Barker College

Barker Student Number:________________

2008
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

Staff Involved:

- GDH*
- RMH*
- JM
- WMD
- GIC
- LJP

70 copies

PM THURSDAY 14 AUGUST

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages
- Board-approved calculators may be used
- A table of standard integrals is provided on page 13
- ALL necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged working

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

Question 1 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) (i) Sketch $y = \dfrac{1}{x - 1} + 3$, clearly showing all intercepts and asymptotes. **3**

(ii) Show that $\dfrac{1}{x - 1} + 3 = \dfrac{3x - 2}{x - 1}$ **1**

(iii) Hence, or otherwise, solve $\dfrac{3x - 2}{x - 1} \geq 0$ **2**

(b) Find $\displaystyle\int \frac{x}{\sqrt{9 - x^4}}\,dx$ using the substitution $u = x^2$ **3**

(c) (i) Show that $\dfrac{1}{\sqrt{x + h} + \sqrt{x}} = \dfrac{\sqrt{x + h} - \sqrt{x}}{h}$ **1**

(ii) Using $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$,

find $f'(x)$ from first principles if $f(x) = \sqrt{x}$ **2**

Question 2 (12 marks) [BEGIN A NEW PAGE]
Marks

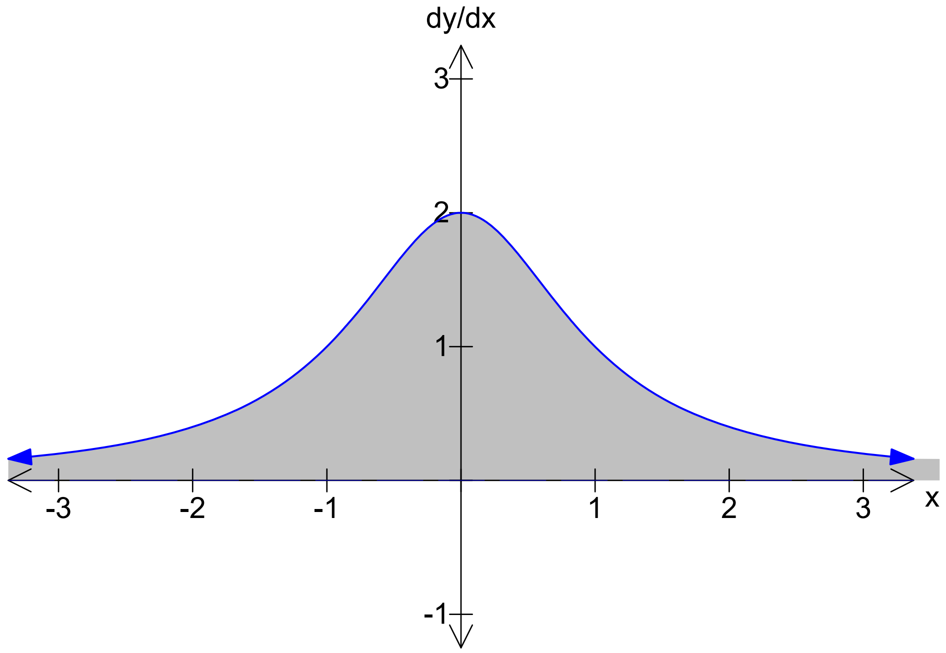
(a) (i) How many arrangements can be made from the letters of the word EXCESSIVE? **1**

(ii) How many of the words found in part (i) have the consonants and vowels in alternating positions? **2**

(b) The curve below is the <u>derivative</u> of $y = 2\tan^{-1} x$

The curve has an asymptote at $y = 0$

The shaded area 'goes forever' but its value has a limit.



(i) Find the exact value(s) of x when $\dfrac{dy}{dx} = \dfrac{1}{3}$ **2**

(ii) Find the limit that the value of the shaded area approaches. **2**

(c) (i) By letting $t = \tan\dfrac{\theta}{2}$, prove that $\dfrac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\dfrac{\theta}{2}$ **2**

(ii) Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1 - \cos\theta}{1 + \cos\theta}\, d\theta$, giving your answer in exact simplified form. **3**

