BARKER COLLEGE

TRIAL HIGHER SCHOOL CERTIFICATE
1999

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

BHC
EH
BJR
LJP
GJR
RMH
CLK

PM TUESDAY 17 AUGUST

TIME ALLOWED : TWO HOURS
[Plus 5 minutes reading time]

130 copies

DIRECTIONS TO STUDENTS:

- Write your Barker Student Number on EACH AND EVERY page.
- Students are to attempt ALL questions.
  ALL questions are of equal value. [12 marks]
- The questions are not necessarily arranged in order of difficulty.
  Students are advised to read the whole paper carefully at the start of the examination.
- ALL necessary working should be shown in every question.
  Marks may be deducted for careless or badly arranged work.
- Begin your answer to each question on a NEW page. The answers to the questions in this paper are to be returned in SEVEN SEPARATE BUNDLES.
  Write on ONLY ONE SIDE of each page.
- Approved calculators and geometrical instruments may be used.
- A table of Standard Integrals is provided at the end of the paper.

* * * *

QUESTION 1. (Start a <u>NEW</u> page) **Marks**

(a) Find $\lim\limits_{x \to 0} \dfrac{\sin 5x}{2x}$ **1**

(b) Evaluate (i) $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x} + 1}\, dx$ **2**

(ii) $\displaystyle\int_0^4 \frac{3}{\sqrt{16 - x^2}}\, dx$ **2**

(c) Solve $\dfrac{2x}{x - 1} > 1$ for all real x. **2**

(d) A and B are the points (4, 5) and (8, -1) respectively. **2**
Find the point P which divides the interval AB externally in the ratio 3 : 5.

(e) Find the acute angle between the curves $y = \log_e x$ and $y = 1 - x^2$ **3**
at the point P (1, 0).

QUESTION 2. (Start a **NEW** page) **Marks**

(a) (i) Write down the expansion of $\cos(\alpha + \beta)$. **3**

(ii) Hence, or otherwise, find the exact value of $\cos 105^\circ$.

(b) A debating team consists of 12 students, 8 of whom are girls. **3**
If three students are chosen at random, what is the probability of selecting

(i) no girls at all

(ii) exactly one girl

(iii) at least two girls ?

(c) Prove that $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$ **2**

(d) Use the substitution $u = 1 - x$ to find the exact value of the integral **4**

$$\int_0^1 x\sqrt{1-x}\, dx$$

QUESTION 3. (Start a NEW page) **Marks**

(a) Melinda invites eleven guests to dinner to celebrate her birthday. Everyone is randomly seated about a round table. Find **3**

(i) the number of seating arrangements that are possible.

(ii) the probability that a particular couple, Stuart and Rachael, sit together.

(b) (i) State the domain and range of the function $f(x) = \cos^{-1} 2x$. **3**

(ii) Draw a neat sketch of the function $f(x) = \cos^{-1} 2x$, clearly labelling all essential features.

(c) (i) Find the exact value of $\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(-1)$. **3**

(ii) Hence, or otherwise, find the area bounded by the curve $y = \dfrac{1}{4 + x^2}$, the x-axis and the ordinates $x = -2$ and $x = 2\sqrt{3}$.

(c) Prove by Mathematical Induction that $7^n - 1$ is divisible by 6 for all positive integers of n. **3**

QUESTION 4. (Start a NEW page) **Marks**

(a) Given that $\sin x > 0$, differentiate $y = \sin^{-1}(\cos x)$, simplifying your answer fully. **2**

(b) Find the term independent of x in the expansion of $\left(x + \dfrac{1}{2x^2}\right)^6$. **3**

(c) Solve the equation $3\sin x + 4\cos x = 2$ for $0 \leq x \leq 2\pi$. **3**

(d) (i) Given the function $f(x) = x - \sin x - 2$ is a continuous function, determine the nature of any stationary points in the domain $0 \leq x \leq 4\pi$ and show that this function inflects at $x = n\pi$. (where n is any integer) **4**

(ii) Hence, or otherwise, draw a neat sketch of the function $f(x) = x - \sin x - 2$ over the domain $0 \leq x \leq 4\pi$.

QUESTION 5: (Start a NEW page) **Marks**

(a) Newton's Law of Cooling can be expressed in the form $\frac{dT}{dt} = -k(T - T_o)$ where T_o is the temperature of the surrounding medium and t is the time and k is a constant. 5

(i) Verify, by substitution or otherwise, that $T = T_o + Ae^{-kt}$ (where A is a constant) is the solution to the above differential equation.

