# BAULKHAM HILLS HIGH SCHOOL

YEAR 12 TRIAL HSC EXAMINATION

# 2004

# MATHEMATICS EXTENSION 1

*Time Allowed : Two hours*

*(Plus five minutes reading time)*

## QUESTION 1

**(a)** Find the co-ords of the point P that divides the interval A(–3, 4) and B(2, – 3) externally in the ratio 1 : 2.

**(b)** Solve $\frac{4}{x-3} < 1$.

**(c)** Evaluate $\lim_{x \to 0} \frac{\sin 2x \cos 2x}{3x}$.

**(d)** A curve has parametric equations $x = 2t - 2$, $y = t^2 + 1$. Find the cartesian equation for this curve.

**(e)** Use the substitution $u = 2 + x$ to evaluate $\int_{-2}^{2} x\sqrt{2+x}\,dx$.

## QUESTION 2

**(a)** Find **(i)** $\int \tan x\,dx$.

**(ii)** $\int_{-3/2}^{3/4} \frac{1dx}{\sqrt{9-4x^2}}$.

**(b)** Find the term independent of x in the binomial expansion $\left(x^2 + \frac{1}{x}\right)^9$.

**(c)** **(i)** Express $\sin 4t + \sqrt{3}\cos 4t$ in the form $R\sin(4t + \alpha)$, where $\alpha$ is in radians.

**(ii)** Hence solve $\sin 4t + \sqrt{3}\cos 4t = 0$ for $0 \le t \le \pi$.

**QUESTION 3**

(a) Prove by induction $9^{n+2} - 4^{n}$ is divisible by 5 for $n \geq 1$.

(b) Consider the function $f(x) = 2\tan^{-1}x$.

(i) State the range of the function $y = f(x)$.

(ii) Sketch the graph of $y = f(x)$.

(iii) Find the gradient of the tangent to the curve $y = f(x)$ at $x = \frac{1}{\sqrt{3}}$.

(c) (i) By equating the coefficients of sin x and cos x, or otherwise, find constan satisfying the identity.

$$A(2\sin x + \cos x) + B(2\cos x - \sin x) \equiv \sin x + 8\cos x.$$

(ii) Hence evaluate $\int \frac{\sin x + 8\cos x}{2\sin x + \cos x}\, dx$.

**QUESTION 4**

(a) If $x^3 - 8x^2 + kx - 12 = 0$ has one root equal to the sum of the other two; find k.

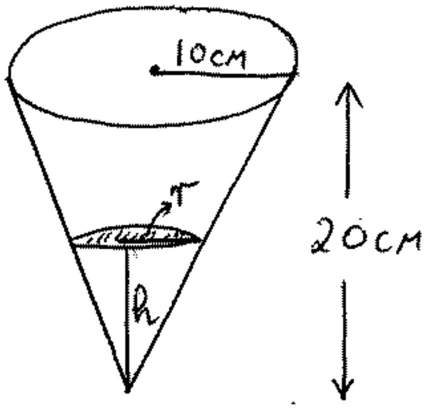
(b) Taking $x = 0.5$ as the first approximation, use Newtons method to find a second ap to the root of:

$$x - 3 + e^{2x} = 0.$$

Write your answer to 2 significant figures.

**QUESTION 4 (Continued)**

**(c)**




Water is poured into a conical vessel at a rate of $30cm^3/s$.

(i) What is the rate of increase of the radius of water when $r = 5$.

(ii) Hence find the rate of increase of the area of the surface of the liquid when $r = 5$.

**(d)** Using $\sin 3\theta = \sin(2\theta + \theta)$. Prove $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

**QUESTION 5**

**(a)** A particle moves in a straight line such that its position x from a fixed point 0 at time 't' is given by $x = 5 + 8\sin 2t + 6\cos 2t$.