Question 3 (12 marks) [BEGIN A NEW PAGE]
Marks

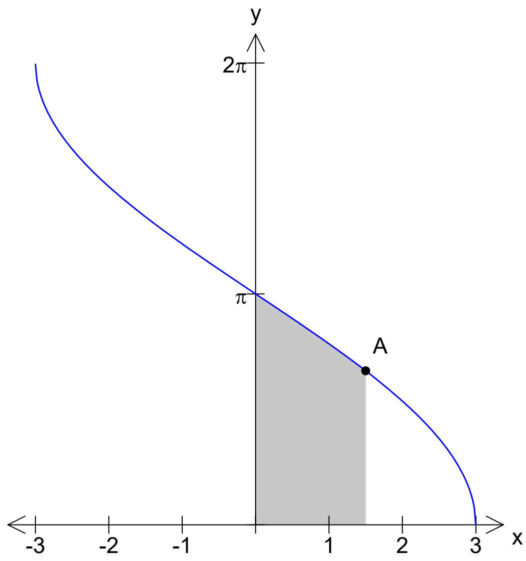
(a) The curve on the right has equation

$f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$.

The point A has coordinates $\left(\frac{3}{2}, \frac{2\pi}{3}\right)$.

Hilary wanted to find the shaded area.
However she realised that she could not integrate $f(x)$.

Her friend Barack suggested she use the inverse function.



(i) Find the inverse function $f^{-1}(x)$ **2**

(ii) Sketch the inverse function $f^{-1}(x)$ and shade the corresponding area.
You must be sure your sketch clearly shows the domain and range of $f^{-1}(x)$ **2**

(iii) Hence evaluate the shaded area. **3**

(b) $\binom{16}{10}$ is the coefficient of x^r in the expansion of $(1 + x)^n$.

Write down the possible values of n and r. **2**

(c) (i) Verify that $n^4 + 5n$ is an even number for $n = 1$ and $n = 2$. **1**

(ii) Prove, by mathematical induction, that $n^4 + 5n$ is divisible by 2 for all positive integers n **2**

Question 4 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) Find the value of the term independent of x in the expansion of $\left(x^2 - \frac{1}{x}\right)^{12}$ **3**

(b) A particle is moving in a straight line and its position x is given by the equation $x = 2\sin^2 t$.

(i) Express the acceleration of the particle in terms of x in the form

$\ddot{x} = -n^2(x - a)$ **3**

(ii) Hence state the period and centre of motion. **2**

(iii) What are the extremities of the motion? **1**

(c) (i) Using $\cos(2ax) = 1 - 2\sin^2(ax)$, find the simplest expression for

$\int_0^{\frac{\pi}{2}} \sin^2(ax)\,dx$ **2**

(ii) If a is an integer, $\int_0^{\frac{\pi}{2}} \sin^2(ax)\,dx$ always gives the same number.

Find this number. **1**

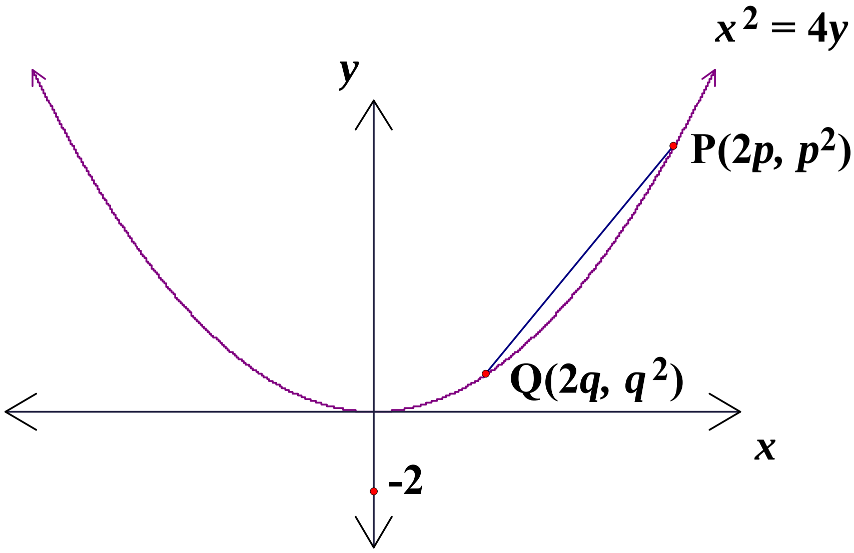
Question 5 (12 marks) [BEGIN A NEW PAGE]
Marks

(a) A particle moves such that $\ddot{x} = x - 1$. Initially, $x = 2$ and $v = 1$

(i) Show that $v = x - 1$ **3**

(ii) Find x as a function of t. **3**

(b)



(i) Show that the equation of the normal at P is $x + py = 2p + p^3$ **2**

It is given that the equation of PQ is $y = \left(\frac{p+q}{2}\right)x - pq$ [Do not prove this]

For the remainder of the question, assume that PQ passes through the point $(0, -2)$.