(ii) A body whose temperature is $150^{\circ}C$ is immersed in a liquid kept at a constant temperature of $70^{\circ}C$. In 40 minutes, the temperature of the immersed body falls to $90^{\circ}C$. How long altogether will it take for the temperature of the body to fall to $76^{\circ}C$?

(b) The rate $\frac{dV}{dt}$ at which a balloon is pumped up is given by $\frac{dV}{dt} = 1000e^{-2t}$ 7

(i) Prove that the volume V of air present in the balloon at time t seconds is given by $V = 500(1 - e^{-2t})$.

(ii) How many seconds does it take before there is 400 cubic units of air in the balloon ?

(iii) What is the maximum volume of air which the balloon can hold ?

(iv) Assuming the balloon is spherical, find the rate at which the radius of the balloon is increasing when the balloon contains 400 cubic units of air.

QUESTION 6. (Start a **NEW** page) **Marks**

(a) Using the fact that $(1 + x)^{m+n} = (1 + x)^m(1 + x)^n$, show that **3**

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}$$

(b) A particle moves in such a way that its displacement x cm from an origin O at any time t seconds is given by the function $x = \sqrt{3}\cos 3t - \sin 3t$. **4**

(i) Show that the particle is moving in simple harmonic motion.

(ii) Find the period of the motion.

(iii) Find when the particle first passes the origin.

(c) Rambo is at the top P of a 100 metre vertical cliff PQ. A flat plain extends horizontally from the base Q of the cliff. A Sherman tank is situated somewhere on this plain at point T. Rambo fires a mortar shell from P with an initial velocity of $\frac{190}{\sqrt{3}} ms^{-1}$ at an angle of θ to the horizontal and the shell lands on the tank 20 seconds later. **5**

(i) Taking the acceleration due to gravity to be $10 ms^{-2}$, show that $\theta = 60°$.

(ii) Find the maximum height above the plain that the mortar shell reaches.

QUESTION 7. (Start a **NEW** page) **Marks**

(a) P and Q are two points on the parabola $x^2 = 4ay$ with coordinates $(2ap,\ ap^2)$ and $(2aq,\ aq^2)$ respectively. The tangents at P and Q meet at T which is situated on the parabola $x^2 = -4ay$. **6**

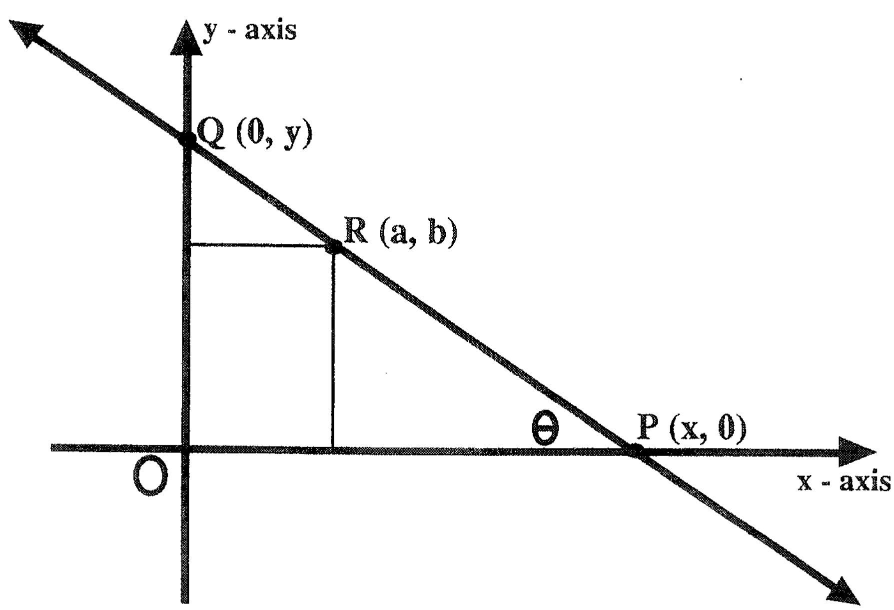
(i) Write down the equations of the tangents at P and Q.

(ii) Show that T is the point $(a(p+q),\ apq)$.

