Ascham School

Mathematics Extension 1

Trial HSC Examination

Wednesday 29th July 2020

2 hours

General Instructions
- Reading time – 10 minutes
- Working time – 2 hours
- Write using black non-erasable pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/ or calculations

Total marks: 70

Section I – 10 marks (pages 2–3)
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 4–7)
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

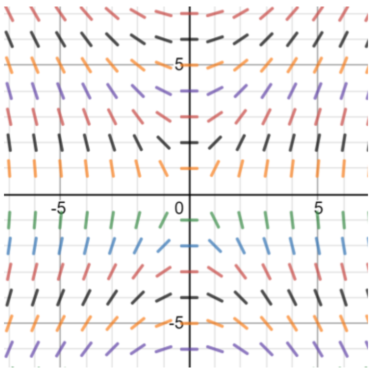
10 marks

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 The slope field below could represent which of the following differential equations?



A. $\frac{dy}{dx}=\frac{2x}{y}$ B. $\frac{dy}{dx}=\frac{2y}{x}$ C. $\frac{dy}{dx}=\frac{x^2}{y^2}$ D. $\frac{dy}{dx}=\frac{y^2}{x^2}$

2 Each of the letters A, A, C, H, M, and S is written on a separate card. The cards are drawn at random from a hat and placed next to each other to form a word. What is the probability that the word ASCHAM appears?

A. $\frac{1}{60}$ B. $\frac{1}{360}$ C. $\frac{1}{720}$ D. $\frac{1}{50}$

3 The solution for the inequality $\frac{3}{x-1}<5$ can be expressed as:

A. $x\in\left[\frac{8}{5},\infty\right)$

B. $x\in(-\infty,1]\cup\left[\frac{8}{5},\infty\right)$

C. $x\in\left(\frac{8}{5},\infty\right)$

D. $x\in(-\infty,1)\cup\left(\frac{8}{5},\infty\right)$

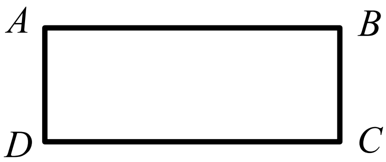
4 A possible solution for $\tan\frac{\theta}{2}$ given $2\cos\theta+\sin\theta=-1$ is:

A. -3 B. $-\frac{1}{3}$ C. $\frac{1}{3}$ D. 3

5 $\int \frac{x}{\sqrt{1-x^2}}\,dx =$

A. $\sin^{-1} x + c$ B. $-x\cos^{-1} x + c$ C. $-\frac{1}{2}\sqrt{1-x^2} + c$ D. $-\sqrt{1-x^2} + c$

6 Consider the rectangle below.



Which of the following statements is **false**?

A. $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ B. $\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{DB}$

C. $\overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{BD}$ D. $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$

7 A projectile has the equation of path $y = -3x^2 + 2x + 4$. How far will it have travelled horizontally before it returns to its original height?

A. $\frac{1}{3}$ unit   B. $\frac{2}{3}$ unit   C. 2 units   D. 4 units

8 The coefficient of the fourth term in the expansion of $(3x-4)^6$ is:

A. 34560   B. -34560   C. 25920   D. -25920

9 The graph of the function $y = \cos^{-1}(2x)$ is dilated horizontally by a dilation factor of 4 and then translated vertically by 3 units. What is its new equation?

A. $y = \cos^{-1}\left(\frac{2x}{4} + 3\right)$ B. $y = \cos^{-1}\left(\frac{2x}{4}\right) + 3$

C. $y = \cos^{-1}(8x) + 3$ D. $y = \cos^{-1}(8x+3)$

10 5 boys and 5 girls are seated at a round table. In how many ways can this happen with Robert and Hannah sitting together?

A. 725760   B. 362880   C. 80640   D. 40320

Section II

60 marks

Attempt Questions 11–14

Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Given $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(i) Find $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ **2**

(ii) Hence, find $(\underset{\sim}{a} + \underset{\sim}{b}) \bullet (\underset{\sim}{a} - \underset{\sim}{b})$. State if $\underset{\sim}{a} + \underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$ and justify your answer with a reason. **2**

(b) The polynomial $P(x) = x^3 - 4x^2 + tx + 10$ has roots $\alpha, -\alpha$ and β. Find the three roots and hence find the value of *t*. **3**

(c) Zara is testing a batch of homemade bullets on her shooting range.
She holds her rifle at shoulder height (170 cm above the ground) and shoots horizontally with an initial speed of 360 m/s at a target 100m away.
Assume acceleration due to gravity is 10m/s^2.

