Barker College

**2003**
**TRIAL**
**HIGHER SCHOOL**
**CERTIFICATE**

# Mathematics Extension 1

**Staff Involved:**

- CFR*
- HG*
- DOK
- RMH
- MRB
- BJR
- VAB

**PM THURSDAY 14 AUGUST**

**90 copies**

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 9
- ALL necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged working

**Total marks – 84**

- Attempt Questions 1 – 7
- All questions are of equal value

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**Attempt Questions 1 – 7**
**ALL questions are of equal value**

Answer each question on a SEPARATE sheet of paper

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**Marks**

**Question 1** (12 marks) **[BEGIN A NEW PAGE]**

(a) If $f(x) = x^3$ and $g(x) = -\sqrt{x}$, what is the value of $f(g(9)) - g(f(9))$? **2**

(b) $y = f(x)$ is a linear function with slope $\frac{1}{2}$

(i) Find an expression for the inverse function of $y = f(x)$ **2**

(ii) Hence find the slope of $y = f^{-1}(x)$ **1**

(c) Find $\int \frac{2}{3\sqrt{16 - x^2}}\,dx$ **1**

(d) Find the coordinates of the point that divides the interval $AB$, where $A$ is $(-1, 3)$ and $B$ is $(2, 8)$, externally in the ratio of $3 : 2$. **2**

(e) If $\sin 2A = \frac{1}{2}$, what is the value of $\frac{1}{\sin A \cos A}$? **2**

(f) If $0 \leq t \leq 1$, find the Cartesian equation of the curve whose parametric equations are $y = t^2$ and $x = \sqrt{t}$ **2**

Marks

**Question 2** (12 marks) **[BEGIN A NEW PAGE]**

(a) Consider the function $y = 2\sin^{-1}\frac{x}{3}$

(i) State the domain and range of $y = f(x)$ **2**

(ii) Hence sketch the graph of $y = f(x)$ **1**

(b) From the top, C, of a vertical cliff, 200 m high, two ships P and Q are observed at sea level. A is the foot of the cliff at sea level. P is due south of A and the angle of elevation of C from P is $45°$. Q is S 50° W of A and the angle of elevation of C from Q is $60°$.

(i) Draw a diagram showing this information. **1**

(ii) Find the distance PQ (to nearest metre). **3**

(c) Consider the curve whose equation is $y = \dfrac{x^2}{1 - x^2}$

(i) Find any vertical asymptotes. **1**

(ii) Find $\lim\limits_{x \to \pm\infty} y$ **1**

(iii) Show that the curve is an even function. **1**

(iv) Hence (without using calculus), sketch the curve, showing all main features. **2**

**Marks**

**Question 3** (12 marks) **[BEGIN A NEW PAGE]**

(a) Differentiate $x\cos^{-1}x$ **2**

(b) Find $\int_0^{\pi}\sin^3 x\,dx$ using the result $\sin 3x = 3\sin x - 4\sin^3 x$ **3**

(c) A boat is attached by a rope to a jetty 2 m above the bow of the boat.
The rope is being pulled in at the rate of 1 m $s^{-1}$.
At what rate is the boat approaching the jetty when 3 m of rope still remains to be pulled in? (Answer correct to 1 decimal place) **4**

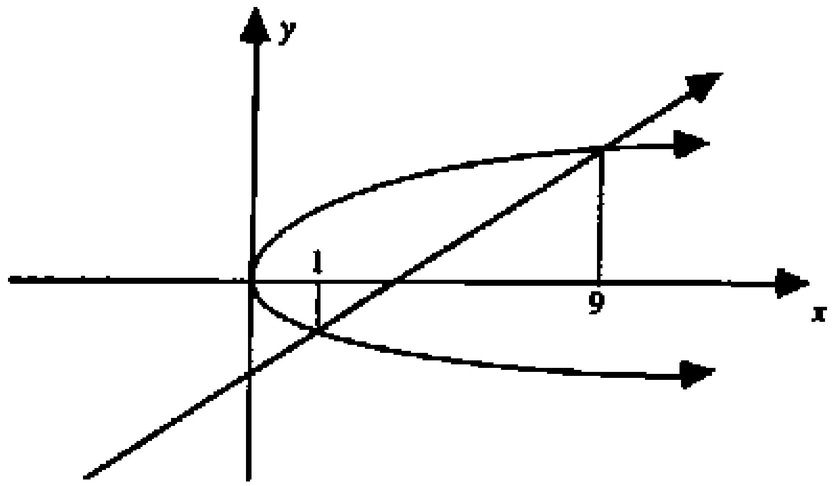
(d) (i) Express $x^2 + x + 1$ in the form $(x - A)^2 + B$ where $A$, $B$ are constants. **1**