(iii) Prove that $p^2 + q^2 = -6pq$.

(iv) Find the equation of the locus of the midpoint of PQ.

(b) The point $R(a,\ b)$ lies in the positive quadrant of the number plane. A line through R meets the positive x and y axes at P and Q respectively and makes an angle θ with the x-axis. **6**



(i) Show that the length of PQ is equal to $\dfrac{a}{\cos\theta} + \dfrac{b}{\sin\theta}$.

(ii) Hence, show that the minimum length of PQ is equal to $\left(a^{2/3} + b^{2/3}\right)^{3/2}$.

Suggested Marking Scheme as a result of markers meeting

ar 12 3 Unit Trial HSC Barker College 1999 - Solutions

estion 1

$\lim\limits_{x\to0} \frac{\sin 5x}{2x} = \lim\limits_{x\to0} \frac{\sin 5x}{5x} \times \frac{5}{2}$

$= 1 \times \frac{5}{2} = \frac{5}{2}$ 1

(i) $\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$

$\frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}\,dx$

$\frac{1}{2}\left[\ln(e^{2x}+1)\right]_0^1$ 1

$\frac{1}{2}\left[\ln(e^2+1)-\ln(e^0+1)\right]$

$\frac{1}{2}\left[\ln(e^2+1)-\ln(1+1)\right]$ 1

$\frac{1}{2}\left[\ln(e^2+1)-\ln 2\right] = \frac{1}{2}\ln\left(\frac{e^2+1}{2}\right)$

$\int_0^4 \frac{3}{\sqrt{16-x^2}}\,dx$

$3\left[\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4$ 1

$\left[\sin^{-1}(1)-\sin^{-1}(0)\right]$

$\left(\frac{\pi}{2}-0\right)$

$\frac{3\pi}{2}$ 1

$\frac{2x}{x-1} > 1$

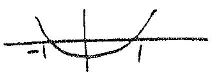
$-1)^2 \times \frac{2x}{(x-1)} > 1.(x-1)^2$

$x(x-1) > (x-1)^2$

$2x^2-2x > x^2-2x+1$ 1

$x^2-1>0$

$x-1)(x+1)>0$



$x<-1$ or $x>1$ 1

Extend $\Rightarrow$ $k:l = -3:5$

(e) For $y=\log_e x$, $y' = \frac{1}{x}$

when $x=1$, $m_1 = 1$

For $y = 1-x^2$, $y' = -2x$ 1

When $x=1$, $m_2 = -2$

$\therefore \tan\theta = \left|\frac{1+2}{1+1\times-2}\right| = \left|\frac{3}{-1}\right| = 3$ 1

$\therefore \theta \doteqdot 71°34'$ 1

Question 2

(a)(i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ 1

(ii) $\cos 105° = \cos(60°+45°)$

$= \cos 60°\cos 45° - \sin 60°\sin 45°$ 1

$= \frac{1}{2}\times\frac{1}{\sqrt2} - \frac{\sqrt3}{2}\times\frac{1}{\sqrt2}$

$= \frac{1-\sqrt3}{2\sqrt2} \times \frac{\sqrt2}{\sqrt2}$ 1

$= \frac{\sqrt2-\sqrt6}{4}$

(b)(i) $P(\text{no girls}) = \frac{{}^4C_3}{{}^{12}C_3} = \frac{4}{220} = \frac{1}{55}$ 1

(ii) $P(\text{exactly 1 girl}) = \frac{{}^8C_1 \times {}^4C_2}{{}^{12}C_3} = \frac{8\times6}{220} = \frac{12}{55}$ 1

(iii) $P(\text{at least 2 girls}) = 1 - P(\text{No girls or 1 girl})$

$= 1 - \left(\frac{1}{55}+\frac{12}{55}\right)$ 1

$= \frac{42}{55}$

(c) $LHS = \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)}$ 1

$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$

$= \frac{\sin\theta}{\cos\theta}$ 1

$= \tan\theta = RHS$

$$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 -(1-u)\sqrt{u}\,du \quad (1)$$

$$= \int_1^0 -u^{1/2}(1-u)\,du$$

$$= \int_1^0 -u^{1/2}+u^{3/2}\,du$$

$$= \left[-\frac{2u^{3/2}}{3}+\frac{2u^{5/2}}{5}\right]_1^0 \quad (1)$$

$$= 0+0-\left(-\frac{2}{3}+\frac{2}{5}\right)$$

$$= \frac{2}{3}-\frac{2}{5}$$

$$= \frac{4}{15} \quad (1)$$

uestion 3

i) 12 people

No. of outcomes $= (12-1)!$

$= 11!$ (1)