(ii) Show that $pq = 2$ **1**

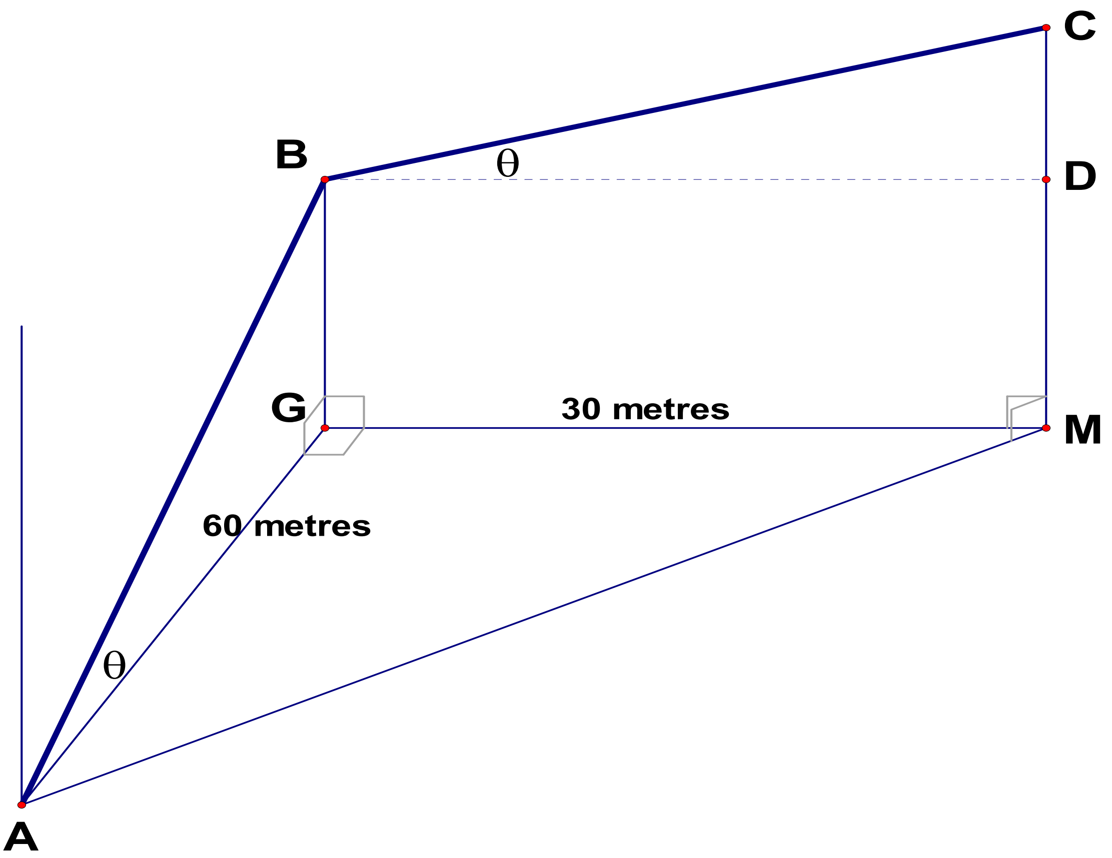
(iii) The normals at P and Q intersect at T.

T has coordinates $\left[-pq(p+q),\ p^2 + pq + q^2 + 2\right]$ [Do not prove this]

Show that the locus of T is the original parabola. **3**

Question 6 (12 marks) [BEGIN A NEW PAGE]
Marks

(a)



The diagram represents part of the inside of two of the walls (built vertically and at right angles) of a parking station.
AGM represents the (horizontal) ground floor. AG is 60 metres and GM is 30 metres.
AB and BC represent the ramp climbing up the two walls.
The ramp climbs at a constant angle θ, where $\tan\theta = \frac{1}{6}$.

