opposite signs

$\therefore$ Curve cuts x-axis between $x=0$, $x=1$

$\therefore$ root between $x=0$, $x=1$

$f(x) = e^x - 3\cos x$

$f'(x) = e^x + 3\sin x$

$f(1) = 1.097, \quad f'(1) = 5.2427$

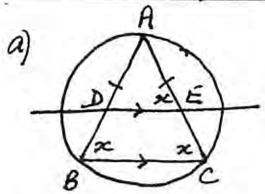
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\doteqdot 1 - \frac{1.097}{5.2427}$

$\doteqdot 0.79075\ldots$

$\doteqdot 0.79$ (2 d p)

3. a)



Let $\angle ABC = x$

$\angle ACB = \angle ABC$ (angles opposite equal sides in isosc. Δ)

$\therefore \angle ACB = x$

$\angle AED = \angle ACB$ (corresp. $\angle$s $BC \parallel DE$)

$\therefore \angle AED = x$

$\therefore \angle AED = \angle ABC$

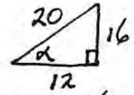
$\therefore BCED$ is a cyclic quad (exterior $\angle$ = int. opp $\angle$)

3 b) i) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$12\cos\theta + 16\sin\theta = R\cos(\theta-\alpha)$

$\therefore R\cos\alpha = 12$

$R\sin\alpha = 16$



$\therefore R = 20$ and $\tan\alpha = \frac{16}{12}$

$\therefore \alpha = 0.9273$

$\therefore 12\cos\theta + 16\sin\theta$

$= 20\cos(\theta - 0.9273)$

ii) $20\cos(\theta-0.9273) = 10$

for $-0.9273 \leq \theta - 0.9273 \leq 2\pi - 0.9273$

$\cos(\theta-0.9273) = \frac{1}{2}$

$\therefore \theta - 0.9273 = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{3} + 0.9273,\ \frac{5\pi}{3} + 0.9273$

$\doteqdot 1.97,\ 6.16$ (2 d p)

c) $x^2 - 1 = (x-1)(x+1)$

$P(x) = 3x^3 - 4x^2 + ax + b$

$P(1) = 3 - 4 + a + b = 0$

$\therefore a + b = 1$ — (1)

$P(-1) = -3 - 4 - a + b = 0$

$\therefore -a + b = 7$ — (2)

(1)+(2) $\quad 2b = 8$

$b = 4$

$\therefore a = 1 - b$

$a = -3$

$\therefore a = -3$ and $b = 4$

3 (d) $LHS = \dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A}$

$= \dfrac{\sin A + 2\sin A\cos A}{\cancel{1} + \cos A + 2\cos^2 A \cancel{-1}}$

$= \dfrac{\sin A(1 + \cancel{2\cos A})}{\cos A(1 + \cancel{2\cos A})}$

$= \tan A \quad = RHS.$

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4 a) $\alpha\beta\gamma = \dfrac{-d}{a}$ $\quad \therefore -3\times 4\times\gamma = \dfrac{-12}{3}$

$\therefore$ third root is $\frac{1}{3}$ //

b) i) $n = 1$ $\quad LHS = \dfrac{1}{1\times 4} = \dfrac{1}{4}$ $\quad RHS = \dfrac{1}{(3-2)(3+1)} = \dfrac{1}{4}$ $\quad \therefore$ true $n=1$

Assume true $n = k$

Assume $\dfrac{1}{1\times 4} + \dfrac{1}{4\times 7} + \cdots + \dfrac{1}{(3k-2)(3k+1)} = \dfrac{k}{3k+1}$

Now prove true $n = k+1$

ie. prove $\dfrac{1}{1\times 4} + \dfrac{1}{4\times 7} + \cdots + \dfrac{1}{(3k-2)(3k+1)} + \dfrac{1}{(3k+1)(3k+4)} = \dfrac{k+1}{3k+4}$

$LHS = \dfrac{k}{3k+1} + \dfrac{1}{(3k+1)(3k+4)}$ (using the assumption)

$= \dfrac{k(3k+4) + 1}{(3k+1)(3k+4)}$

$= \dfrac{3k^2 + 4k + 1}{(3k+1)(3k+4)}$

$= \dfrac{\cancel{(3k+1)}(k+1)}{\cancel{(3k+1)}(3k+4)}$

$= \dfrac{k+1}{3k+4} \quad = RHS$

$\therefore$ True for $n = k+1$ if true for $n = k$

$\therefore$ By principal of mathematical induction, true for all integral $n \geq 1$

b) ii)