$= 39916800$

i)

leaving 10 people

i.e. no. of outcomes $= 10!$

But can have SR or RS, thus (1) for method

no. of outcomes $= 2\times 10!$

$$P(S \text{ and } R \text{ together}) = \frac{2\times 10!}{11!} = \frac{2}{11} \quad (1)$$

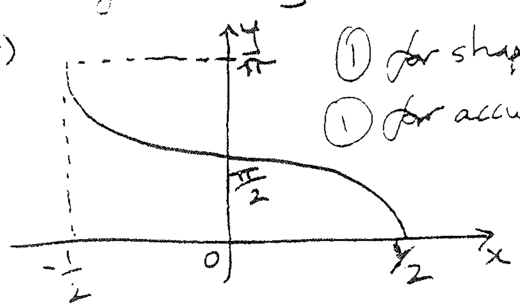
(i) Domain $= -1 \le 2x \le 1$

$\therefore -\frac{1}{2} \le x \le \frac{1}{2}$ (1)

Range $= 0 \le y \le \pi$

ii) (1) for shape

(1) for accuracy



)(i) $\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1) = \frac{\pi}{3} - \left(-\frac{\pi}{4}\right)$

$= 7\pi$ (1)

$$\therefore \text{Area} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2\sqrt{3}}$$

$$= \frac{1}{2}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(-1)\right]$$

$$= \frac{1}{2}\times\frac{7\pi}{12}$$

$$= \frac{7\pi}{24} \text{ units}^2 \quad (1)$$

(d) If $n=1$, $7^1-1=6$ which is divisible by 6

$\therefore$ Statement is true for $n=1$ (1)

Assume statement is true for $n=k$

i.e. $\frac{7^k-1}{6} = M$ (where M is an integer)

i.e. $7^k-1 = 6M$

i.e. $7^k = 6M+1$

Now, $7^{k+1}-1 = 7^k \times 7^1 - 1$

$= (6M+1)7-1$

$= 42M+7-1$

$= 42M+6$

$= 6(7M+1)$ which is divisible by 6. (1)

$\therefore$ If statement is true for $n=k$, then statement is true for $n=k+1$.

Thus, since statement is true for $n=1$, it is true for $n=2,3,4$, etc.

Thus, statement is true for all $n \ge 1$ (where n is an integer) (1)

Question 4

(a) $y = \sin^{-1}(\cos x)$

$$\therefore \frac{dy}{dx} = \frac{-\sin x}{\sqrt{1-\cos^2 x}} \quad (1)$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}}$$

$$= \frac{-\sin x}{\sin x}$$

$$= -1 \quad (\text{if } \sin x > 0) \quad (1)$$

(b) General term $= {}^6C_r\, x^{6-r}\left(\frac{1}{2x^2}\right)^r$

$= {}^6C_r\, x^{6-r}\left(\frac{1}{2}\right)^r\left(x^{-2}\right)^r$

...rm independent of x occurs when $6-3r=0$

$\therefore 3r=6$ i.e. when $r=2$ (1)

Term $= {}^6C_2 \times \left(\frac{1}{2}\right)^2$

$= 15 \times \frac{1}{4}$

$= \frac{15}{4}$ (1)

OR

(2)

(1)

$3\sin x + 4\cos x = A\sin(x+y)$

$= A\sin x\cos y + A\sin y\cos x$

$A\cos y = 3$ and $A\sin y = 4$

$\cos y = \frac{3}{A}$ and $\sin y = \frac{4}{A}$

$\therefore A = \sqrt{3^2+4^2}$

$= 5$ (1)

OR using t-method

(1) Subst.