(ii) Hence find $\int \frac{dx}{x^2 + x + 1}$ **2**

**Marks**

**Question 4** (12 marks) **[BEGIN A NEW PAGE]**

(a) The curves $y^2 = 16x$ and $y = 2x - 6$ intersect at the points where $x = 1$ and $x = 9$.




Find the acute angle between the two curves at the point where $x = 1$ **3**

(b) If $\tan\dfrac{\theta}{2} = t$ and $\theta$ is acute, express $\sqrt{\dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}}$ in terms of $t$ **3**

(c) Evaluate in exact form $\cos 105°$ **2**

(d) Solve $\sqrt{2}\cos x - \sin x = \dfrac{3}{2}$ for $0° \le x \le 360°$ **4**

Marks

**Question 5** (12 marks) **[BEGIN A NEW PAGE]**

(a) A body is cooling in a room of constant temperature $15°C$.
At time $t$ minutes its temperature, $T$, decreases according to the equation

$$\frac{dT}{dt} = -k(T - 15)$$

where $k$ is a positive constant.
The initial temperature of the body is $75°C$, and it cools to $55°C$ after 10 minutes.
What is the temperature of the body after a further 5 minutes?
(Answer correct to 1 decimal place) 4

(b) (i) Show that the relation $v^2 = -kx^2 + c$, where $k$ and $c$ are constants, is satisfied by the equation $\frac{d}{dx}\left(\frac{v^2}{2}\right) = -kx$ 1

(ii) A pendulum, $P$, swings so that it oscillates about its centre of motion according to the equation $\frac{d^2x}{dt^2} = \frac{-x}{9}$, where $x$ is the distance of $P$ from its centre of oscillation at any time $t$ seconds.
Show that $v^2 = \frac{1}{9}(4 - x^2)$, given that its maximum displacement is 2 cm.
Hence find the maximum speed of $P$. 4

(c) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$ using $u = \cos x$ 3

**Marks**

**Question 6** (12 marks) **[BEGIN A NEW PAGE]**

(a) Write down the value of ${}^nC_j - {}^nC_{n-j}$ **1**

(b) Find the term independent of $x$ in the expansion of $\left(x^3 + \dfrac{5}{x}\right)^8$ **3**

(c) By considering the identity $(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$, show that

$$\sum_{k=1}^{2n} k \binom{2n}{k} = n4^n$$ **4**

(d) What is the greatest coefficient in the expansion of $(2 + 3x)^{20}$ ? **4**

**Question 7** (12 marks) **[BEGIN A NEW PAGE]**

(a) Given that $y = \sin x$, and using the result $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, it can be shown that :

$$\frac{dy}{dx} = \cos x$$

$$= \sin\left(x + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dx^2} = \cos\left(x + \frac{\pi}{2}\right)$$

$$= \sin\left[\left(x + \frac{\pi}{2}\right) + \frac{\pi}{2}\right]$$

$$= \sin\left[x + \frac{2\pi}{2}\right]$$

Similarly :

$$\frac{d^3y}{dx^3} = \sin\left(x + \frac{3\pi}{2}\right)$$

Therefore :

$$\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$$

Prove, by induction, that the generalisation given above, i.e. $\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right)$, is correct for all positive integers $n$ when $y = \sin x$ 5

(b) A particle is projected under gravity with speed $u$ m s$^{-1}$ and at an angle $\frac{\pi}{4}$, from a point $O$ on horizontal ground. It strikes the ground at $P$, where $OP = R$.