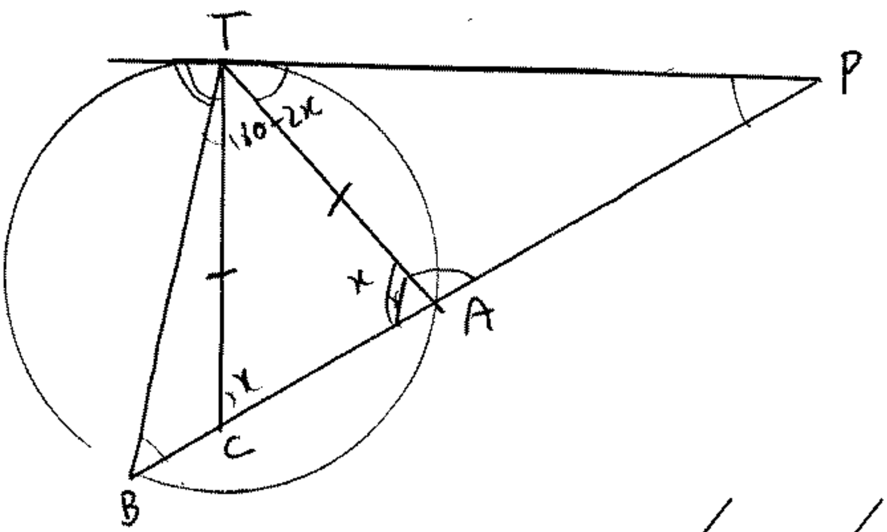
(i) Prove the motion is simple harmonic motion.

(ii) Find the period and amplitude of the motion.

(iii) Find the greatest speed of the particle.

**(b)** State the largest positive domain for which $y = x^2 - 4x + 7$ has an inverse function.

**(c)**




PT is a tangent and PAB is a secant. TC = TA. Prove $\angle BTC = \angle TPA$

**(d)** By using the expansion $(1 + x)^n$. Prove $\sum_{k=0}^{n} 2^{3k}\binom{n}{k} = 3^{2n}$.

## QUESTION 6

(a) A particle is projected horizontally with velocity $V ms^{-1}$, from a point h metres above the ground. Take $g\ ms^{-2}$ as the acceleration due to gravity.

(i) Taking the origin as the point on the ground immediately below the projection point, find expressions for x and y, the horizontal and vertical displacements of the particle at time 't' secs.

(ii) Show the equation of the path is given by $y = \dfrac{2hV^2 - gx^2}{2V^2}$.

(iii) Find the range of the particle.

(b) Assume that the rate at which a body warms in air is proportional to the difference between its temperature T and the constant temperature A of the surrounding air. The rate can be expressed as:

$$\frac{dT}{dt} = K(T - A) \text{ where 't' is in minutes and K is constant.}$$

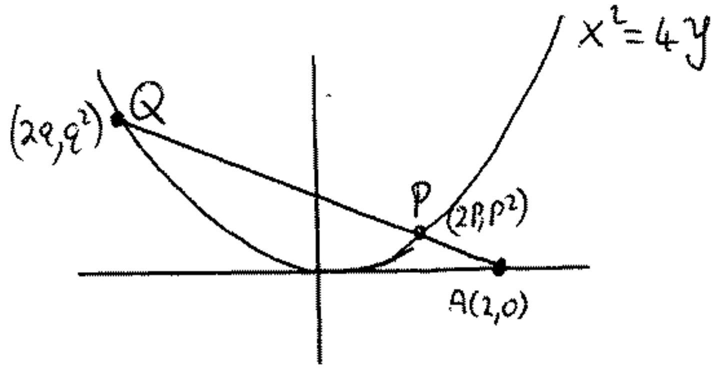
(i) Show $T = A + Ce^{kt}$ (where C is constant) is a solution of the differential equation.

(ii) A cooled body warms from $5^0C$ to $10^0C$ in 20 minutes. The air temperature is $20^0C$. Find the temperature of the body after a further 30 minutes have elapsed.

(iii) Explain the behaviour of T as t becomes large.

(c) Differentiate from 1st Principles $f(x) = x^2 - 2x + 1$.

## QUESTION 7




(a) The chord PQ joining the points $P(2p,p^2)$ and $Q(2q,q^2)$ on $x^2 = 4y$ always passes through the point A(2,0) when produced.