(1) Getting quad

(1) Answers

$y = \cos^{-1}\left(\frac{3}{5}\right) \doteqdot 0.9273$

$3\sin x + 4\cos x = 5\sin(x+0.9273)$ (1)

$\therefore 5\sin(x+0.9273) = 2$

$\sin(x+0.9273) = 2/5$

$x + 0.9273 \doteqdot 0.~~4115~~, 2.7301, 6.6947$

$\therefore x \doteqdot 1.8028, 5.7674$ (1) for two solns

(ii) $f(x) = x - \sin x - 2$

$f'(x) = 1 - \cos x$

$f'(x) = 0 \Rightarrow \cos x = 1$

$\therefore x = \ldots, \cancel{-2\pi}, 0, 2\pi, 4\pi, \ldots$ (1)

$y = \ldots, \cancel{\times}, -2, 2\pi-2, 4\pi-2, \ldots$

x	-1	0	1
y'	+	0	+
	/	—	/

$\therefore$ Horizontal pt of inflexion at $(0,-2)$

x	6	2π	7
y'	+	0	+
	/	—	/

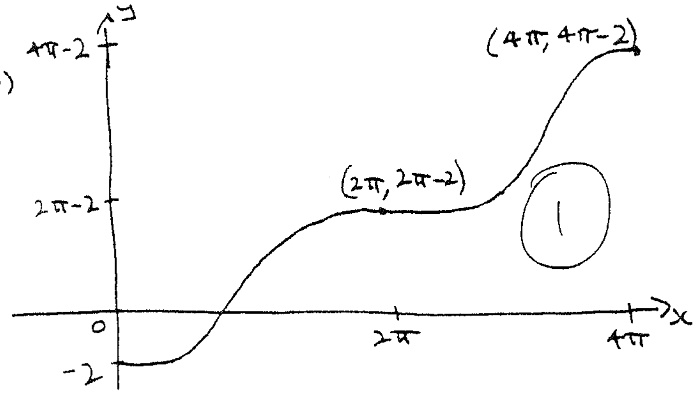
$\therefore$ Horizontal pt of inflexion at $(2\pi, 2\pi-2)$

x	12	4π	13
y'	+	0	+
	/	—	/

$\therefore$ Horizontal pt of inflexion at $(4\pi, 4\pi-2)$ (1)

$f''(x) = \sin x$

(iii)



(1)

Question 5

(a)(i) If $T = T_0 + Ae^{-kt}$, then $T - T_0 = Ae^{-kt}$

Now, $\frac{dT}{dt} = 0 - kAe^{-kt}$

$= -k(Ae^{-kt})$

$= -k(T - T_0)$ (1)

(ii) $T_0 = 70$ and thus $T = 70 + Ae^{-kt}$

When $t=0$, $T=150$

$\therefore 150 = 70 + Ae^0$

$\therefore 150 = 70 + A \quad \therefore A = 80$ (1)

When $t = 40$, $T = 90$

$\therefore 90 = 70 + 80e^{-40k}$

$\therefore 20 = 80e^{-40k}$

$\therefore e^{-40k} = \frac{20}{80} = \frac{1}{4}$ (1)

$\therefore -40k = \log_e(1/4)$

$\therefore k = \frac{\ln(1/4)}{-40} = \frac{\ln 4^{-1}}{-40} = \frac{\ln 4}{40} \doteqdot 0.0347$ (1)

$T = 76 \Rightarrow 76 = 70 + 80e^{-kt}$

$\therefore 6 = 80e^{-kt}$

$\therefore e^{-kt} = \frac{6}{80} = \frac{3}{40}$

$\therefore -kt = \log_e(3/40)$

$\therefore t = \frac{\ln(3/40)}{-\frac{\ln 4}{40}} \doteqdot 74.74$ minutes (1)

(b)(i) $\frac{dV}{dt} = 1000e^{-2t}$

$\therefore V = \frac{1000e^{-2t}}{-2} + c$

$\therefore V = -500e^{-2t} + c$ (1)

But, when $t=0$, $V=0$

$\therefore 0 = -500e^0 + c \quad \therefore c = 500$

$\therefore V = 500 - 500e^{-2t}$ (1)

$e^{-2t} = 1 - \frac{4}{5} = \frac{1}{5}$

$-2t = \log_e(1/5)$ (1)

$t = \frac{\ln(1/5)}{-2} \doteq 0.8047$ seconds

) As $t \to \infty$, $e^{-2t} \to 0$

$\therefore (1 - e^{-2t}) \to 1$

$\therefore 500(1 - e^{-2t}) \to 500$ (1)