(i) Find the length MC. **1**

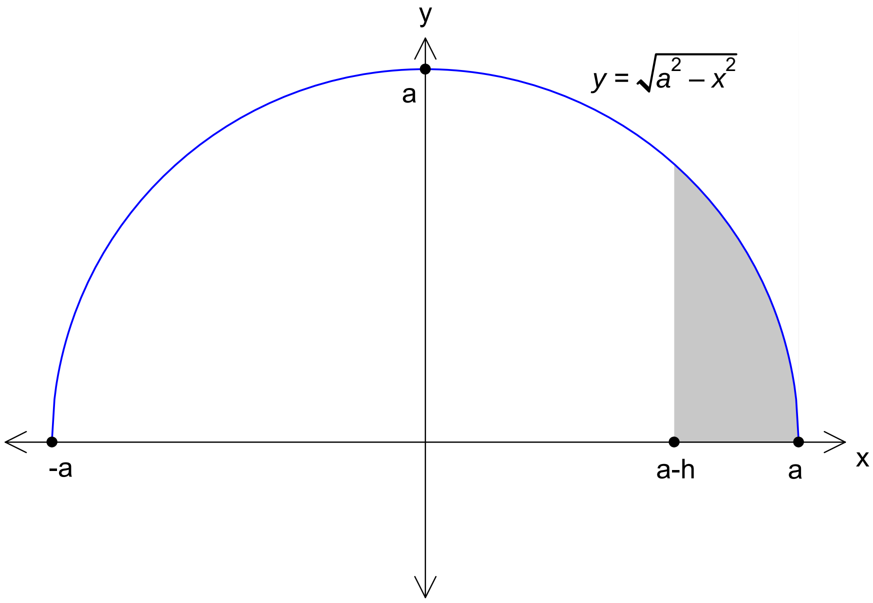
(ii) Hence show that the distance AC is $\sqrt{4725}$ metres. **1**

(iii) By first finding distances AB and BC, find $\angle ABC$ to the nearest minute. **3**

Question 6 continues on the next page

Question 6 (continued)
Marks

(b)



(i) Show that the volume of the spherical cap formed when the shaded area is rotated about the x-axis is $V = \pi h^2\left(a - \dfrac{h}{3}\right)$ **4**

(ii) Water is poured into a hemispherical bowl of radius $5cm$ at the rate of $22cm^3/\sec$.
Find the exact rate at which the depth of the water is increasing when the depth is $3cm$. **3**

Question 7 (12 marks) [BEGIN A NEW PAGE]
Marks

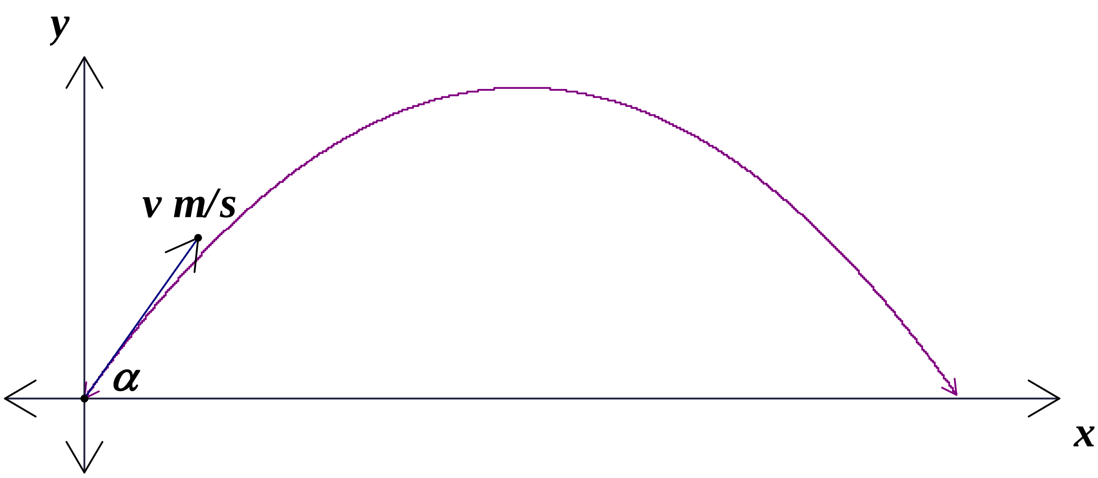
(a) In March this year, the 8 quarterfinalists of the 2008 Champions League Football competition were randomly drawn into 4 quarterfinals.

