ASCHAM SCHOOL
2004

MATHEMATICS EXTENSION 1: FORM VI

Time Allowed: 2 hours plus 5 minutes' reading time
Examination Date: Thursday 29 July

Instructions

All questions may be attempted
All questions are of equal value (12 marks)
All necessary working must be shown
Marks may not be awarded for careless work.
Approved calculators and templates may be used.

Collection

Start each question in a new booklet
If you use a second booklet for a question, staple it to the first.
Write your name, teacher's name and question number on each booklet

DKG

Question 1

a) Express $\dfrac{7\pi}{18}$ radians as degrees [1]

b) Solve the inequation: $\dfrac{2x+1}{x-2} \geq 1$ [3]

c) Differentiate: $\cos^{-1}\dfrac{1}{x}$ [2]

d) Evaluate: $\displaystyle\int_0^{\pi} \sin^2 x\, dx$ [3]

e) (i) On the same number plane, sketch the graphs of $y = |2x-1|$ and $y = |x+1|$ [2]

(ii) Hence or otherwise, solve $|2x-1| \leq |x+1|$ [1]

Question 2

a) If $\tan\dfrac{\theta}{2} = \dfrac{1}{2}$, find the value of $\cos 2\theta$ in exact form. [3]

b) For the parabola $x^2 = 12y$

(i) Derive the equation of the tangent at $(6t\ ,\ 3t^2)$ [3]

(ii) find the equations of the two tangents that pass through the point (5 , - 2) [3]

c) Evaluate: $\sin\left(2\sin^{-1}\dfrac{3}{4}\right)$ [3]

Question 3

a) (i) Find the domain and range of the function $y = 3\sin^{-1}(x-1)$

(ii) Sketch the graph of the function $y = 3\sin^{-1}(x-1)$ [2]

b) The volume, V, of a sphere of radius r is increasing at a constant rate of 200 mm^3 per second.

(i) Find $\dfrac{dr}{dt}$ in terms of r. [3]

(ii) Determine the rate of increase of the surface area, S of the sphere when the radius is 50 mm. (NOTE: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$] [2]

(c) A vertical flagpole CD of height h metres stands with its base C on horizontal ground. A is a point on the ground due south of C and B is a point on the ground on a bearing of 120° from C such that the distance AB is 70 metres. The angles of elevation of D from A and B are α and β respectively, where $\tan\alpha = \frac{1}{5}$ and $\tan\beta = \frac{1}{8}$. Find the exact value of h. [5]

Question 4

a) (i) Prove that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$ [2]

(i) A bug moving in a straight line has an acceleration given by $\ddot{x} = x(8-3x)$ where x is the displacement in metres from a fixed point O. Initially the bug is at the origin O and has a speed of 4 m/s. Find its speed when it is 1 m on the positive side of O. [2]

b) A particle is oscillating in simple harmonic motion such that its displacement x metres from the origin is given by the equation $\ddot{x} = -16x$ where t is the time in seconds. When $t = 0$, $v = 4$ m/s, and $x = 5$ m.

(i) Show that $x = a\cos(4t+\varepsilon)$ is a solution for this equation. (a and ε are constants) [2]

(ii) Find the period of the motion. [1]

(iii) Show that the amplitude of the oscillation is $\sqrt{26}$. [3]

(iv) What is the maximum speed of the particle? [2]

Question 5

a) Prove that $\frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x} = \sec x$ [2]

b) Prove that if a and b are both positive, then $\frac{a+b}{2} \geq \sqrt{ab}$ [2]

c) (i) Show that $x^3 - 3x + 1 = 0$ has a root α between $x = 0$ and $x = 0.5$ [2]