(i) Find expressions for $\dot{y}$ and $\dot{x}$. **2**

(ii) Find expressions for y and x. **2**

(iii) Hence find the equation of path as y in terms of x. **1**

(iv) Assuming the ground is horizontal at her range, how far above the ground will the bullet hit the target? Give your answer correct to the nearest cm. **1**

(v) When does the bullet hit her target? Find the impact speed (correct to 5 significant figures). **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Find the particular solutions to the differential equation $\frac{dy}{dx} = \frac{2x-1}{y^2}$ given $y = \sqrt[3]{3}$ when $x = 0$. **3**

(b) Solve $3\cos 3x - 4\sin 3x = 5$ for $0 \le x \le \frac{2\pi}{3}$. **3**

(c) The area bounded by the line $y = x - 1$ and the curve $y = 4(x-1)^2$ is rotated about the x-axis.

(i) Sketch the bounded area, noting any intercepts. **2**

(ii) Find the volume generated. **3**

(d) Given $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$,

(i) Find $proj_{\underset{\sim}{u}}\underset{\sim}{v}$ **2**

(ii) Find $\underset{\sim}{w} = \underset{\sim}{v} - proj_{\underset{\sim}{u}}\underset{\sim}{v}$ **1**

(iii) What is the angle between $\underset{\sim}{u}$ and $\underset{\sim}{w}$? **1**

End of Question 12

<u>Question 13</u> (15 marks) Use the Question 13 Writing Booklet.

(a) Prove by mathematical induction: **3**

$$a+ar+ar^2+ar^3+...+ar^n=\frac{a(r^{n+1}-1)}{r-1} \quad \text{for } n\geq 1$$

(b) Use the substitution $u=x\ln x$ to evaluate $\int_1^2 \frac{\ln x^2+2}{(x\ln x-1)^2}dx$. **3**

(c) A population of 1200 feral cats is released into Darling Point. The rate of increase of the feral cat population is: $\frac{dP}{dt}=\frac{P}{30}\left(1-\frac{P}{10000}\right)$ where P is the feral cat population and t is the number of months.

(i) Show that $\frac{10000}{P(10000-P)}=\frac{1}{P}+\frac{1}{10000-P}$ **1**

(ii) Hence solve the differential equation $\frac{dP}{dt}=\frac{P}{30}\left(1-\frac{P}{10000}\right)$ for P in terms of t. **3**

(iii) Hence find the limiting feral cat population. **1**

(iv) How many months does it take for the population to double? **1**

(d) Without the use of calculus, find the maximum and minimum points on the curve of $y=\frac{5}{2+\sin x+\cos x}$ for $0\leq x\leq 2\pi$ **3**

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Solve $\sin 7x + \sin x = \sin 4x$ for $0 \le x \le \frac{\pi}{2}$. **3**

(b) Find $\int_0^{\frac{\pi}{8}} \cos^4 x \, dx$. **3**

(c) Given that $f(x) = \cos^{-1}(2x-1) - 2\cos^{-1}\sqrt{x}$ for $0 \le x \le 1$, show that $f'(x) = 0$. **3**

(d) The ferry at Double Bay is being pulled into the wharf by a rope at a speed of 25m/min. The rope is attached to a point on the ferry 6m vertically below the level of the wharf. At what rate is the rope being drawn in, when the ferry is 28m from the wharf? **3**