(i) Show $(p + q) = pq$.
(ii) Find the co-ordinates of M, the mid-point of PQ.
(iii) Find the equation of the locus of M as P and Q vary on the parabola.

**QUESTION 7 (Continued)**

(b) Two circles $C_1$ and $C_2$ are members of a set of circles defined by the equation: $x^2 + y^2 - 6x + 2ky + 3k = 0$ where k is real. The centre of $C_1$ lies on the line $x - 3y = 0$ and $C_2$ touches the x – axis. Find the equations of $C_1$ and $C_2$.

(c) Use Simpson's Rule with 3 function values to approximate the volume when $y = \ln x$ is rotated about the x – axis between $x = 1$ and $x = 3$.

(1)

YEAR 12 TRIAL EXT 1

MARKING SCALE

QUESTION 1

$x = \frac{1\times 2 + -2\times -3}{1-2}$

$= -8$ ①

$y = \frac{1\times -3 + 4\times -2}{-1-2}$ ①

$\therefore = 11$ $\boxed{2}$

---

$\frac{4}{x-3} < 1$

$4(x-3) < (x-3)^2$ ①

$0 < (x-3)(x-3-4)$

$0 < (x-3)(x-7)$ ①

$\longleftarrow$ 3    7 $\longrightarrow$

$x < 3, \quad x > 7$. ① $\boxed{3}$

---

$\lim_{x\to 0} \frac{\sin 2x \cos 2x}{\sin 3x}$

$\lim_{x\to 0} \frac{\frac{1}{2}\sin 4x}{\sin 3x}$

$= 2/3$. $\boxed{1}$

---

$x = 2t-2, \quad y = t^2+1$.

$t = \frac{x+2}{2}$ ①

$y = \left(\frac{x+2}{2}\right)^2 + 1$ ① $\boxed{2}$

---

e) $\int_{-2}^{2} x\sqrt{2+x}\,dx$    $U = 2+x$

$\frac{dU}{dx} = 1$

$dU = dx$

CHANGE LIMITS

$x = -2$, $U = 0$.

$x = 2$, $U = 4$

$= \int_0^4 (U-2)\sqrt{U}\,dU$. ①

$= \int_0^4 U^{\frac{3}{2}} - 2U^{\frac{1}{2}}\,dU$

$= \left[\frac{2}{5}U^{5/2} - \frac{4}{3}U^{\frac{3}{2}}\right]_0^4$ ①

$= \left(\frac{2}{5}\cdot 32 - \frac{4}{3}\cdot 8\right) - (0)$

$= 2\,^2\!/_{15}$. ① $\boxed{3}$

11

---

Question 2.

a) i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$= -\ln(\cos x) + C$ ① ① $\boxed{2}$

ii) $\int_{-3/2}^{3/4} \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int_{-3/2}^{3/4} \frac{dx}{\sqrt{9/4 - x^2}}$ ①

$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{3}\right]_{-3/2}^{3/4}$ ①

$= \frac{1}{2}\left[\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(-1)\right]$

$= \frac{1}{2}\left[\frac{\pi}{6} - -\frac{\pi}{2}\right]$

$= \pi/3$. ① $\boxed{3}$

b) $\left(x^2 + \frac{1}{x}\right)^9$

$T_{K+1} = {}^9C_K (x^2)^{9-K}\left(\frac{1}{x}\right)^K$ ①

$= {}^9C_K\, x^{18-3K}$.

$\therefore 18 - 3K = 0$

$K = 6$ ①

$T_7 = {}^9C_6$ ①

$= 84$. $\boxed{3}$

c) i) $\sin 4t + \sqrt{3}\cos 4t = R\sin(4t+\alpha)$

$= R[\sin 4t\cos\alpha + \cos 4t\sin\alpha]$

$\therefore R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}$

$\therefore \tan\alpha = \sqrt{3}$

$\therefore \alpha = \pi/3$ ①

$R = 2$. ①

$\therefore \sin 4t + \sqrt{3}\cos 4t = 2\sin\left(4t + \frac{\pi}{3}\right)$

ii) $2\sin\left(4t + \frac{\pi}{3}\right) = 0$

$\therefore 4t + \frac{\pi}{3} = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$

$4t = -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}, \frac{11\pi}{3}$ ①

$t = -\frac{\pi}{3}, \frac{\pi}{6}, \frac{5\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}$

but $t \neq -\pi/3$ ① $\boxed{4}$

12

## Question 3

STEP 1: Prove True for $N=1$

$9^3 - 4 = 725$

which is divisible by 5.