(i) Taking the $x$ and $y$ axes through 0, show that the equation of the trajectory is given by $y = x - g\frac{x^2}{u^2}$ 2

(ii) Hence, or otherwise, show that $R = \frac{u^2}{g}$ 2

(iii) A ball is fired from $O$ with velocity 30 m s$^{-1}$ at an angle $\frac{\pi}{4}$ to the horizontal. Find the speed of the ball when it has travelled a horizontal distance of 15 m from its starting point. (Take $g = 10$ m s$^{-2}$)
(Answer correct to 1 decimal place) 3

**End of Paper**

## YEAR 12 EXT. 1 MATHS TRIAL 2003, SOLUTIONS.

**QUESTION 1:**

(a) $f[g(9)] - g[f(9)] = (-\sqrt{9})^2 - \sqrt{9^2}$ ①
$= 9 + 9 = 18$ ①

(b)(i) $y = \frac{1}{2}x + b$
Inv. $x = \frac{1}{2}y + b$ ①
$2x = y + c$
$\therefore y = 2x - c$ ①
(ii) $m_{inv.} = 2$ ①

(c) $\int \frac{2}{3\sqrt{16 - x^2}}\,dx = \frac{2}{3}\int \frac{1}{\sqrt{4^2 - x^2}}\,dx$
$= \frac{2}{3}\sin^{-1}\frac{x}{4} + C$ ①

(d) $A(-1,3)$ $B(2,8)$
$-3 : 2$
Point $= \left(\frac{2\times -1 + -3\times 2}{-3+2}, \frac{2\times 3 + -3\times 8}{-3+2}\right)$ ①
$= (8, 18)$ ①

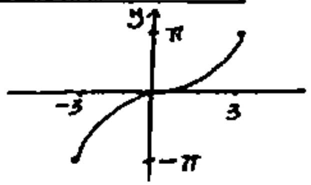
(e) $\frac{1}{\sin A\cos A} = \frac{1}{\frac{1}{2}\sin 2A}$ ①
$= \frac{1}{\frac{1}{2}\times\frac{1}{2}}$
$= 4$ ①

(f) $y = t^2$, $x = \sqrt{t}$
$\therefore t = x^2$ ①
$\therefore y = (x^2)^2 = x^4$ ①

**QUESTION 2:**

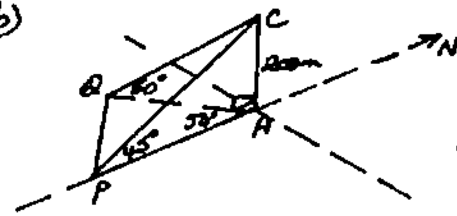
(a)(i) $y = 2\sin^{-1}\frac{x}{3}$
Dom. $-1 \le \frac{x}{3} \le 1$
$\therefore -3 \le x \le 3$ ①
Range: $-\frac{\pi}{2} \le \sin^{-1}\frac{x}{3} \le \frac{\pi}{2}$
$\therefore -\pi \le y \le \pi$ ①

(ii)

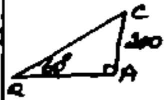



①

(b)




①




$QA = \frac{200}{\tan 60°}$
$\doteqdot 115.47$

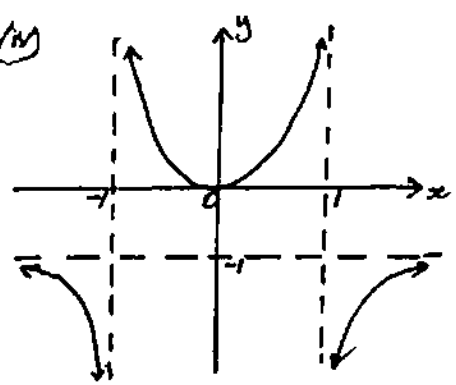

$PA = 200$ (isos. ∠ Δ)


$QP^2 = (115.47)^2 + 200^2 - 2\times 115.47\times 200\times\cos 50°$ ①
$= 23644.24678$
$\therefore QP \doteqdot 154$ m (nearest m) ①

(c) $y = \frac{x^2}{1 - x^2}$
(i) Vert. asym. $x \neq \pm 1$ ①
(ii) $\lim_{x\to\pm\infty}\frac{x^2}{1-x^2} = \lim_{x\to\pm\infty}\frac{1}{\frac{1}{x^2} - 1}$
$= -1$ ①
(iii) $f(x) = \frac{x^2}{1 - x^2}$
$f(-x) = \frac{(-x)^2}{1 - (-x)^2}$ ①
$= \frac{x^2}{1 - x^2} = f(x)$.
$\therefore$ Even



2(c)(iv)