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \dfrac{n}{3n+1}$

$= \lim_{n\to\infty} \dfrac{1}{3 + \frac{1}{n}}$

$= \dfrac{1}{3+0}$

$= \dfrac{1}{3}$

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4 c) i) $y = \dfrac{2 - \sin\theta}{\cos\theta}$, $0 \leq \theta \leq \dfrac{\pi}{4}$

$\dfrac{dy}{d\theta} = \dfrac{\cos\theta(-\cos\theta) - (2-\sin\theta)\times -\sin\theta}{\cos^2\theta}$

$= \dfrac{-\cos^2\theta + 2\sin\theta - \sin^2\theta}{\cos^2\theta}$

$= \dfrac{2\sin\theta - 1}{\cos^2\theta}$

$= \sec^2\theta(2\sin\theta - 1)$

ii) $\dfrac{dy}{d\theta} = 0$ $\quad \therefore \sec^2\theta = 0$ or $\sin\theta = \frac{1}{2}$ $\quad \left(0 \leq \theta \leq \frac{\pi}{4}\right)$

no solution $\quad \theta = \dfrac{\pi}{6}$

Test

θ	0.5	$\pi/6$	0.6
$\frac{dy}{d\theta}$	$-$	0	+

$\left(\dfrac{\pi}{6} \doteqdot 0.523\right)$

$\therefore$ Minimum occurs at $\theta = \pi/6$

Min value is $\dfrac{2 - \sin\pi/6}{\cos\pi/6} = \dfrac{2 - \frac{1}{2}}{\sqrt{3}/2}$

$= \dfrac{3}{2} \times \dfrac{2}{\sqrt{3}}$

$= \sqrt{3}$ //

iii) Max will occur at an endpoint

$\theta = 0$, $y = \dfrac{2 - \sin 0}{\cos 0}$

$= 2$

$\theta = \dfrac{\pi}{4}$, $y = \dfrac{2 - \sin\pi/4}{\cos\pi/4}$

$= \dfrac{2 - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

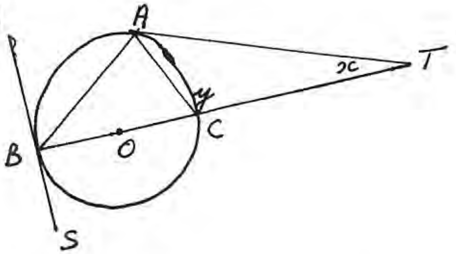
$= \dfrac{2\sqrt{2} - 1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{1}$

$= 2\sqrt{2} - 1$

$\doteqdot 1.828\ldots$

$\therefore$ Max value is 2 //

5. a) i)



In $\triangle ACT$, $\angle CAT = 180° - (x+y)°$ (angle sum of triangle

$\angle BAC = 90°$ (angle in a semi circle)

$\angle ABC = \angle CAT = 180° - (x+y)°$ (angle between tangent & chord drawn to point of contact equals angle in the alternate segment)

$\therefore$ In $\triangle ABT$, $\angle ABT + \angle BTA + \angle TAB = 180°$ ($\angle$ sum of $\triangle$)

$\therefore 180° - (x+y) + x + \cancel{180}° - (x+y) + 90° = \cancel{180}°$

$270° - \cancel{x} - y + \cancel{x} - x - y = 0$

$\therefore \quad 270° = x + 2y$

ii) $\angle OBS = 90°$ ($\angle$ between tangent and radius)

$\therefore \angle ABS = \angle OBS + \angle ABO$

$= 90° + 180° - x - y$

In part (i) $x = 270° - 2y$

$\therefore \angle ABS = 90° + 180° - (270° - 2y) - y$

$= y.$

b) $N = \dfrac{900}{1+be^{-ct}}$

i) $\dfrac{dN}{dt} = \dfrac{1+be^{-ct} \times 0 - 900 \times -cbe^{-ct}}{(1+be^{-ct})^2}$

$= \dfrac{900cbe^{-ct}}{(1+be^{-ct})^2}$

$\therefore \dfrac{dN}{dt} = \dfrac{cN}{900}(900-N)$

$\dfrac{cN}{900}(900-N)$

$= \dfrac{c\,\cancel{900}}{\cancel{900}(1+be^{-ct})}\left(900 - \dfrac{900}{1+be^{-ct}}\right)$

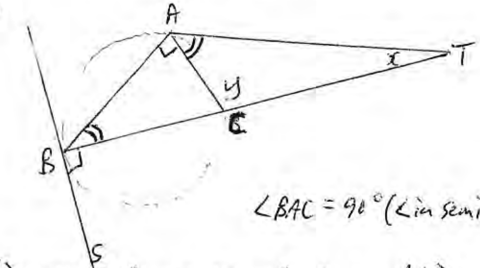
$= \dfrac{c}{1+be^{-ct}}\left[\dfrac{900(1+be^{-ct}) - 900}{1+be^{-ct}}\right]$

$= \dfrac{c}{1+be^{-ct}} \times \dfrac{900be^{-ct}}{1+be^{-ct}}$

$= \dfrac{900cbe^{-ct}}{(1+be^{-ct})^2}$

Alternative solution

5a)



$\angle BAC = 90°$ ($\angle$ in semi-circle)

(i) $y = 90° + \angle ABC$ (ext. $\angle$ of $\triangle$)

$\angle ABC = 180 - (90 + 180 - y)$

$= y - 90$

$\angle CAT = \angle ABC$ ($\angle$ between tangent + chord drawn to point of contact equals angle in the alternate segment).