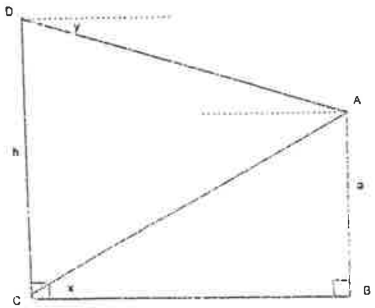
(ii) Taking $x = 0.1$ as a first approximation, use one application of Newton's method to find a closer approximation of α, giving your answer correct to four decimal places. [2]

d) If the three roots of $x^3 - 6x^2 + 3x + k = 0$ form an arithmetic series, find the value of k. [4]

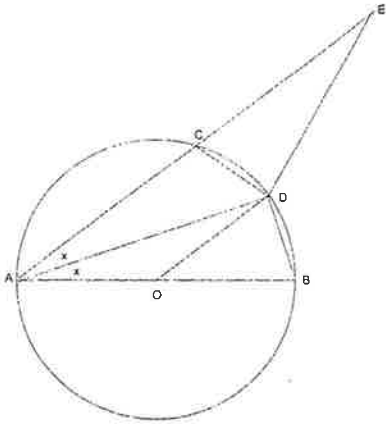
Question 6

a) From the foot of a tower CD, the angle of elevation of a building AB a metres high is x. From the top D of the tower, the angle of depression to the top A of the building is y.

Show that the height h of the tower is given by
$$h = \frac{a\sin(x+y)}{\sin x \cos y}$$ [6]



b) In the diagram, AD = DE, and DA bisects angle CAB. O is the centre of the circle.



i) The diagram is reproduced on page 5. **Staple** this into your book and **work on the diagram** on page 5.

ii) Prove: OD || AC [2]

iii) Prove: $\angle BDC = \angle ADE$ [3]

iv) Prove that $\angle CDE = 90^\circ$. [1]

Question 7

a) Use the substitution $u = x^2 + 1$ to evaluate $\int x^3\left(x^2+1\right)^3\,dx$ [4]

b) A particle is projected from a point O with an initial velocity of V and with an angle θ of elevation. (Air resistance is ignored)

(i) Given that $\ddot{x} = 0$ and $\ddot{y} = -g$, derive the equations for x and y as functions of time. [2]

(ii) If the particle is projected at an angle of 60° with a velocity of $\sqrt{2gl}$ and it passes through the point $P(l\ ,\ h)$, prove that $\frac{h}{l} = \sqrt{3} - 1$. (You may assume that the equation of the path of the projectile is $y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$). [2]

c) (i) Give the expansion for $\cos(A+B)$. [1]

(ii) Prove by mathematical induction that for integers $n \geq 1$,
$\cos \pi n = (-1)^n$ [3]

End of examination

2004 : Extension 1 Ascham Trial Exam Solutions

Question 1

a) $70°$

b) $\boxed{\times (x-2)^2}$, $x \neq 2$

$\therefore (2x+1)(x-2) \geq x^2-4x+4$

$\therefore x^2+x-6 \geq 0$

$(x+3)(x-2) \geq 0$

$\therefore x \leq -3$ or $x > 2$

c) $\frac{d}{dx}\cos^{-1}\frac{1}{x} = \frac{-\frac{1}{x^2}}{-\sqrt{1-\frac{1}{x^2}}} =$

$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$= \frac{1}{x\sqrt{x^2-1}}$

d) $\int_0^{\pi} \sin^2 x \, dx$

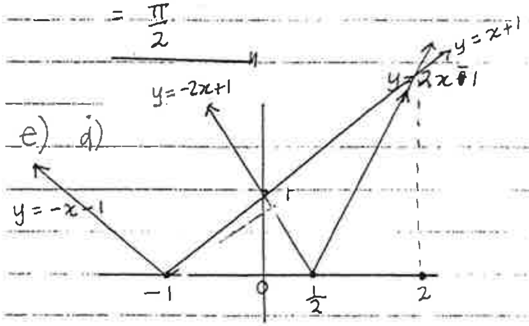
$= \frac{1}{2}\int_0^{\pi}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$