(e) (i) Sketch the graph of $y = x^2 - 2x - 4$ **1**

(ii) Hence sketch the graph of $y = \frac{1}{|x^2 - 2x - 4|}$ **2**

End of Question 14

END OF EXAM

Ascham Maths Ext 1 Trial 2020

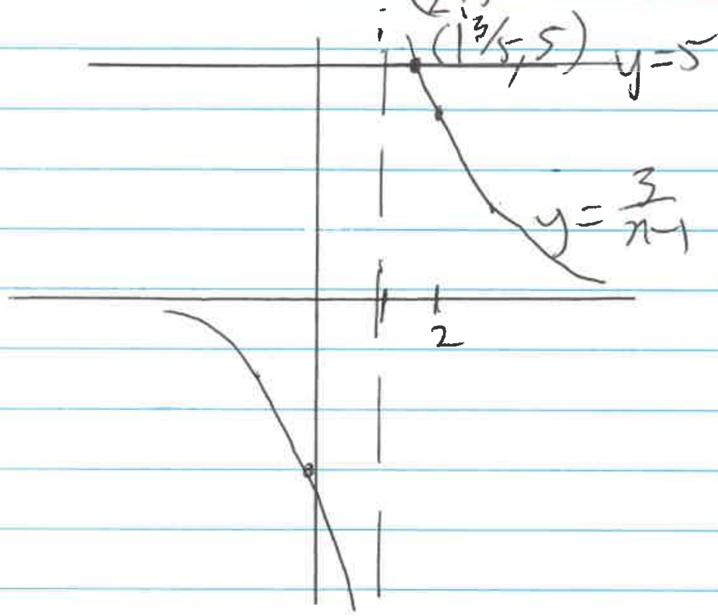
SOLUTIONS AND MARKING ALLOCATION

Section 1 Multiple Choice

1. A

2. B $\quad P(ASCHAM) = \dfrac{1}{\left(\frac{6!}{2!}\right)} = \dfrac{1}{360}$

3.



let $\dfrac{3}{x-1} = 5$

$\dfrac{3}{5} = x-1$

$x = 1\frac{3}{5}$

D

4. $t = \tan\frac{\theta}{2} \qquad \cos\theta = \dfrac{1-t^2}{1+t^2}, \quad \sin\theta = \dfrac{2t}{1+t^2}$

$2\cos\theta + \sin\theta = -1$

$\dfrac{2(1-t^2)}{1+t^2} + \dfrac{2t}{1+t^2} = -1$

D

$2(1-t^2) + 2t = -(1+t^2)$

$2 - 2t^2 + 2t = -1 - t^2$

$0 = t^2 - 2t - 3$

$0 = (t+1)(t-3)$

$\therefore t = -1$ or $t = 3$

5 $$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int -2x\,(1-x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{1}{2}\int f'(x)\,(f(x))^{-\frac{1}{2}}\,dx$$

D

$$= -2\times\frac{1}{2}\,(f(x))^{\frac{1}{2}} + C$$

$$= -\sqrt{1-x^2} + C$$

6 C

7 original height when $y=4$

$$-3x^2+2x+4=4$$
$$-3x^2+2x=0$$
$$-x(3x-2)=0$$
$$x=0 \quad \text{or} \quad x=\frac{2}{3}$$

B.

8. $(3x-4)^6 = {}^6C_0\,(3x)^6(-4)^0 + \ldots + {}^6C_3\,(3x)^3(-4)^3$

coefficient $= {}^6C_3 \times 3^3 \times (-4)^3$
$= -34560$

B.

9. B

10. $8! \times 2! = 80640$

C

Section II

11 (a) $\underset{\sim}{a} + \underset{\sim}{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$ ✓

$\underset{\sim}{a} - \underset{\sim}{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ ✓

(ii) $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = \begin{pmatrix} 7 \\ 9 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -5 \end{pmatrix}$

$= -7 + -45$
$= -52$ ✓
$\neq 0$

$\therefore$ $\underset{\sim}{a} + \underset{\sim}{b}$ is not perpendicular to $\underset{\sim}{a} - \underset{\sim}{b}$ ✓

(b) $y = \sin^{-1}(2x)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \times 2 = \frac{2}{\sqrt{1-4x^2}}$ ✓

when $x_1 = \frac{1}{4}$, $y_1 = \sin^{-1}(2 \times \frac{1}{4}) = \sin^{-1}(\frac{1}{2})$

$= \frac{\pi}{6}$

$m = \frac{2}{\sqrt{1-4(\frac{1}{4})^2}} = \frac{2}{\sqrt{\frac{3}{4}}} = \frac{4}{\sqrt{3}}$ ✓

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{6} = \frac{4}{\sqrt{3}}(x - \frac{1}{4})$ ✓