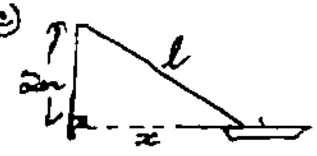



**QUESTION 3:**

(a) Let $y = x\cos^{-1}x$ ① rule ① ans.
$\frac{dy}{dx} = \cos^{-1}x + x\times\frac{-1}{\sqrt{1 - x^2}}$
$= \cos^{-1}x - \frac{x}{\sqrt{1 - x^2}}$

(b) $\sin 3x = 3\sin x - 4\sin^3 x$
$\therefore \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ ①
$\int_0^{\pi}\sin^3 x\,dx = \frac{1}{4}\int_0^{\pi}(3\sin x - \sin 3x)\,dx$
①
$= \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\pi}$
$= \frac{1}{4}\left\{(-3\cos\pi + \frac{1}{3}\cos 3\pi) - (-3\cos 0 + \frac{1}{3}\cos 0)\right\}$
$= \frac{1}{4}\left\{(3 - \frac{1}{3}) - (-3 + \frac{1}{3})\right\}$
$= \frac{4}{3}$. ①

(c)




$l = \sqrt{x^2 + 4}$, $\frac{dl}{dt} = -1$, $l = 3$, $x = \sqrt{5}$ ①
$\frac{dl}{dx} = \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}\cdot 2x$
$= \frac{x}{\sqrt{x^2 + 4}}$ ①

$\frac{dl}{dt} = \frac{dl}{dx}\times\frac{dx}{dt}$
$-1 = \frac{\sqrt{5}}{\sqrt{5 + 4}}\,\frac{dx}{dt}$ ①
$\therefore \frac{dx}{dt} = -\frac{3}{\sqrt{5}}$
$\doteqdot -1.3416$
$\therefore$ Boat app. jetty at $1.3\,ms^{-1}$ ①

(d)(i) $x^2 + x + 1 = x^2 + x + (\frac{1}{2})^2 + 1 - (\frac{1}{2})^2$
$= (x + \frac{1}{2})^2 + \frac{3}{4}$ ①
OR
$x^2 + x + 1 \equiv x^2 - 2Ax + A^2 + B$
$\therefore -2A = -1 \Rightarrow A = -\frac{1}{2}$
$\therefore (-\frac{1}{2})^2 + B = 1 \Rightarrow B = \frac{3}{4}$
$\therefore x^2 + x + 1 \equiv (x + \frac{1}{2})^2 + \frac{3}{4}$
(ii) $\int\frac{dx}{x^2 + x + 1} = \int\frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}}$
$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2(x + \frac{1}{2})}{\sqrt{3}} +$

**QUESTION 4:**

(a) At $x = 1$, $y = -4\sqrt{x}$
$\frac{dy}{dx} = -2x^{-\frac{1}{2}}$
$\therefore m_1 = -2$ (at $x = 1$)
$y = 2x - 6$
$m_2 = 2$
$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$
$= \left|\frac{-2 - 2}{1 + -2\times 2}\right|$ ①
$= \left|\frac{-4}{-3}\right| = \frac{4}{3}$
$\therefore \theta \doteqdot 53°8'$ ①

4.(b) $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = \sqrt{\dfrac{1-(1-2\sin^2\theta)}{1+(1-2\sin^2\theta)}}$ (1)

$= \sqrt{\dfrac{2\sin^2\theta}{2-2\sin^2\theta}}$

$= \sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}}$ (1)

$= \tan\theta$

$= \dfrac{2t}{1-t^2}$ (1)

(c) $\cos 105° = \cos(60°+45°)$ (1)

$= \cos 60° \cos 45° - \sin 60° \sin 45°$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \dfrac{1-\sqrt{3}}{2\sqrt{2}}$ (1)

(d) $\sqrt{2}\cos x - \sin x \equiv R\cos(x+\alpha)$

$R = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3}$ (1)

$\cos\alpha = \sqrt{2}/\sqrt{3}$

$\therefore \alpha = 35°16'$ (1)

$\therefore \sqrt{3}\cos(x+35°16') = \frac{3}{2}$

$\cos(x+35°16') = \frac{\sqrt{3}}{2}$ (1)

$\therefore x + 35°16' = 30°, 330°, 390°, \ldots$

$\therefore x = 294°44'$ or $354°44'$ (1)