$= \frac{1}{2}[(\pi - 0) - (0-0)]$

$= \frac{\pi}{2}$

e) i)



ii) sol^n of $|2x-1| \leq |x+1|$

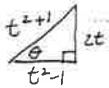
$x+1 = 2x-1$

$\therefore x = 2$

soln : $0 \leq x \leq 2$

Question 2

a) $\tan\frac{\theta}{2} = \frac{1}{2}$



let $\tan\frac{\theta}{2} = t = \frac{1}{2}$

so $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{t^2-1}{t^2+1}\right)^2 - \left(\frac{2t}{t^2+1}\right)^2$

$= \frac{t^4-2t^2+1-4t^2}{(t^2+1)^2}$

$= \frac{t^4-6t^2+1}{(t^2+1)^2}$

but $t = \frac{1}{2}$

so $\cos 2\theta = \frac{\left(\frac{1}{2}\right)^4 - 6\left(\frac{1}{2}\right)^2+1}{\left(\left(\frac{1}{2}\right)^2+1\right)^2}$

$= \frac{\frac{1}{16}-\frac{6}{4}+1}{\left(\frac{5}{4}\right)^2}$

$= -\frac{7}{25}$

b) i) $x^2 = 12y$

$\therefore y = \frac{x^2}{12}$

$\therefore y' = \frac{x}{6}$

at $x = 6t$, $y' = t$

so grad of tang $= t$

so eq^n of tang

$y - 3t^2 = t(x-6t)$

$y = tx - 3t^2$

ii) tang thru $(5,-2)$

$\therefore -2 = 5t - 3t^2$

$3t^2 - 5t - 2 = 0$

$(3t+1)(t-2) = 0$

so $t = -\frac{1}{3}$ or $t = 2$

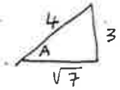
so eq^{n}s are

$y = 2x-12$ or $y = -\frac{1}{3}x - \frac{1}{3}$

c) $\sin\left(2\sin^{-1}\frac{3}{4}\right)$

let $\sin^{-1}\frac{3}{4} = A$

$\therefore \sin A = \frac{3}{4}$



$\therefore \sin\left(2\sin^{-1}\frac{3}{4}\right)$

$= \sin(2A)$

$= 2\sin A \cos A$

$= 2 \times \frac{3}{4} \times \frac{\sqrt{7}}{4}$

$= \frac{3\sqrt{7}}{8} \doteq 0.992156$

Question 3

a) i) $y = 3\sin^{-1}(x-1)$

$\therefore \frac{y}{3} = \sin^{-1}(x-1)$

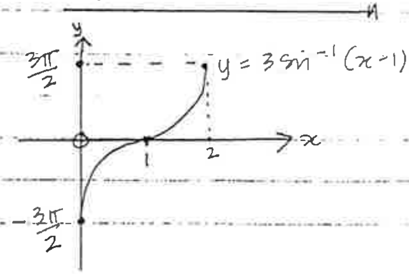
Domain: $-1 \leq (x-1) \leq 1$

$\therefore 0 \leq x \leq 2$

Range: $-\frac{\pi}{2} \leq \frac{y}{3} \leq \frac{\pi}{2}$

$-\frac{3\pi}{2} \leq y \leq \frac{3\pi}{2}$

ii)