$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{6} - \frac{1}{\sqrt{3}}$

(b) $P(x) = x^3 - 4x^2 + tx + 10$

$\alpha + (-\alpha) + \beta = 4$

$\beta = 4$ ✓

$\alpha(-\alpha)\beta = -10$

$-\alpha^2 \times 4 = -10$

$\alpha^2 = \frac{5}{2}$

$\alpha = \sqrt{\frac{5}{2}}$ ✓ and $-\alpha = -\sqrt{\frac{5}{2}}$

$\alpha\beta + \alpha(-\alpha) + (-\alpha)\beta = t$

$-\alpha^2 = t$

$\therefore t = -\frac{5}{2}$

(c) (i) $\ddot{x} = 0$

$\dot{x} = \int 0 \, dt$

$= c_1$

when $t=0$, $\dot{x} = 360$

$\therefore \dot{x} = 360$ ✓

$\ddot{y} = -10$

$\dot{y} = \int -10 \, dt$

$= -10t + c_2$

when $t=0$, $\dot{y} = 0$

$\therefore c_2 = 0$

$\therefore \dot{y} = -10t$ ✓

ii) when $t=0$, $x=0$

$x = \int 360 \, dt$

$= 360t + c_3$

when $t=0$, $x=0$ $\therefore c_3 = 0$

and $x = 360t$ (1) ✓

$y = \int (-10t) \, dt$

$= -5t^2 + c_4$

when $t=0$, $y = 1.7$m

$\therefore c_4 = 1.7$

and $y = -5t^2 + 1.7$ (2) ✓

iii) from (1) $t = \frac{x}{360}$ (3) sub (3) into (2):

$y = -5\left(\frac{x}{360}\right)^2 + 1.7$

$= -\frac{x^2}{25920} + 1.7$ ✓ (4)

iv) when $x = 100$, $y = \frac{-100^2}{25920} + 1.7$

$= 1.31$ m (nearest cm)

(v) From ① above, when $x = 100$,

$100 = 360t$

$\therefore t = \frac{100}{360} = \frac{5}{18}$ second ✓

when $t = \frac{5}{18}$, $\dot{x} = 360$, $\dot{y} = -10 \times \frac{5}{18} = -\frac{50}{18}$

speed $= \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{360^2 + \left(-\frac{50}{18}\right)^2}$

$= 360.01$ m/s (5 sig figs) ✓

12 (a) $\frac{dy}{dx} = \frac{2x-1}{y^2}$

$\int y^2 dy = \int (2x-1) dx$

$\frac{y^3}{3} = x^2 - x + c$ ✓

when $x = 0$, $y = \sqrt[3]{3}$

so $\frac{(\sqrt[3]{3})^3}{3} = c$ or $c = 1$ ✓

$\frac{y^3}{3} = x^2 - x + 1$ ✓

$y^3 = 3x^2 - 3x + 3$

$y = \sqrt[3]{3x^2 - 3x + 3}$

(b) $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$

let $R\cos(3x+\alpha) = R\cos\alpha\cos 3x - R\sin\alpha\sin 3x$

$= 3\cos 3x - 4\sin 3x$

$R^2 = 3^2 + 4^2$

$= 5^2 \qquad \therefore R = 5$ ✓

$5\cos\alpha = 3 \qquad 5\sin\alpha = 4$

$\therefore \alpha = \cos^{-1}\left(\frac{3}{5}\right)$ ✓

$\therefore 5\cos\left(3x + \cos^{-1}\frac{3}{5}\right) = 5$

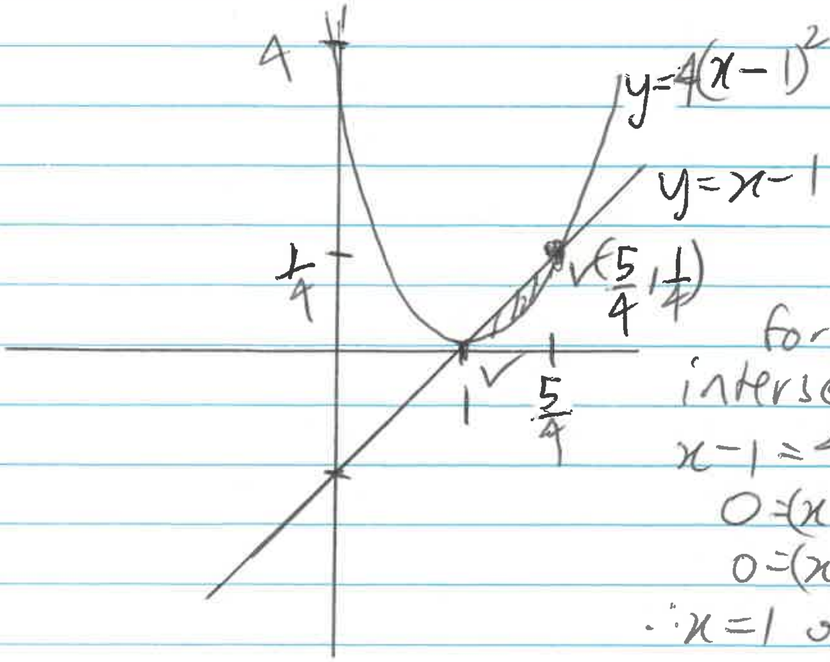
$\cos\left(3x + \cos^{-1}\frac{3}{5}\right) = 1$