(b)(i) $v^2 = -kx^2 + c$

$\frac{1}{2}v^2 = -\frac{k}{2}x^2 + c_1$ (1)

$\therefore \frac{d}{dx}\left(\frac{v^2}{2}\right) = -kx$

(ii) $\dfrac{d^2x}{dt^2} = -\dfrac{x}{9}$

i.e. $\dfrac{d(\frac{1}{2}v^2)}{dx} = -\dfrac{x}{9}$

$\therefore \frac{1}{2}v^2 = -\frac{x^2}{18} + c$ (1)

$v=0$, $x=2$: $\therefore 0 = -\frac{4}{18} + c$ (1)

$\therefore c = \frac{4}{18}$

$\therefore \frac{1}{2}v^2 = -\frac{x^2}{18} + \frac{4}{18}$

$\therefore v^2 = -\frac{x^2}{9} + \frac{4}{9}$ (1) meth.

$= \frac{1}{9}(4-x^2)$

Max $v$ when $x=0$ $\therefore v^2 = \frac{4}{9}$

$v = \pm\frac{2}{3}$

$\therefore$ Max speed $\frac{2}{3}$ cms$^{-1}$ (1)

(c) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = \frac{\pi}{2}, u = 0$ (1)

$= \displaystyle\int_{\frac{1}{2}}^{0} -u^2 \, du$

$= \left[-\frac{u^3}{3}\right]_{\frac{1}{2}}^{0}$ (1)

$= 0 - -\frac{\frac{1}{8}}{3} = \frac{1}{24}$ (1)

QUESTION 5:

(a) $T = 15 + Ae^{-kt}$

$t=0$, $T=75$: $\therefore 75 = 15 + Ae^0$

$\therefore A = 60$ (1)

$t=10$, $T=55$: $\therefore 55 = 15 + 60e^{-10k}$ 47.1

$\therefore \frac{2}{3} = e^{-10k}$

$k = -\frac{1}{10}\ln\frac{2}{3}$ (1)

$\left(\text{or } \frac{1}{10}\ln\frac{3}{2}\right)$

$T = 15 + 60e^{\frac{t}{10}\ln\frac{2}{3}}$

$t=15$, $T = 15 + 60e^{\frac{15}{10}\ln\frac{2}{3}}$ (1)

$= 47.659 \doteq 47.7°$ (1)

QUESTION 6:

(a) ${}^nC_j - {}^nC_{n-j} = 0$ (1)

(b) $\left(x^3 + \frac{5}{x}\right)^8$

$T_{r+1} = {}^8C_r \, x^{3(8-r)} \left(\frac{5}{x}\right)^r$ (1)

$= \binom{8}{r} 5^r x^{24-4r}$ (1)

$\therefore 24 - 4r = 0$ for $T$ indep. of $x$

$\therefore r = 6$

$\therefore T_7 = \binom{8}{6} 5^6$ (1)

$= 437\,500$



6.(c) $(1+x)^{2n} = \displaystyle\sum_{k=0}^{2n} \binom{2n}{k} x^k$

Differentiating both sides wrt $x$:

$2n(1+x)^{2n-1} = \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k} x^{k-1}$ (2) 1 each side

Let $x=1$, $2n \times 2^{2n-1} = \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k}$ (1)

$\therefore n 2^{2n} = \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k}$

$\therefore \displaystyle\sum_{k=1}^{2n} k\binom{2n}{k} = n4^n$ (1)

(d) $(2+3x)^{30}$

$\dfrac{\text{coeff. } T_{r+1}}{\text{coeff. } T_r} = \dfrac{n-r+1}{r} \times \dfrac{b}{a}$

$= \dfrac{31-r}{r} \times \dfrac{3}{2}$ (1)

If $T_{r+1} > T_r$ then:

$93 - 3r > 2r$

$5r < 93$

$r < 18\frac{3}{5}$ (2) method

$\therefore r = 18, 17, 16, \ldots$

$\therefore T_{19} > T_{18} > T_{17} > \ldots$

If $T_{r+1} < T_r$ then:

$93 - 3r < 2r$

$5r > 93$

$r > 18\frac{3}{5}$

$\therefore r = 19, 20, \ldots$

$\therefore T_{20} < T_{19}$, $T_{21} < T_{20}, \ldots$

$\therefore T_{19}$ coeff. is greatest.