b) $\frac{dV}{dt} = 200\ m^3/s$

$V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dr} = 4\pi r^2$

i) $\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \cdot 200$

$= \frac{50}{\pi r^2}$

ii) rate of incr of SA $= \frac{dS}{dt}$

$S = 4\pi r^2$

$\therefore \frac{dS}{dr} = 8\pi r$

and $\frac{dS}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

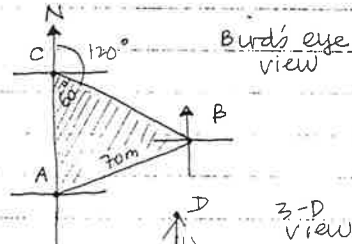
$= 8\pi r \times \frac{50}{\pi r^2}$

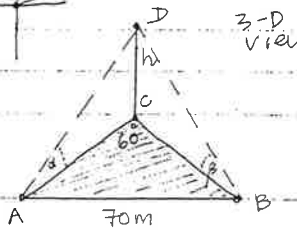
when $r = 50$,

$\frac{dS}{dt} = 8 \times \pi \times 50 \times \frac{50}{\pi \times 50^2}$

$= 8\ mm^2/s$

c)





$\angle ACB = 60^\circ$ ($\angle$'s on str line)

now $\tan\alpha = \frac{h}{AC}$

so $\frac{h}{AC} = \frac{1}{5}$

$\therefore AC = 5h$

and $\tan\beta = \frac{h}{BC}$

so $\frac{h}{BC} = \frac{1}{8}$

$\therefore BC = 8h$

in ΔABC using cosine rule:

$AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos 60^\circ$

$70^2 = 25h^2 + 64h^2 - 80h^2 \times \frac{1}{2}$

$49h^2 = 4900$

$h^2 = 100$

so $h = 10$ m

Question 4

a) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$

$= v \times \frac{dv}{dx}$

$= \frac{dx}{dt} \times \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \ddot{x}$ QED.

ii) $\ddot{x} = x(8-3x)$

$\therefore \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x - 3x^2$

$\therefore \frac{1}{2}v^2 = 4x^2 - x^3 + \frac{1}{2}c$

$\therefore v^2 = 8x^2 - 2x + c$

at $x=0$, $v=4 \Rightarrow c = 16$

$\therefore$ so $v^2 = 8x^2 - 2x + 16$

when $x = 1$

$v^2 = 8 - 2 + 16$

$v^2 = 22$

$v = \pm\sqrt{22}$

so speed is $\sqrt{22}$ m/s when $x = 1$

b) $\ddot{x} = -16x$ $t=0$, $v=4$ $x=5$

i) $x = a\cos(4t+\varepsilon)$

$\therefore \dot{x} = -4a\sin(4t+\varepsilon)$

$\therefore \ddot{x} = -16a\cos(4t+\varepsilon)$

$= -16x$

so it is a sol^n to eq^n QED.

ii) period $= \frac{2\pi}{4}$

$= \frac{\pi}{2}$

iii) $x = 5$, $v = 4$ and $t = 0$

so $x = a\cos(4t+\varepsilon)$

$\therefore 5 = a\cos\varepsilon$ — ①

and $4 = -4a\sin\varepsilon$

$-1 = a\sin\varepsilon$ — ②

squaring and adding ①&②

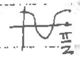
$26 = a^2\cos^2\varepsilon + a^2\sin^2\varepsilon$