$3x + \cos^{-1}\frac{3}{5} = 0,\ 2\pi$

$3x = -\cos^{-1}\frac{3}{5},\ 2\pi - \cos^{-1}\frac{3}{5}$

$x = \frac{2\pi}{3} - \frac{1}{3}\cos^{-1}\frac{3}{5}$ ✓

(c) (i)



for point of intersection;

$x-1 = 4(x-1)^2$

$0 = (x-1)(4(x-1)-1)$

$0 = (x-1)(4x-5)$

$\therefore x = 1$ or $x = 5/4$

when $x=1$, $y = 1-1 = 0$

$x = \frac{5}{4}$, $y = \frac{5}{4} - 1 = \frac{1}{4}$

(ii) $V = \pi\int_1^{\frac{5}{4}} (x-1)^2 dx - \pi\int_1^{\frac{5}{4}} (4(x-1)^2)^2 dx$ ✓

$= \pi\left[\frac{(x-1)^3}{3}\right]_1^{\frac{5}{4}} - 16\pi\left[\frac{(x-1)^5}{5}\right]_1^{\frac{5}{4}}$ ✓

$= \pi\left[\frac{(\frac{1}{4})^3}{3}\right] - 16\pi\left[\frac{(\frac{1}{4})^5}{5}\right]$

$= \frac{\pi}{480} u^3$ ✓

(d) i) $\text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = \frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} \times \underset{\sim}{u}$ $\quad$ $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$

$= \frac{(-2\times2+3\times4)}{(2^2+3^2)}(2\underset{\sim}{i} + 3\underset{\sim}{j})$

$= \frac{8}{13}(2\underset{\sim}{i} + 3\underset{\sim}{j})$

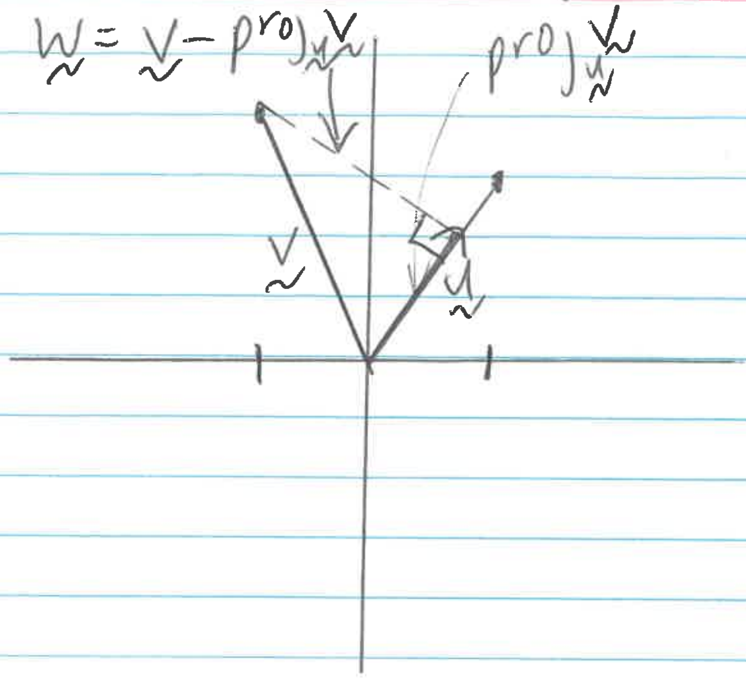
$= \frac{16}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}$ ✓ ✓

ii) $\underset{\sim}{w} = \underset{\sim}{v} - \text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - \left(\frac{16}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}\right)$

$= -\frac{42}{13}\underset{\sim}{i} + \frac{28}{13}\underset{\sim}{j}$

✓

(iii)



$\therefore \angle$ between $\underset{\sim}{w}$ and $\underset{\sim}{u} = 90°$ ✓

13 (a) $a + ar + ar^2 + ar^3 + \dots + ar^n = \dfrac{a(r^{n+1} - 1)}{r-1}$ ①

Step 1 Prove ① holds for $n = 1$

$LHS = a + ar$

$RHS = \dfrac{a(r^2 - 1)}{r-1} = \dfrac{a\cancel{(r-1)}(r+1)}{\cancel{(r-1)}}$ ✓

$= a(r+1)$

$= a + ar = LHS$

$\therefore$ ① holds for $n = 1$

Step 2 Assume ① holds for $n = k$ and prove ① holds for $n = k+1$.

i.e.