Max possible volume $= 500$ units^3

Assuming sphere $\Rightarrow V = \frac{4}{3}\pi r^3$

when $V = 400$,

$400 = \frac{4}{3}\pi r^3$

$\therefore 300 = \pi r^3$

$\therefore r^3 = \frac{300}{\pi}$ $\therefore r = \sqrt[3]{\frac{300}{\pi}} \doteq 4.5708$ (1)

ow, if $V = \frac{4}{3}\pi r^3$, then

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{4\pi r^2}$

en $V = 400$, $r = \sqrt[3]{\frac{300}{\pi}}$, $t = \frac{\ln(1/5)}{-2}$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$ (1)

$\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 1000e^{-2t}$

$= \frac{1}{4\pi\left(\sqrt[3]{\frac{300}{\pi}}\right)^2} \times 1000e^{-2\times\frac{\ln(1/5)}{-2}}$

$\doteq 0.7618$ (1)

stion 6

$(1+x)^{m+n} = 1 + {}^{m+n}C_1 x + {}^{m+n}C_2 x^2 + \dots$

oefficient of $x^2 = {}^{m+n}C_2 = \binom{m+n}{2}$ (1)

$(1+x)^m(1+x)^n$

$(1 + {}^mC_1 x + {}^mC_2 x^2 + \dots)(1 + {}^nC_1 x + {}^nC_2 x^2 + \dots)$

s containing x^2 will be (1)

$\dots x^2 + {}^mC_1 x \times {}^nC_1 x + {}^mC_2 x^2 \times 1$

(1)

$\therefore$ Comparing coefficients of x^2 on both sides

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + \binom{m}{1}\binom{n}{1}$$

(b)(i) $x = \sqrt{3}\cos 3t - \sin 3t$

$\dot{x} = -3\sqrt{3}\sin 3t - 3\cos 3t$

$\ddot{x} = -9\sqrt{3}\cos 3t + 9\sin 3t$

$= -9(\sqrt{3}\cos 3t - \sin 3t)$ (1)

$\therefore \ddot{x} = -9x$ which is the form $\ddot{x} = -n^2 x$

$\therefore$ Motion is SHM.

(ii) Period $= \frac{2\pi}{3}$ (1)

(iii) When $x = 0$,

$0 = \sqrt{3}\cos 3t - \sin 3t$ (1)

$\therefore \sin 3t = \sqrt{3}\cos 3t$

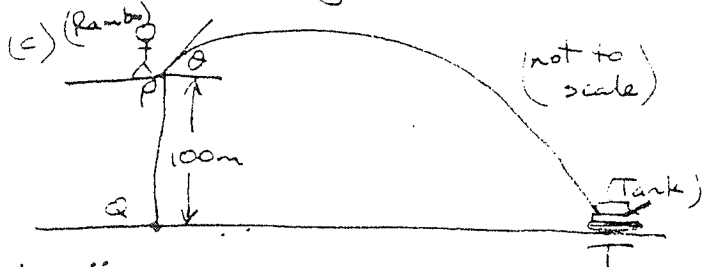
$\therefore \tan 3t = \sqrt{3}$

$\therefore 3t = \frac{\pi}{3}, \frac{4\pi}{3}, \dots$

$\therefore t = \frac{\pi}{9}, \frac{4\pi}{9}, \dots$ (1)

$\therefore$ First passes origin at $t = \frac{\pi}{9}$ seconds

(c)



(i) $\ddot{x} = 0$ $\quad \ddot{y} = -10$

$\therefore \dot{x} = c_1$ $\quad \dot{y} = -10t + c_2$

when $t=0$, $\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta$

$\therefore \dot{x} = V\cos\theta$ $\quad \dot{y} = -10t + V\sin\theta$

$\therefore x = Vt\cos\theta + c_3$ $\quad y = -5t^2 + Vt\sin\theta + c_4$

Let P be origin, thus when $t=0$, $x=0$ and $y=0$

$\therefore c_3 = c_4 = 0$

$\therefore x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$ (1)

Now, $V = \frac{190}{\sqrt{3}}$ and when $t = 20$, $y = -100$

$\therefore -100 = -5 \times 20^2 + \frac{190}{\sqrt{3}} \times 20 \times \sin\theta$ (1)

$\therefore -100 = -2000 + \frac{3800\sin\theta}{\sqrt{3}}$

(iii) Max height occurs when $\dot{y} = 0$

$\therefore 0 = -10t + \frac{190}{\sqrt{3}}\sin 60°$ 1

$\therefore 10t = \frac{190}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = 95$