$\therefore a^2 = 26$

$a = \sqrt{26}$ since $a > 0$

iv) speed is max at $\ddot{x} = 0$

$\therefore 0 = -16\sqrt{26}\cos(4t+\varepsilon)$

$\therefore \cos(4t+\varepsilon) = 0$ 

So $4t + \varepsilon = \frac{\pi}{2}$

at $(4t+\varepsilon) = \frac{\pi}{2}$, sub $\to \dot{x}$

$\dot{x} = -4\sqrt{26}\sin\frac{\pi}{2}$

$= -4\sqrt{26} \times 1$

$= -4\sqrt{26}$

so max speed $= 4\sqrt{26}$ m/s

Question 5

a) LHS $= \frac{\sin 2x}{\sin x} - \frac{\cos 2x}{\cos x}$

$= \frac{2\sin x\cos x}{\sin x} - \frac{\cos^2 x - \sin^2 x}{\cos x}$

$= \frac{2\cos^2 x - \cos^2 x + \sin^2 x}{\cos x}$

$= \frac{\sin^2 x + \cos^2 x}{\cos x}$

$= \frac{1}{\cos x}$

$= \sec x$

$=$ RHS QED

b) To pr: $\frac{a+b}{2} \geq \sqrt{ab}$

$\Leftrightarrow \frac{a+b}{2} - \sqrt{ab} \geq 0$

LHS $= \frac{a - 2\sqrt{ab} + b}{2}$

$= \frac{(\sqrt{a} - \sqrt{b})^2}{2}$

now $(\sqrt{a}-\sqrt{b})^2 > 0$ for $a \neq b$

and $(\sqrt{a}-\sqrt{b})^2 = 0$ for $a = b$

So $\frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0$

i.e $\frac{a+b}{2} \geq \sqrt{ab}$ QED

c) i) let $f(x) = x^3 - 3x + 1$

$\therefore f(0) = 1$

$f(0.5) = (\frac{1}{2})^3 - 3(\frac{1}{2}) + 1$

$= -0.375$

so root lies b/w

$x = 0$ and $x = 0.5$

since $f(x)$ is continuous

ii) $x_0 = 0.1$

by NM, $x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$

$f'(x) = 3x^2 - 3$

$f'(0.1) = 3(0.1)^2 - 3 = -2.97$

$f(0.1) = (0.1)^3 - 3(0.1) + 1 = 0.70$

$\therefore x_1 = 0.1 + \frac{0.701}{-2.97}$

$= 0.3360269\ldots$

So $\alpha = 0.3360$ (to 4 d.p)

d) $x^3 - 6x^2 + 3x + k = 0$

let roots be $a-d$, a, $a+d$

sum of roots $= \frac{6}{1}$

so $3a = 6$

$a = 2$

sum of prod of pairs $= \frac{c}{a}$

so $a(a-d) + (a-d)(a+d) + a(a+d) =$

$a^2 - ad + a^2 - d^2 + a^2 + ad = 3$

$3a^2 - d^2 = 3$

$\therefore 12 - d^2 = 3$

$d^2 = 9$

$\therefore d = \pm 3$

prod of roots $= \frac{-k}{a}$

$a(a-d)(a+d) = -k$

$2(2-d)(2+d) = -k$

$\therefore 8 - 2d^2 = -k$

when $d = 3$ $\therefore -k = 8 - 18$

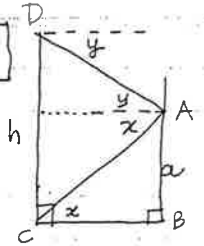
$\therefore k = 10$

when $d = -3$ $-k = 8 - 18$

$k = 10$

Question 6

a)



$\angle DAC = x + y$

$\sin x = \dfrac{a}{CA}$

$\therefore \quad CA = \dfrac{a}{\sin x}$

$\angle CDA = 90 - y$

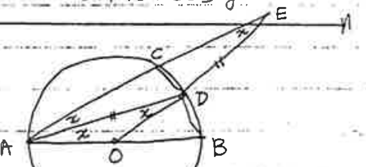
in Δ DAC

$\dfrac{h}{\sin(x+y)} = \dfrac{CA}{\sin(90-y)}$

$\therefore h = \dfrac{CA \cdot \sin(x+y)}{\cos y}$

$= \dfrac{a \sin(x+y)}{\sin x \cdot \cos y}$ QED.

b)



i) $\angle ADO = x$ $(AO = OD)$

$= \text{alt } \angle DAC$

so $OD \parallel AC$ QED.

ii) $\angle CED = x$ $(AD = DE)$

$\therefore \angle ADE = 180 - 2x$

($\angle$ sum ΔADE)

$\angle CDB = 180 - 2x$

(opp $\angle$ cyclic quad ACDB)

ergo $\angle BDC = \angle ADE$ QED.