4 of the quarterfinalists were English teams:
Manchester United, Liverpool, Chelsea, Arsenal.

The other 4 quarterfinalists were from mainland Europe:
Roma, Barcelona, Schalke, Fenerbahce.

What was the probability that <u>at least one</u> quarterfinal was between two English teams? **3**

(b)



A golf ball is hit with initial velocity v m/s at an angle of projection α.

The ball's horizontal displacement from the origin, t seconds after being hit is given by the equation $x = vt\cos\alpha$ [Do not prove this].

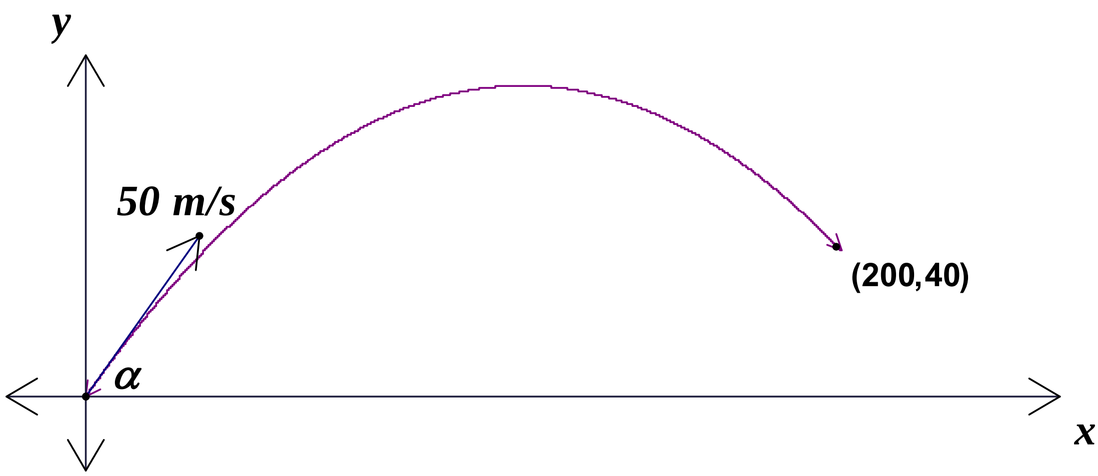
The ball's vertical displacement from the origin, t seconds after being hit is given by the equation $y = vt\sin\alpha - \frac{1}{2}gt^2$ [Do not prove this].

(i) Show that the time of maximum height occurs when $t = \dfrac{v\sin\alpha}{g}$ **2**

(ii) Find the simplest expression for the x-value when the maximum height is reached.

Your answer must be in terms of 2α **2**

Question 7 continues on the next page



A golfer hits the ball at 50m/s to an elevated green.

The hole is 200m away horizontally and 40m up vertically. Take $g = 10m/s^2$.

(iii) Find the 2 angles that the golfer can hit the ball in order to hit the hole. **3**

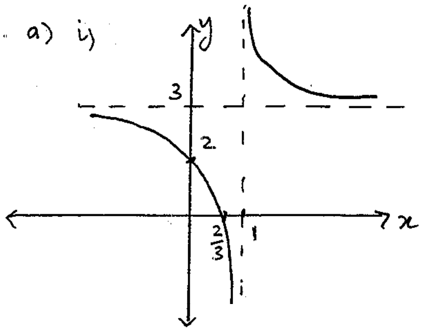
(iv) The golfer's caddy makes the following statement:
"Hitting the ball at one of these angles will result in hitting the hole on the way up.
Since the green is elevated, this will be physically impossible.
Thus you must hit the ball at the higher angle in order for the ball to land <u>in</u> the hole."