$a + ar + ar^2 + \dots + ar^k = \dfrac{a(r^{k+1} - 1)}{r-1}$ ②

$\underline{\text{RTP}}$ $a+ar+ar^2+\ldots ar^k+ar^{k+1}=\dfrac{a(r^{k+1+1}-1)}{r-1}$.

$$\begin{aligned}\text{LHS} &= a+ar+ar^2+\ldots+ar^k+ar^{k+1}\\ &= \frac{a(r^{k+1}-1)}{r-1}+ar^{k+1} \quad (\text{from } (2)) \checkmark\\ &= \frac{a(r^{k+1}-1)}{r-1}+\frac{ar^{k+1}(r-1)}{r-1}\\ &= \frac{ar^{k+1}-a+ar^{k+2}-ar^{k+1}}{r-1}\\ &= \frac{ar^{k+2}-a}{r-1} \quad \checkmark\\ &= \frac{a(r^{k+1+1}-1)}{r-1}\\ &= \text{RHS} \quad \text{as required.}\end{aligned}$$

$\therefore$ (1) holds for $n=k+1$ given it holds for $n=k$.

$\underline{\text{Step 3}}$ It follows by mathematical induction from Steps 1 and 2 that (1) holds for $n\geq 1$.

(b) $u=x\ln x$

$$\begin{aligned}\frac{du}{dx} &= x\left(\frac{d}{dx}(\ln x)\right)+\ln x\left(\frac{d}{dx}(x)\right)\\ &= x\times\frac{1}{x}+(\ln x)\times 1\\ &= \ln x+1\end{aligned}$$

$\therefore du=(\ln x+1)dx$ $\checkmark$

When $x=1$, $u=1\ln 1=0$

$x=2e$, $u=2\times\ln 2=2(\ln 2)=2\ln 2$

$$\int_1^2 \frac{\ln x^2+2}{(x\ln x-1)^2}\,dx=\int_1^2\frac{2\ln x+2}{(x\ln x-1)^2}\,dx$$

$$=2\int_1^2\frac{(\ln x+1)\,dx}{(x\ln x-1)^2}$$

$$=2\int_0^{2\ln 2}\frac{du}{(u-1)^2}$$

$$=2\int_0^{2\ln 2}(u-1)^{-2}\,du$$

$$=2\left[\frac{(u-1)^{-1}}{-1}\right]_0^{2\ln 2}$$

$$=-2\left[\frac{1}{2\ln 2-1}-\frac{1}{0-1}\right]$$

$$=-2\left[1+\frac{1}{2\ln 2-1}\right]$$

$$=-2\left[\frac{2\ln 2-1}{2\ln 2-1}+\frac{1}{2\ln 2-1}\right]$$

$$=\frac{-2(2\ln 2)}{2\ln 2-1}$$

$$=\frac{4\ln 2}{1-2\ln 2}$$

(c) (i) $\dfrac{10\,000}{P(10000-P)} = \dfrac{1}{P} + \dfrac{1}{10000-P}$

$$\text{RHS} = \frac{1}{P} + \frac{1}{10000-P} = \frac{(10000-P)+P}{P(10000-P)} \quad ✓$$

$$= \frac{10000}{P(10000-P)}$$

$= \text{LHS}$

(ii) $\dfrac{dP}{dt} = \dfrac{P}{30}\left(1 - \dfrac{P}{10000}\right) = \dfrac{P}{30}\left(\dfrac{10000-P}{10000}\right)$

$$\int \frac{10000}{P(10000-P)}\,dP = \int \frac{1}{30}\,dt \quad ✓$$

$$\int \left(\frac{1}{P} + \frac{1}{10000-P}\right)dP = \frac{1}{30}t + c$$

$$\ln P - \ln(10000-P) = \frac{1}{30}t + c \quad ✓$$

when $t=0$, $P = 1200$

$$\ln 1200 - \ln(10000-1200) = c$$

$$\therefore c = \ln\left(\frac{1200}{8800}\right)$$

$$= \ln\frac{3}{22}$$

$$\ln P - \ln(10000-P) = \frac{1}{30}t + \ln\frac{3}{22}$$

$$\ln\left(\frac{P}{10000-P}\right) = \frac{1}{30}t + \ln\frac{3}{22} \quad ✓$$

$$\frac{P}{10000-P} = e^{\left(\frac{1}{30}t + \ln\left(\frac{3}{22}\right)\right)}$$