$\therefore t = 9.5$ sec

when $t = 9.5$,

$y = 100 + \left(-5\times 9.5^2 + \frac{190}{\sqrt{3}}\times 9.5\times \sin 60°\right)$

$y = 100 - 451.25 + 902.5$

Max height $= 551.25$ m 1

estion 7

(i) $y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

$P,\quad m = \frac{2ap}{2a} = p$

n of ~~tangent at~~ P is

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - ap^2$

$Q,\quad m = \frac{2aq}{2a} = q$

n of tangent at Q is

$y - aq^2 = q(x - 2aq)$

$y - aq^2 = qx - 2aq^2$

$\therefore y = qx - aq^2$ 1

$\left.\begin{array}{l} y = px - ap^2 \\ y = qx - aq^2 \end{array}\right\} \therefore px - ap^2 = qx - aq^2$

$\therefore px - qx = ap^2 - aq^2$

$\therefore x(p-q) = a(p-q)(p+q)$

$\therefore x = a(p+q)$

$y = ap(p+q) - ap^2$

$= ap^2 + apq - ap^2$

$= apq$

$\equiv (a(p+q), apq)$ 1

T lies on parabola $x^2 = -4ay$

$a^2(p+q)^2 = -4a^2pq$

(iv) Midpt of $PQ = \left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$= \left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$

$\therefore x = a(p+q)$ and $y = \frac{a}{2}(p^2+q^2)$ 1

$\therefore \frac{x}{a} = p+q \qquad y = \frac{a}{2}\times -6pq = -3apq$

$\therefore \frac{y}{-3a} = pq$

Now, if $p^2+q^2 = -6pq$, then

$p^2 + 2pq + q^2 = -4pq$

$\therefore (p+q)^2 = -4pq$

using these 1

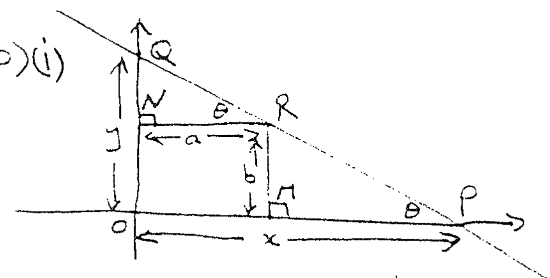
Thus, $\left(\frac{x}{a}\right)^2 = -4\times\frac{y}{-3a}$

$\therefore \frac{x^2}{a^2} = \frac{4y}{3a}$

$\therefore$ Eqn of locus of midpt of PQ is

$y = \frac{3x^2}{4a}$ 1

(b)(i)



From ΔRPM, $\tan\theta = \frac{b}{x-a}$

$\therefore \cot\theta = \frac{x-a}{b}$

$\therefore b\cot\theta = x - a$

$\therefore x = a + b\cot\theta$

From ΔQRN, $\tan\theta = \frac{y-b}{a}$

$\therefore a\tan\theta = y - b$

$\therefore y = b + a\tan\theta$

2 for method

Now, length of $PQ = \sqrt{x^2+y^2}$

$\therefore l^2 = x^2 + y^2$

$= (a + b\cot\theta)^2 + (b + a\tan\theta)^2$

$= a^2 + 2ab\cot\theta + b^2\cot^2\theta + b^2 + 2ab\tan\theta + a^2\tan^2\theta$

$= a^2 + a^2\tan^2\theta + b^2 + b^2\cot^2\theta + 2ab(\tan\theta + \cot\theta)$

$= a^2(1+\tan^2\theta) + b^2(1+\cot^2\theta) + 2ab\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$l^2 = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta + 2ab\left(\frac{1}{\sin\theta\cos\theta}\right)$

$= a^2\sec^2\theta + 2ab\sec\theta\operatorname{cosec}\theta + b^2\operatorname{cosec}^2\theta$

$l^2 = (a\sec\theta + b\operatorname{cosec}\theta)^2$

$l = a\sec\theta + b\operatorname{cosec}\theta \quad$ (since $l>0$)

$l = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$

$l = a(\cos\theta)^{-1} + b(\sin\theta)^{-1}$

$l' = -a(\cos\theta)^{-2} \times -\sin\theta - b(\sin\theta)^{-2}\cos\theta$

$= \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} \Leftarrow$ 1

$\therefore l' = 0 \Rightarrow \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0$