Using parts (i) or (ii) and showing mathematical calculations, do you agree? **2**

End of Paper

YEAR 12 EXT 1 TRIAL 2008

Q1. a) i)



ii) LHS = $\frac{1}{x-1} + \frac{3(x-1)}{x-1}$

$= \frac{1+3x-3}{x-1}$

$= \frac{3x-2}{x-1}$

= RHS

iii) $\frac{1}{x-1} + 3 \geq 0$

from graph, $x \leq 2/3,\ x > 1$

b) $u = x^2$ $\quad I = \frac{1}{2}\int \frac{du}{\sqrt{9-u^2}}$

$du = 2x\ dx$ $\quad = \frac{1}{2}\sin^{-1}\left(\frac{u}{3}\right) + c$

$\frac{1}{2}du = x\ dx$

$= \frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right) + c$

c) i) $\frac{1}{\sqrt{x+h}+\sqrt{x}} \times \frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}-\sqrt{x}}$

$= \frac{\sqrt{x+h}-\sqrt{x}}{(x+h)-x}$

$= \frac{\sqrt{x+h}-\sqrt{x}}{h}$

ii) $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

Q2. a) i) $\frac{9!}{3!\,2!} = 30240$

ii) $\frac{5! \times 4!}{2! \times 3!} = 240$

b) i) $\frac{dy}{dx} = 2 \times \frac{1}{1+x^2}$

$\therefore \frac{1}{3} = \frac{2}{1+x^2}$

$1 + x^2 = 6$

$x = \pm\sqrt{5}$

ii) $A = 2\int_0^\infty \frac{2}{1+x^2}\,dx$

$= 2\left[2\tan^{-1}x\right]_0^\infty$

$= 4\left[\tan^{-1}\infty - \tan^{-1}0\right]$

$= 4\left[\frac{\pi}{2} - 0\right]$

$= 2\pi$ units^2

c) i) LHS = $\dfrac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}$

$= \frac{1+t^2-(1-t^2)}{1+t^2+(1-t^2)}$

$= \frac{2t^2}{2}$

$= t^2$

$= \tan^2 \theta/2$ = RHS

ii) $\int_0^{\pi/2} \tan^2 \theta/2\ d\theta$

$= \int_0^{\pi/2} (\sec^2 \theta/2 - 1)\,d\theta$

$= \left[2\tan \theta/2 - \theta\right]_0^{\pi/2}$

$= \left[(2\tan \pi/4 - \pi/2) - (2\tan 0 - 0)\right]$

$= 2 - \pi/2$

Q3. a) i) $y = 2\cos^{-1}\left(\frac{x}{3}\right)$

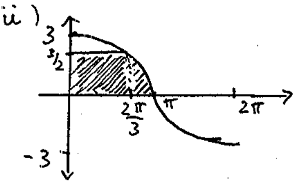
$x = 2\cos^{-1}(y/3)$

$\frac{x}{2} = \cos^{-1}(y/3)$

$\cos(x/2) = y/3$

$\therefore f^{-1}(x) = 3\cos(x/2)$

ii)



iii) $A = \frac{2\pi}{3} \times \frac{3}{2} + \int_{2\pi/3}^{\pi} 3\cos(x/2)\,dx$

$= \pi + 6\left[\sin(x/2)\right]_{2\pi/3}^{\pi}$

$= \pi + 6\left[\sin \pi/2 - \sin \pi/3\right]$

$= \pi + 6 - 3\sqrt{3}$ units^2

b) $(1+x)^n$ gen term $\binom{n}{r}x^r$

$\therefore n = 16,\ r = 10$ (or 6)

c) i) $1^4 + 5(1) = 6 \quad 6 \div 2 = 3$ ✓

$2^4 + 5(2) = 26 \quad 26 \div 2 = 13$ ✓

ii) STEP 1 true for $n=1$ (from i)

STEP 2 Assume true for $n=k$

ie $k^4 + 5k = 2M$ (M + integer)

Prove true for $n = k+1$

ie $(k+1)^4 + 5(k+1) = 2M$

LHS $= k^4 + 4k^3 + 6k^2 + 4k + 1 + 5k + 5$

$= k^4 + 5k + 4k^3 + 6k^2 + 4k + 6$

$= 2M + 2(2k^3 + 3k^2 + 2k + 3)$

$= 2N$ (N + integer)

Proven true for $n=1$, from step 2, it is true for $n = 2, 3, 4\ldots$ and all positive integers n.