$\therefore$ TRUE (1)

Step 2: Assume True $N = K$

$9^{K+2} - 4^K = 5A$ [A is INTEGER]

Step 3: Prove True $N = K+1$

$9^{K+3} - 4^{K+1} = 5B$ [B is INTEGER]

$LHS = 9^{K+3} - 4^{K+1}$

$= 9.9^{K+2} - 4.4^K$

$= 9(5A + 4^K) - 4.4^K$ (2)

$= 45A + 5.4^K$

$= 5(9A + 4^K)$

Since A is integer, K is integer

$\therefore LHS = 5.B$ [B is integer]

Step 4: If true $N=K$, true $N=K+1$

$\therefore$ Since True $N=1$, True $N=2$

& True $N=2$, True $N=3$ etc

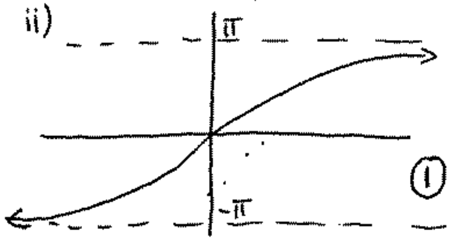
$\therefore$ True for all $N$. (1)

$\boxed{4}$

b) i) $f(x) = 2\tan^{-1}x$.

Range: $-\pi < y < \pi$. (1)

ii)




(1)

iii) $y = 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{2}{1+x^2}$ (1)

at $x = \frac{1}{\sqrt{3}}$

$\frac{dy}{dx} = \frac{2}{1+1/3}$

$= 3/2$.

$\therefore$ Gradient of tangent $= 3/2$. (1) $\boxed{4}$

c) $2A\sin x + A\cos x + 2B\cos x - B\sin x$

$= \sin x + 8\cos x$

$\therefore 2A - B = 1$

$A + 2B = 8$. (1)

$\therefore A = 2, \; B = 3$ (1)

$\therefore 2(2\sin x + \cos x) + 3(2\cos x - \sin x)$

$= \sin x + 8\cos x$

$\therefore \int \frac{\sin x + 8\cos x}{2\sin x + \cos x}dx = \int 2 + 3\left(\frac{2\cos x - \sin x}{2\sin x + \cos x}\right)dx$ (1)

$= 2x + 3\ln(2\sin x + \cos x) + C$. (1)

$\boxed{4}$

## Question 4:

a) $x^3 - 8x^2 + Kx - 12 = 0$

Let roots be $\alpha, \beta, \gamma$

where $\alpha = \beta + \gamma$.

$\alpha + \beta + \gamma = 8$

$\therefore 2\alpha = 8$

$\alpha = 4$. (1)

$\therefore 64 - 8(16) + 4K - 12 = 0$.

$K = 19$. (1) $\boxed{2}$

b) $Z_2 = Z_1 - \frac{f(Z_1)}{f'(Z_1)}$.

$f(x) = x - 3 + e^{2x}$

$f(0.5) = 0.218$ (1)

$f'(x) = 1 + 2e^{2x}$

$f'(0.5) = 6.437$. (1)

$Z_2 = 0.5 - \frac{0.218}{6.437}$

$= 0.47$. (1)

$\boxed{3}$

c) $\frac{dV}{dt} = 30$.

$V = \frac{1}{3}\pi r^2 h$.