$= y - 90$

$\therefore$ in $\triangle ACT$

$y + y - 90 + x = 180$ ($\angle$ sum of $\triangle$)

i.e $\underline{2y + x = 270}$

(ii) $\angle CBS = 90°$ (tangent line $\perp$ to diameter,

$\angle ABC = y - 90$ (from (i))

$\therefore \angle ABS = \angle CBS + \angle ABC$ (adj angles)

$= 90 + y - 90$

$= \underline{y.}$

5b) (i) $N = \frac{900}{1+be^{-ct}}$

Alternative solution

$\Rightarrow N + Nbe^{-ct} = 900$

$Nbe^{-ct} = 900 - N \quad -(1)$

also $1 + be^{-ct} = \frac{900}{N} \quad (2)$

$\frac{dN}{dt} = \frac{u'v - v'u}{v^2}$

$= \frac{--cbe^{-ct}(900)}{(1+be^{-ct})^2}$

$= \frac{900\,cbe^{-ct}}{(1+be^{-ct})(1+be^{-ct})}$

$= N\frac{cbe^{-ct}}{1+be^{-ct}}$ (sub (1) & (2) here gives:)

$= c \cdot \frac{N}{900}(900 - N)$

$=$ RHS

alternative:

(i) $N = \frac{900}{1+be^{-ct}}$

$1 + be^{-ct} = \frac{900}{N}$

$be^{-ct} = \frac{900}{N} - 1$

$-cbe^{-ct} = \frac{-900}{N^2}\frac{dN}{dt}$

$\therefore \frac{dN}{dt} = \frac{-cbe^{-ct}N^2}{-900}$

$= \frac{cN}{900}(Nbe^{-ct})$

$= \frac{cN}{900}(900 - N)$

5 b) ii) $N = \frac{900}{1+be^{-ct}}$

$t = 0,\ N = 300 \quad \therefore 300 = \frac{900}{1+be^{0}}$

$1 + b = \frac{900}{300} \quad \therefore b = 2$

$\frac{dN}{dt} = \frac{CN}{900}(900 - N)$

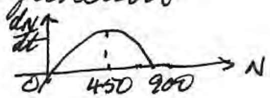
$t = 0,\ \frac{dN}{dt} = 20 \quad \therefore 20 = \frac{C \times 300}{900}(900 - 300)$

$N = 300 \quad 20 = \frac{C}{3} \times 600 \quad \therefore C = 0.1$

iii) $\frac{dN}{dt} = \frac{0.1N}{900}(900 - N)$

$= \frac{N}{9000}(900 - N)$

This is a concave down quadratic function and $\frac{dN}{dt} = 0$ at $N = 0$ or 900.



$\therefore$ Max occurs when $N = 450$

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6 a) i) $y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

at $P(2ap, ap^2)$, $y' = \frac{4ap}{4a}$

$= p$

Equation of tangent is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

ii) $R:\ y = 0$

$\therefore 0 = px - ap^2$

$x = \frac{ap^2}{p}$

$x = ap$

$T:\ x = 0$

$\therefore y = 0 - ap^2$

$= -ap^2$

R is $(ap, 0)$, T is $(0, -ap^2)$

iii) For Q: $x = ap$, $y = -ap^2$

$p = \frac{x}{a}$

$\therefore$ Locus is $y = -a\left(\frac{x}{a}\right)^2$

$y = -\frac{x^2}{a}$ or $x^2 = -ay$

6 b) i) $y = Vt\sin\alpha - \frac{1}{2}gt^2$

$V = 75,\ \alpha = 30^\circ,\ g = 10$

$\therefore y = 75t \times \sin 30^\circ - \frac{1}{2}\times 10t^2$

$y = \frac{75}{2}t - 5t^2$

$\dot{y} = \frac{75}{2} - 10t$

$\dot{y} = 0$ for max

$\therefore 10t = \frac{75}{2}$

$t = \frac{15}{4}$

$y_{max} = \frac{75}{2}\times\frac{15}{4} - 5\times\left(\frac{15}{4}\right)^2$

$= 70.3125\ m.$

ii) $y = x\tan\alpha - \frac{10x^2\sec^2\alpha}{2\times 75^2}$

$x = 150,\quad y = 50$

$\therefore 50 = 150\tan\alpha - \frac{10\times 150^2\sec^2\alpha}{2\times 75^2}$