$$P = \left(e^{\left(\frac{1}{30}t\right)} \cdot e^{\ln\left(\frac{3}{22}\right)}\right)(10000-P)$$

$$= \frac{3}{22} e^{\frac{1}{30}t}(10000) - \frac{3}{22} e^{\frac{1}{30}t}(P)$$

$$P + \frac{3}{22} e^{\frac{1}{30}t} P = \frac{3}{22} e^{\frac{1}{30}t}(10000)$$

$$P\left(1 + \frac{3}{22} e^{\frac{1}{30}t}\right) = \frac{30000}{22} e^{\frac{1}{30}t}$$

$$P = \frac{30000}{22} \frac{e^{\frac{1}{30}t}}{\left(1 + \frac{3}{22} e^{\frac{1}{30}t}\right)} \quad \begin{matrix} \div e^{\frac{1}{30}t} \\ \div e^{\frac{1}{30}t} \end{matrix}$$

$$P = \frac{30000}{22\left(e^{-\frac{1}{30}t} + \frac{3}{22}\right)} \checkmark$$

iii) as $t \to \infty$, $P \to \frac{30000}{22 \times \frac{3}{22}} = 10000$ ✓

iv) when $t = 0$, $P = 1200$

population doubles when $P = 2400$

$$2400 = \frac{30000}{22\left(e^{-\frac{1}{30}t} + \frac{3}{22}\right)}$$

$$e^{-\frac{1}{30}t} + \frac{3}{22} = \frac{30000}{22 \times 2400} = \frac{25}{22 \times 2}$$

$$e^{-\frac{1}{30}t} = \frac{25}{22\times 2} - \frac{6}{22\times 2}$$

$$= \frac{19}{44}$$

$$e^{\frac{1}{30}t} = \frac{44}{19}$$

$$\frac{1}{30}t = \ln\left(\frac{44}{19}\right)$$

$$t = 30\ln\left(\frac{44}{19}\right)$$

$$= 25.19\ldots$$

$= 25$ months (to the nearest month) ✓

(d) let $\cos x + \sin x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\cos\alpha\sin\alpha$

$R\sin\alpha = 1$ $\qquad$ $R\cos\alpha = 1$

$R = \sqrt{1^2+1^2} = \sqrt{2}$ and $\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}}$

$\therefore \alpha = \frac{\pi}{4}$

$\therefore$ $2 + \sin x + \cos x$

$= 2 + \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ has a max of $2+\sqrt{2}$ (when $x = \pi/4$) and a min of $2-\sqrt{2}$ (when $x = 5\pi/4$)

$\therefore$ $\frac{5}{2+\sin x+\cos x}$ has a max at $\frac{5}{2-\sqrt{2}}$ when $x = \frac{5\pi}{4}$ ✓

and a min at $\frac{5}{2+\sqrt{2}}$ when $x = \frac{\pi}{4}$ ✓

14 (a) $\sin 7x + \sin x = \sin 4x$ ✓

now $\frac{1}{2}(\sin(4x+3x) + \sin(4x-3x)) = \sin 4x \cos 3x$

$\therefore \sin 7x + \sin x = 2\sin 4x \cos 3x$

and $\sin 4x = 2\sin 4x \cos 3x$

$0 = 2\sin 4x \cos 3x - \sin 4x$

$0 = \sin 4x (2\cos 3x - 1)$

$\therefore \sin 4x = 0$ or $2\cos 3x - 1 = 0$

$4x = 0, \pi, 2\pi, 3\pi, 4\pi$

$\cos 3x = \frac{1}{2}$

$\therefore x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$ ✓

$3x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$3x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{9}, \frac{5\pi}{9}$ ✓

(b) Recall $\cos 2x = 2\cos^2 x - 1$

So $2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$\cos^4 x = \left(\frac{1}{2}(\cos 2x + 1)\right)^2$ ✓