$\frac{r}{10} = \frac{h}{20}$

$\therefore h = 2r$ (1

$V = \frac{1}{3}\pi r^2 . 2r$

$= \frac{2}{3}\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$30 = 2\pi r^2 . \frac{dr}{dt}$ (1)

$r = 5$

$30 = 50\pi \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{5\pi}$. (1)

(4)

1.

$$\frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt}.$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{3}{5\pi} \quad (1)$$

$$= 10\pi \cdot \frac{3}{5\pi}$$

$$= 6\ \text{cm}^2/\text{s}. \quad (1) \quad \boxed{5}$$

$$\sin 3\theta = \sin(2\theta + \theta)$$

$$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$$

$$= 2\sin\theta\cos\theta.\cos\theta \quad (1)$$

$$+ (1 - 2\sin^2\theta)\sin\theta \quad (1)$$

$\frac{7}{12}$

$$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta \quad (1)$$

$$= 3\sin\theta - 4\sin^3\theta \quad \boxed{2}$$

## Question 5:

$$x = 5 + 8\sin 2t + 6\cos 2t$$

$$\dot{x} = 16\cos 2t - 12\sin 2t$$

$$\ddot{x} = -32\sin 2t - 24\cos 2t$$

$$= -4(8\sin 2t + 6\cos 2t) \quad (1)$$

$$= -4(x-5) \quad (1)$$

since $\ddot{x} = -n^2(x-b)$

$\therefore$ SHM

ii) Period $= \frac{2\pi}{n} = \pi.$ (1)

$$x = 5 + 8\sin 2t + 6\cos 2t$$

but $8\sin 2t + 6\cos 2t = 10\sin(2t+\alpha)$ (1)

$\therefore$ AMP $= 10.$ (1)

iii) $\dot{x} = 16\cos 2t - 12\sin 2t$

$= 20\cos(2t+\alpha)$ (1)

$\therefore$ greatest speed $= 20$ m/s. $\boxed{6}$

b) $x \geq 2.$ $\boxed{1}$

c) $\angle TCA = \angle TAC = y°$ (base $\angle$s of ISOS $\triangle$)

$\angle PTA = \angle TBC = x°$ ($\angle$ between tang & CHORD = $\angle$ in ALT. segment). (1)

$\therefore \angle BTC = y - x$ (EXT $\angle$ of $\triangle$ = SUM of interior OPP $\angle$'s)

$\angle TAP = (180 - y)$ ($\angle$ sum of st line) (1)

$\therefore \angle TPA = y - x$ ($\angle$ sum of $\triangle$ = 180°).

$\therefore \angle TPA = \angle BTC.$ $\boxed{3}$

5d)

$$(1+x)^N = {}^NC_0 + {}^NC_1 x + {}^NC_2 x^2 + \dots + {}^NC_N x^N.$$

RTP:

$${}^NC_0 + 8\,{}^NC_1 + 64\,{}^NC_2 + \dots 2^{3N}\,{}^NC_N = 3^{2N} \quad (1)$$

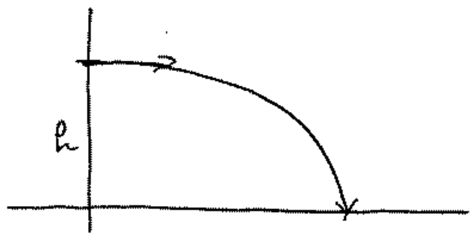
Let $x = 8$

$$9^N = {}^NC_0 + 8\,{}^NC_1 + 64\,{}^NC_2 + \dots \quad (1)$$

$$\therefore 3^{2N} = {}^NC_0 + 8\,{}^NC_1 + 64\,{}^NC_2 + \dots \quad \boxed{2}$$

12

## Question 6:




$\ddot{x} = 0$ | $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ | $\dot{y} = -gt + V\sin\alpha$