Q4. a) $T_{r+1} = \binom{12}{r}(x^2)^{12-r}\left(-\frac{1}{x}\right)^r$

$= \binom{12}{r} x^{24-2r} \frac{(-1)^r}{x^r}$

independent of x: $x^{24-2r-r} = x^0$

$\therefore r = 8$

$T_9 = \binom{12}{8}(-1)^8 = \binom{12}{8}$

b) i) $\dot{x} = 2 \times 2 \sin t \cos t$

$= 4 \sin t \cos t$

$\ddot{x} = 4\left[\sin t \cdot -\sin t + \cos t \cdot \cos t\right]$

$= 4\left[\cos^2 t - \sin^2 t\right]$

$= 4\left[1 - 2\sin^2 t\right]$

$= 4[1 - x]$

$= -4(x-1)$

ii) period $= \pi$, centre $x = 1$.

iii) $0 \le x \le 2$

c) i) $I = \frac{1}{2}\int_0^{\pi/2}(1 - \cos 2ax)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2ax}{2a}\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin a\pi}{2a}\right) - 0\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sin a\pi}{2a}\right]$

ii) since $\sin a\pi = 0$

then $\frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$

Q5. a) i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$x-1 = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\int (x-1)\,dx = \frac{1}{2}v^2$

$\frac{x^2}{2} - x + c = \frac{1}{2}v^2$

when $x = 2, v = 1$

$\frac{2^2}{2} - 2 + c = \frac{1}{2}(1)^2 \Rightarrow c = \frac{1}{2}$

$\frac{x^2}{2} - x + \frac{1}{2} = \frac{1}{2}v^2$

$v^2 = x^2 - 2x + 1$

$v^2 = (x-1)^2$

$v = \pm(x-1)$

when $x = 2, v = 1$

$\therefore v = (x-1)$.

ii) $\frac{dx}{dt} = x - 1$

$\frac{dt}{dx} = \frac{1}{x-1}$

$t = \int \frac{1}{x-1}\,dx$

$t = \ln(x-1) + c$

when $t = 0, x = 2$

$0 = \ln 1 + c \Rightarrow c = 0$

$t = \ln(x-1)$

$e^t = x - 1$

$x = e^t + 1$

b) i) $x^2 = 4y \Rightarrow y = x^2/4$

$y' = \frac{x}{2}$

when $x = 2p$, $y' = p$

$\therefore$ gradient of normal is $-1/p$.

$y - p^2 = \frac{-1}{p}(x - 2p)$

$py - p^3 = -x + 2p$

$x + py = 2p + p^3$

ii) $PQ: \quad y = \left(\frac{p+q}{2}\right)x - pq$

subst $(0,-2)$: $\quad -2 = 0 - pq$

$\therefore pq = 2$

iii) $T\left(-pq(p+q),\ p^2 + pq + q^2 + 2\right)$

$x = -pq(p+q)$, but $pq = 2$

$x = -2(p+q)$

$\therefore p+q = \frac{-x}{2}$

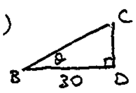
$y = (p+q)^2 - pq + 2$

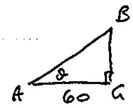
$= \left(\frac{-x}{2}\right)^2 - 2 + 2$

$\therefore y = \frac{x^2}{4}$

$x^2 = 4y$ original parabola

Q6 a) i)



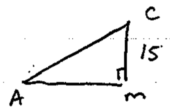


$\tan\theta = \frac{CD}{30} = \frac{1}{6}$ $\qquad$ $\tan\theta = \frac{BG}{60} = \frac{1}{6}$

$\therefore CD = 5m$ $\qquad$ $\therefore BG = 10m$

$MC = 5 + 10 = 15m$

ii)



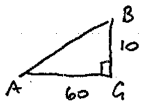
$AM = \sqrt{60^2 + 30^2}$

$= \sqrt{4500}$

$AC = \sqrt{4500 + 15^2}$

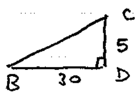
$= \sqrt{4725}$

iii)



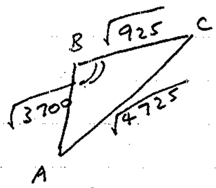
$AB = \sqrt{60^2 + 10^2}$

$= \sqrt{3700}$



$BC = \sqrt{5^2 + 30^2}$

$= \sqrt{925}$



$\cos B = \frac{3700 + 925 - 4725}{2\sqrt{3700}\sqrt{925}}$

$= -0.027027$

$B = 91°33'$

b) i) $V = \pi \int_{a-h}^{a} (a^2 - x^2)\,dx$

$= \pi \left[a^2x - \frac{x^3}{3} \right]_{a-h}^{a}$

$= \pi \left[a^3 - \frac{a^3}{3} - a^2(a-h) + \frac{(a-h)^3}{3} \right]$

$= \pi \left[a^3 - \frac{a^3}{3} - a^3 + ah + \frac{a^3}{3} - \frac{3a^2h}{3} + \frac{3ah^2}{3} - \frac{h^3}{3} \right]$