$= \frac{1}{4}(\cos^2 2x + 2\cos 2x + 1)$

$= \frac{1}{4}\left(\frac{1}{2}(\cos 4x + 1) + 2\cos 2x + 1\right)$

$$\int_0^{\frac{\pi}{8}} \cos^4 x\, dx = \frac{1}{4}\int_0^{\frac{\pi}{8}}\left(\frac{1}{2}\cos 4x + \frac{1}{2} + 2\cos 2x + 1\right)dx$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{8}} (\cos 4x + 4\cos 2x + 3)\,dx$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin 4x + 2\sin 2x + 3x\right]_0^{\frac{\pi}{8}} \checkmark$$

$$= \frac{1}{8}\left[\left(\frac{1}{4}\sin\frac{\pi}{2} + 2\sin\frac{\pi}{4} + \frac{3\pi}{8}\right) - 0\right]$$

$$= \frac{1}{8}\left(\frac{1}{4} + 2\times\frac{1}{\sqrt{2}} + \frac{3\pi}{8}\right) \checkmark$$

$$= \frac{1}{32}\left(1 + 4\sqrt{2} + \frac{3\pi}{2}\right).$$

(c) $f(x) = \cos^{-1}(2x-1) - 2\cos^{-1}(\sqrt{x})$

$$f'(x) = -\frac{1}{\sqrt{1-(2x-1)^2}}\times\frac{2}{1} \checkmark - 2\,\frac{-1}{\sqrt{1-(\sqrt{x})^2}}\times\frac{1}{2}x^{-\frac{1}{2}} \checkmark$$

$$= \frac{-2}{\sqrt{1-(4x^2-4x+1)}} + \frac{1}{\sqrt{x}\sqrt{1-x}}$$

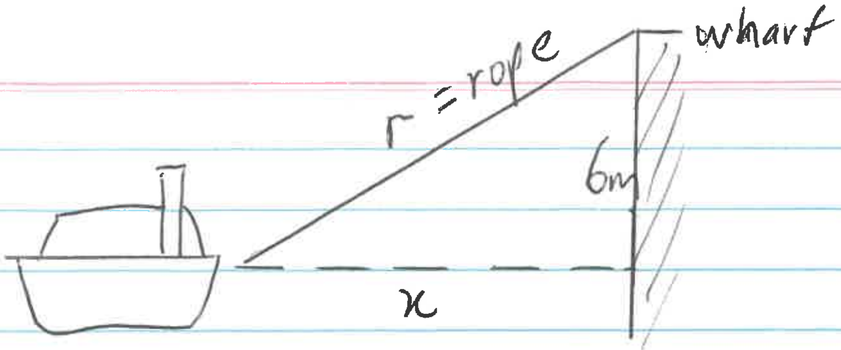
$$= \frac{-2}{\sqrt{4x-4x^2}} + \frac{1}{\sqrt{x}\sqrt{1-x}}$$

$$= \frac{-2}{\sqrt{4x(1-x)}} + \frac{1}{\sqrt{x}\sqrt{1-x}} \checkmark$$

$$= \frac{-\cancel{2}1}{\cancel{2}\sqrt{x}\sqrt{1-x}} + \frac{1}{\sqrt{x}\sqrt{1-x}} = 0$$

as required.

(d)



$$\frac{dx}{dt} = 25 \text{ m/min}$$

$$r^2 = x^2 + 6^2$$

$$r = \sqrt{x^2+36} \qquad \therefore \frac{dr}{dx} = \frac{1}{2} \times 2x\,(x^2+36)^{-\frac{1}{2}}$$

$$= \frac{x}{\sqrt{x^2+36}} \quad \checkmark$$

$$\frac{dr}{dt} = \frac{dr}{dx} \times \frac{dx}{dt}$$

$$= \frac{x}{\sqrt{x^2+36}} \times 25 \quad \checkmark$$

when $x = 28$, $\quad \frac{dr}{dt} = \frac{28}{\sqrt{28^2+36}} \times 25$

$$= \frac{28 \times 25}{\sqrt{820}}$$

$$= \frac{700}{2\sqrt{205}} \quad \checkmark$$

$$= \frac{350\sqrt{205}}{205}$$

$$= \frac{70\sqrt{205}}{41} \text{ m/min}$$

(e) $y = x^2 - 2x - 4$
$= x^2 - 2x + 1 - 5$
$= (x-1)^2 - 5$

when $y = 0$,
$(x-1)^2 = 5$
$x - 1 = \pm\sqrt{5}$
$x = 1 \pm \sqrt{5}$