iii) let $\angle ADC = a$

now $\angle ADB = 90$ ($\angle$ in semi ⊙)

so $\angle BDC = 90 + a$

$= \angle ADE$ (proved above)

so $\angle CDE = 90°$ QED

Question 7

a) $\int x^3 (x^2+1)^3 \, dx$

let $u = x^2 + 1 \quad \therefore x^2 = u - 1$

$\therefore \dfrac{du}{dx} = 2x$

$\therefore dx = \dfrac{du}{2x}$

$\therefore \int x^3 (x^2+1)^3 \, dx = \int x^3 (u)^3 \dfrac{du}{2x}$

$= \frac{1}{2} \int x^2 . u^3 \, du$

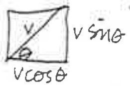
$= \frac{1}{2} \int (u-1) . u^3 \, du$

$= \frac{1}{2} \int (u^4 - u^3) \, du$

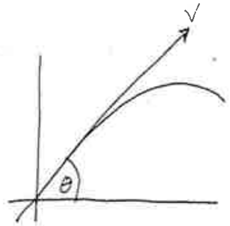
$= \frac{1}{2} \left(\dfrac{u^5}{5} - \dfrac{u^4}{4} \right) + c$

$= \frac{1}{2} \times \dfrac{(x^2+1)^5}{5} - \frac{1}{2} \times \dfrac{(x^2+1)^4}{4} + c$

$= \dfrac{(x^2+1)^5}{10} - \dfrac{(x^2+1)^4}{8} + c$



b)



initially

$\dot{x} = v \cos\theta$

$\dot{y} = v \sin\theta$

$\ddot{x} = 0$

$\dot{x} = C$

$\boxed{\dot{x} = V \cos\theta}$

$x = Vt \cos\theta + D$

when $t = 0$, $x = 0 \rightarrow D = 0$

$\therefore \boxed{x = Vt \cos\theta}$

$\ddot{y} = -g$

$\dot{y} = -gt + E$

when $t = 0$, $\dot{y} = V \sin\theta$

$\Rightarrow E = V \sin\theta$

$\therefore \boxed{\dot{y} = -gt + V \sin\theta}$

$y = -\frac{1}{2} g t^2 + Vt \sin\theta + F$

when $t = 0$, $y = 0 \quad \therefore F = 0$

$\boxed{y = \frac{1}{2} g t^2 + vt \sin\theta}$

ii) path of projectile

$y = x \tan\theta - \dfrac{g x^2}{2V^2} \sec^2\theta$ — [1]

$\theta = 60°$, $V = \sqrt{2gl}$

thru $P(l, h)$

sub → [1]

$\therefore y = x(\sqrt{3}) - \dfrac{g x^2}{2V^2}(2^2)$

$\therefore y = \sqrt{3} x - \dfrac{2 g x^2}{V^2}$

so $h = \sqrt{3} l - \dfrac{2 g l^2}{2gl}$

$h = \sqrt{3} l - l$

$h = l(\sqrt{3} - 1)$

$\therefore \dfrac{h}{l} = \sqrt{3} - 1$ QED

c) i) $\cos(A+B) = \cos A . \cos B - \sin A . \sin B$

ii) to prove: $\cos \pi n = (-1)^n$, $n \geq 1$

for $n = 1$ LHS $= \cos \pi$

$= -1$

$=$ RHS

so statement is true for $n = 1$.

assume true for $n = k$

i.e. $\cos \pi k = (-1)^k$ —— [1]

now to prove statement true for $n = k+1$

i.e $\cos \pi(k+1) = (-1)^{k+1}$

LHS $= \cos(\pi k + \pi)$

$= \cos \pi k . \cos \pi - \sin \pi k . \sin \pi$

$= -\cos \pi k + 0$

$= -\cos \pi k$

$= -(-1)^k$ by induct. hyp [1]

$= (-1)^1 (-1)^k$

$= (-1)^{k+1}$

$=$ RHS

so ∴ it follows by mathematical induction that statement is true for all $n \geq 1$.