$= \pi \left[ah^2 - \frac{h^3}{3} \right] = \pi h^2 (a - h/3)$

b) ii) $r = 5$, $\frac{dV}{dt} = 22$, find $\frac{dh}{dt}$

$V = \pi h^2 (a - h/3)$, $a = 5$

$= \pi h^2 (5 - h/3)$

$= \pi \left[5h^2 - h^3/3 \right]$

$\frac{dV}{dh} = \pi \left[10h - h^2 \right]$

when $h = 3$

$\frac{dV}{dh} = \pi \left[10 \times 3 - 3^2 \right] = 21\pi$

$\frac{dh}{dt} = \frac{dV}{dt} \cdot \frac{dh}{dV}$

$= 22 \times \frac{1}{21\pi} = \frac{22}{21\pi}$ cm/sec

Q7. a) P(at least 1) = 1 − P(not EE)

E − English $\qquad$ M − mainland Eur.

not EE $\quad 4 \times 3 \times 2 \times 1 = 24$

all games $\quad 7 \times 5 \times 3 \times 1 = 105$

$\therefore$ Prob $= 1 - \frac{24}{105} = \frac{81}{105} = \frac{27}{35}$

b) i) max height, $\dot{y} = 0$

$\dot{y} = v\sin\alpha - gt$

$\therefore v\sin\alpha - gt = 0$

$v\sin\alpha = gt$

$\therefore t = \frac{v\sin\alpha}{g}$

ii) $x = vt\cos\alpha$

$= v \times \frac{v\sin\alpha}{g} \times \cos\alpha$

$= v^2 \times \frac{2\sin\alpha\cos\alpha}{2g}$

$= \frac{v^2 \sin 2\alpha}{2g}$

Q7. iii) $v = 50$, $(200, 40)$

$$x = vt\cos\alpha$$

$$200 = 50t\cos\alpha$$

$$\therefore\ t = \frac{4}{\cos\alpha}$$

$$y = vt\sin\alpha = \frac{1}{2}gt^2$$

$$40 = 50 \times \frac{4}{\cos\alpha} \times \sin\alpha - 5\left(\frac{4}{\cos\alpha}\right)^2$$

$$40 = 200\tan\alpha - 80\sec^2\alpha$$

$$1 = 5\tan\alpha - 2\sec^2\alpha$$

$$1 = 5\tan\alpha - 2(1+\tan^2\alpha)$$

$$1 = 5\tan\alpha - 2 - 2\tan^2\alpha$$

$$\therefore\ 2\tan^2\alpha - 5\tan\alpha + 3 = 0$$

$$(2\tan\alpha - 3)(\tan\alpha - 1) = 0$$

$$\tan\alpha = \frac{3}{2},\quad \tan\alpha = 1$$

$$\therefore\ \alpha = 56°19',\ 45°$$

iv) Max height $t = \frac{v\sin\alpha}{g}$

$\therefore$ when $\alpha = 56°19'$

$$t = \frac{50\sin 56°19'}{10}$$

$$= 4.16 \text{ sec}$$

$$x = \frac{v^2\sin 2\alpha}{2g}$$

$$= \frac{50^2 \times \sin(2 \times 56°19')}{2 \times 10}$$

$$= 115.38\text{ m} < 200\text{m}$$

$$y = 50 \times 4.16 \times \sin(56°19') - 5(4.16)^2$$

$$= 86.55\text{ m}$$

when $\alpha = 45°$

$$t = \frac{50\sin 45°}{10}$$

$$= \frac{5}{\sqrt{2}}$$

$$= 3.54 \text{ sec}$$

$$x = \frac{v^2\sin 2\alpha}{2g}$$

$$= \frac{50^2 \times \sin 90°}{2 \times 10}$$

$$= 125\text{ m} < 200\text{m}$$

$$y = 50 \times 3.54 \times \sin 45° - 5 \times (3.54)^2$$

$$= 62.5\text{ m}$$

The caddy is wrong. Both angles can allow the ball to land in the hole.