$\therefore \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta}$

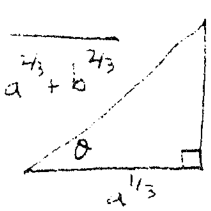
$\therefore a\sin^3\theta = b\cos^3\theta$

$\therefore \tan^3\theta = \frac{b}{a}$

$\therefore \tan\theta = \left(\frac{b}{a}\right)^{1/3}$

$\therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3} \Leftarrow$ 1

$\therefore$ if $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$, Thus



$\cos\theta = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

and $\sin\theta = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$

$l = \frac{a}{\frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}} + \frac{b}{\frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}}$

$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}} + b^{2/3}\sqrt{a^{2/3}+b^{2/3}}$

$= \sqrt{a^{2/3}+b^{2/3}}\left(a^{2/3}+b^{2/3}\right)$

$= \left(a^{2/3}+b^{2/3}\right)^{3/2}$ 1

Alternative soln

(i) From ΔRPM, $\cos\theta = \frac{x-a}{PR} \quad \therefore PR = \frac{x-a}{\cos\theta}$

From ΔQRN, $\sin\theta = \frac{y-b}{QR} \quad \therefore QR = \frac{y-b}{\sin\theta}$

Now $PQ = PR + QR$

$\therefore PQ = \frac{x-a}{\cos\theta} + \frac{y-b}{\sin\theta}$

$\therefore PQ = \frac{x}{\cos\theta} - \frac{a}{\cos\theta} + \frac{y}{\sin\theta} - \frac{b}{\sin\theta}$

(Now, from ΔOPQ, $\sin\theta = \frac{y}{PQ}$ and $\cos\theta = \frac{x}{PQ}$

$\therefore PQ = \frac{y}{\sin\theta}$ and $PQ = \frac{x}{\cos\theta}$)

$\therefore PQ = PQ - \frac{a}{\cos\theta} + PQ - \frac{b}{\sin\theta}$

$\therefore PQ = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$ 2 for part (i) method

OR

(ii) $l' = \frac{a\sin^3\theta - b\cos^3\theta}{\sin^2\theta\cos^2\theta}$

$= \frac{(a^{1/3}\sin\theta - b^{1/3}\cos\theta)(a^{2/3}\sin^2\theta + (ab)^{1/3}\sin\theta\cos\theta + b^{2/3}\cos^2\theta)}{\sin^2\theta\cos^2\theta}$

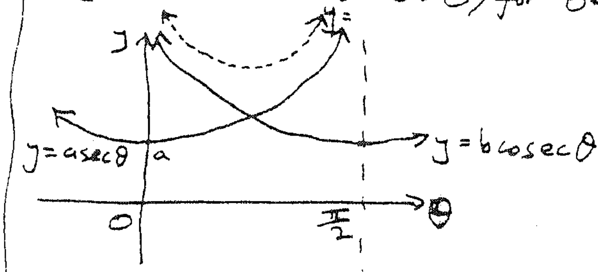
$l' = 0 \Rightarrow a^{1/3}\sin\theta - b^{1/3}\cos\theta = 0$

(since $0<\theta<\frac{\pi}{2}$ and thus $(a^{2/3}\sin^2\theta + (ab)^{1/3}\sin\theta\cos\theta + b^{2/3}\cos^2\theta) > 0$)

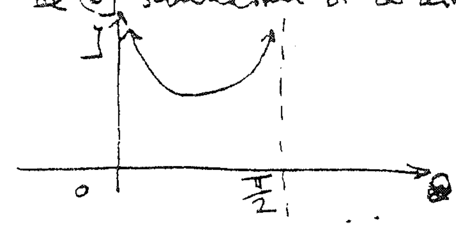
$\therefore a^{1/3}\sin\theta = b^{1/3}\cos\theta$

$\therefore \tan\theta = \frac{b^{1/3}}{a^{1/3}} \quad \therefore \theta = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$

To prove minimum value, investigate the graphs of $y = a\sec\theta$ and $y = b\operatorname{cosec}\theta$ (where $a>0$ and $b>0$) for $0<\theta<\frac{\pi}{2}$.



thus the graph of $y = a\sec\theta + b\operatorname{cosec}\theta$ will be (by summation of ordinates) 1



$\therefore$ Minimum turning pt exists in domain $0<\theta<\frac{\pi}{2}$