$\alpha = 0$ | $\alpha = 0$

$\therefore \dot{x} = V.$ | $\dot{y} = -gt.$

$x = Vt + C.$ | $y = -g\frac{t^2}{2} + C$

$t=0,\ x=0\ \therefore C=0$ | $t=0,\ y=h\ \therefore C=h$

$x = Vt$ (1) | $y = \frac{-gt^2}{2} + h$ (1)

ii) $t = \frac{x}{V}.$ (1)

$$y = \frac{-g\left(\frac{x}{V}\right)^2}{2} + h$$

$$= \frac{-gx^2}{2V^2} + h$$

$$= \frac{2V^2h - gx^2}{2V^2} \quad (1)$$

iii) Range: $x$ when $y = 0$

$$y = \frac{-gt^2}{2} + h$$

$$0 = \frac{-gt^2}{2} + h$$

$$t = \sqrt{\frac{2h}{g}} \quad (1)$$

$$\therefore x = V\sqrt{\frac{2h}{g}} \quad (1)$$

$\boxed{6}$

b) $T = A + Ce^{kt}$

$$\frac{dT}{dt} = kCe^{kt}$$

but $Ce^{kt} = T - A$

$$\frac{dT}{dt} = k(T-A). \quad (1)$$

$$T = 20 + Ce^{kt}$$

$t = 0 \quad T = 5$

$$5 = 20 + Ce^{0}$$

$$\therefore C = -15 \quad (1)$$

$$T = 20 - 15e^{kt}$$

$t = 20, \quad T = 10$

$$10 = 20 - 15e^{20k}$$

$$15e^{20k} = 10$$

$$20k = \ln(2/3)$$

$$k = -0.02. \quad (1)$$

$$T = 20 - 15e^{-0.02\times 50}$$

$$= 14.48^{\circ} \quad (1)$$

$\boxed{5}$

iii) $t \to \infty, \quad T \to 20^{\circ}$ (1)

---

c) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - 1 - (x^2 - 2x + 1)}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} \quad (1)$$

$$= \lim_{h\to 0} 2x + h - 2 \quad (1)$$

$$= 2x - 2.$$

$\boxed{2}$

13

## Question 7.

$$M_{PQ} = \frac{p^2 - q^2}{2p - 2q}$$

$$= \frac{(p-q)(p+q)}{2(p-q)}$$

$$= \frac{p+q}{2}. \quad (1)$$

EQN CHORD PQ

$$y - p^2 = \frac{p+q}{2}(x - 2p) \quad (1)$$

Since $(2,0)$ lies on line

$$-p^2 = \frac{p+q}{2}(2 - 2p)$$

$$-2p^2 = 2p + 2q - 2p^2 - 2pq$$

$$2pq = 2(p+q)$$

$$\therefore pq = p + q. \quad (1)$$

ii) $$X = p + q, \quad Y = \frac{p^2 + q^2}{2} \quad (1)$$

iii) $$p^2 + q^2 = (p+q)^2 - 2pq$$

$$\boxed{2Y = X^2 - 2X} \quad (1)$$

$\boxed{5}$

7b

$$x^2 + y^2 - 6x + 2ky + 3k = 0.$$

$$x^2 - 6x + 9 + y^2 + 2ky = -3k + 9.$$

CIRCLE 1:

$x$ co-ord centre: 3. (1)

Since lies on $x = 3y$

$\therefore y = 1$

$\therefore$ centre $(3, 1)$.

CIRCLE: is: $(x-3)^2 + (y-1)^2 = r^2$

$\therefore +2ky = -2y$

$\therefore k = -1$ (1)

EQN CIRCLE: $x^2 + y^2 - 6x - 2y - 3 = 0$.

CIRCLE 2: $(3, 0)$ lies on the circle.

$$\therefore 9 - 18 + 3k = 0$$

$$k = 3 \quad (1)$$

EQN CIRCLE:

$$x^2 + y^2 - 6x + 6y + 9 = 0. \quad (1)$$

$\boxed{4}$

---

$$V = \pi\int y^2\,dx.$$

$$= \pi\int_1^3 (\ln x)^2\,dx \quad (1)$$

$(\ln x)^2$

| x | 1 | 2 | 3 |
|---|---|---|---|
| y | 0 | 0.48 | 1.21. |

$$V = \pi \cdot \tfrac{1}{3}\left[0 + 4\times 0.48 + 1\ldots\right] \quad (1)$$

$$= 1.04\pi$$

$$= 3.28 \quad (1)$$

$\boxed{3}$

12