coeff. $T_{19} = \binom{30}{18} 2^{30-18} 3^{18}$

$= \binom{30}{18} 2^{12} 3^{18}$ (1)

QUESTION 7:

(a) STEP 1: Prove true for $n=1$

$y = \sin x$

$\dfrac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right)$ (1)

$\therefore$ True for $n=1$

STEP 2: Assume true for $n=k$

i.e. assume $\dfrac{d^k y}{dx^k} = \sin\left(x + \dfrac{k\pi}{2}\right)$ (1)

7. (a) cont. STEP 3: Prove true for $n=k+1$

i.e. Prove $\frac{d^{k+1}y}{dx^{k+1}} = \sin\left[x + \frac{(k+1)\pi}{2}\right]$

Now $\frac{d^{k+1}y}{dx^{k+1}} = \frac{d}{dx}\left(\frac{d^k y}{dx^k}\right)$

$= \frac{d}{dx}\sin\left(x+\frac{k\pi}{2}\right)$ from assump.

$= \cos\left(x+\frac{k\pi}{2}\right)$

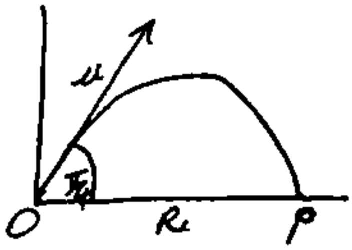
$= \sin\left(x+\frac{k\pi}{2}+\frac{\pi}{2}\right)$

$= \sin\left[x+\frac{(k+1)\pi}{2}\right]$

(3) including 1 for conclusion if it follows from proof

Hence if true for $n=k$, then true for $n=k+1$.
But true for $n=1$ ∴ true for $n=2$, $n=3$ and all positive integers $n$.

(b) (i)




Initially: $\dot{x} = u\cos\frac{\pi}{4} = \frac{u}{\sqrt{2}}$, $\dot{y} = u\sin\frac{\pi}{4} = \frac{u}{\sqrt{2}}$

$x = 0$, $y = 0$

**Horiz.**

$\ddot{x} = 0$

$\dot{x} = C = \frac{u}{\sqrt{2}}$

$x = \frac{u}{\sqrt{2}}t + C_1$

$t=0$, $x=0$ ∴ $C = 0$

∴ $x = \frac{u}{\sqrt{2}}t$ (1)

**Vert.**

$\ddot{y} = -g$

$\dot{y} = -gt + K$

$t=0$, $\dot{y} = \frac{u}{\sqrt{2}}$ ∴ $K = \frac{u}{\sqrt{2}}$

∴ $\dot{y} = -gt + \frac{u}{\sqrt{2}}$

$y = -\frac{gt^2}{2} + \frac{u}{\sqrt{2}}t + K_1$

$t=0$, $y=0$ ∴ $K=0$ ∴ $y = -\frac{gt^2}{2} + \frac{u}{\sqrt{2}}t$ (1)

Now: $t = \frac{\sqrt{2}x}{u}$

∴ $y = -\frac{g}{2}\left(\frac{\sqrt{2}x}{u}\right)^2 + \frac{u}{\sqrt{2}} \times \frac{\sqrt{2}x}{u}$ (1)

$= x - g\frac{x^2}{u^2}$

(ii) At P, $y=0$ ∴ $0 = x\left(1 - \frac{gx}{u^2}\right)$

∴ $x = 0$ or $\frac{u^2}{g}$

∴ $OP = R = \frac{u^2}{g}$

OR (1) for method

$R = \frac{V^2 \sin 2\alpha}{g} = \frac{u^2 \sin\frac{\pi}{2}}{g} = \frac{u^2}{g}$

(iii) From (i) $15 = \frac{30}{\sqrt{2}}t$

∴ $t = \frac{\sqrt{2}}{2}$

$\dot{x} = \frac{30}{\sqrt{2}} = 15\sqrt{2}$

$\dot{y} = -10t + \frac{30}{\sqrt{2}}$
$= -5\sqrt{2} + 15\sqrt{2} = 10\sqrt{2}$

∴ Speed $= \sqrt{(15\sqrt{2})^2 + (10\sqrt{2})^2}$
$= \sqrt{450 + 200}$
$= \sqrt{650}$
$\doteqdot 25.5\ ms^{-1}$