$50 = 150\tan\alpha - 20\sec^2\alpha$

$5 = 15\tan\alpha - 2(1+\tan^2\alpha)$

$2\tan^2\alpha - 15\tan\alpha + 7 = 0$

$(2\tan\alpha - 1)(\tan\alpha - 7) = 0$

$\tan\alpha = \frac{1}{2}$ or 7

$\alpha = 26.56\ldots$ or $81.869\ldots$

$\therefore$ Angles are 27° or 82°

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7. a) $\triangle ADC$, $\cot\beta = \frac{AD}{w}$ $\qquad$ $\triangle BDC$, $\cot\alpha = \frac{BD}{w}$

$\therefore AD = w\cot\beta$ $\qquad$ $\therefore BD = w\cot\alpha$

$AD + BD = 100$ $\qquad$ $\therefore w\cot\beta + w\cot\alpha = 100$

$w(\cot\beta + \cot\alpha) = 100$

$\therefore w = \frac{100}{\cot\alpha + \cot\beta}$

b) $(1+ax)^n = 1 + \binom{n}{1}(ax) + \binom{n}{2}(ax)^2 + \ldots$

$\therefore \binom{n}{1}a = -45$ and $\binom{n}{2}a^2 = 900$

$na = -45$ $\qquad$ $\frac{n(n-1)}{2}a^2 = 900$

$a = \frac{-45}{n}$ $\xrightarrow{\text{sub}}$ $\therefore \frac{n(n-1)}{2}\left(\frac{-45}{n}\right)^2 = 900$

$\frac{n^2-n}{n^2} = \frac{900\times 2}{45\times 45} = \frac{8}{9}$

$9n^2 - 9n = 8n^2$

$n^2 - 9n = 0$

$n(n-9) = 0$

$\therefore n = 9$ (reject $n=0$

$a = \frac{-45}{n} = \frac{-45}{9} = -5$

$\therefore a = -5,\ n = 9.$

c) $\left(\frac{dy}{dx}\right)^2 = \frac{9}{9-x^2}$

$\frac{dy}{dx} = \pm\frac{3}{\sqrt{9-x^2}}$

$\therefore y = \int\frac{dy}{dx}dx = 3\sin^{-1}\left(\frac{x}{3}\right) + c$

or $3\cos^{-1}\left(\frac{x}{3}\right) + c$

$x = 3,\ y = 2\pi$

$\therefore$ (using $y = 3\sin^{-1}\left(\frac{x}{3}\right) + c$)

$2\pi = 3\sin^{-1}1 + c$

$2\pi = 3\times\frac{\pi}{2} + c$ $\qquad$ $\therefore c = \frac{\pi}{2}$

$\therefore y = 3\sin^{-1}\left(\frac{x}{3}\right) + \frac{\pi}{2}$. or

$2\pi = 3\cos^{-1}(1) + c_2$

$2\pi = 3\times 0 + c_2$ $\qquad$ $c_2 = 0$

Using $-3\sin^{-1}\left(\frac{x}{3}\right) + \frac{7\pi}{2}$

d) i) $\sin(A+B) + \sin(A-B) = \sin A\cos B + \cancel{\cos A\sin B} + \sin A\cos B - \cancel{\cos A\sin B}$ $= 2\sin A\cos B$

$\therefore \int_0^t \sin\theta x\cos\theta(t-x)\,dx = \frac{1}{2}\int_0^t \sin(\theta x + \theta(t-x)) + \sin(\theta x - \theta(t-x))\,dx$

$= \frac{1}{2}\int_0^t \sin\theta t + \sin(2\theta x - \theta t)\,dx$

$= \frac{1}{2}\left[x\sin\theta t - \frac{1}{2\theta}\cos(2\theta x - \theta t)\right]_0^t$

$= \frac{1}{2}\left[t\sin\theta t - \frac{1}{2\theta}\cos(2\theta t - \theta t) - \left(0 - \frac{1}{2\theta}\cos(0 - \theta t)\right)\right]$

$= \frac{1}{2}\left[t\sin\theta t - \frac{1}{2\theta}\cos\theta t + \frac{1}{2\theta}\cos(-\theta t)\right]$

$\cos(-x) = \cos x$

$= \frac{1}{2}t\sin\theta t - \cancel{\frac{1}{2\theta}\cos\theta t} + \cancel{\frac{1}{2\theta}\cos\theta t}$

$= \frac{1}{2}t\sin\theta t$