NAME: ______________________________

TEACHER'S NAME: ______________________________

BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2005

MATHEMATICS

EXTENSION 1

Time allowed – Two hours
(Plus five minutes reading time)

GENERAL INSTRUCTIONS:

- Attempt ALL questions.
- Start each of the 7 questions on a new page.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet.
- Marks indicated for each question are only a guide and could change.

QUESTION 1

Marks

(a) Find the acute angle between the lines $y = x$ and $y = \sqrt{3}x$.

(b) If A = (1,4) and B = (6, −12), find the point P(x, y) which divides AB **externally** in the ratio 2:3. **3**

(c) If $\cos 3x = 4\cos^3 x - 3\cos x$, solve $\cos 3x + 2\cos x = 0$ for $0 \le x \le \pi$. **3**

(d) Find the equation of the tangent to the curve $y = \tan^2 x$ at the point $\left(\frac{\pi}{4}, 1\right)$. **3**

(e) Find the co-efficient of x^8 in the expansion $\left(\frac{2}{x} + x^3\right)^{20}$. ...

QUESTION 2

(a) If $f(x) = \sin x + \frac{x}{2} - 1$ has a root near x = 0.6, use one application of Newton's Method to find a better approximation of the root. Give your answer to 2 decimal places.

(b) Evaluate $\int_0^2 \frac{x}{\sqrt{9 - x^2}}\,dx$ using the substitution $u = 9 - x^2$. **4**

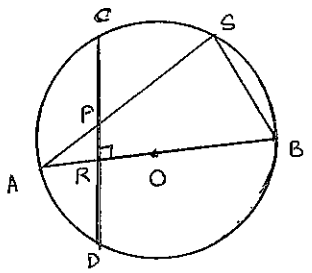
(c) Use mathematical induction to prove $5^n + 2(11)^n$ is divisible by 3 for all positive integer n such that $n \ge 1$. **3**

QUESTION 3

Marks

(a) Find $\int \frac{dx}{9+4x^2}$. **2**

(b)



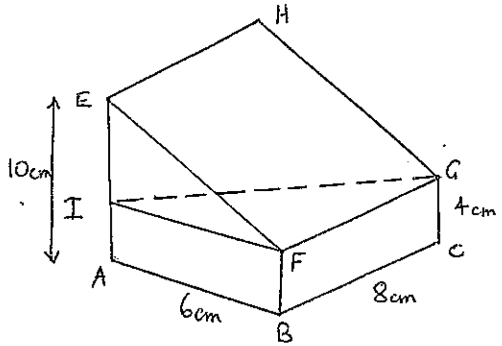
AB is a diameter and CD is perpendicular to AB.

(i) Prove PRBS is a cyclic quadrilatal. **3**

(ii) If AP = 5 and AR = 4 and PS = 8, find BR. **2**

(c) A spherical bubble is expanding so that its volume is increasing at a constant rate of $10mm^3$ per second. What is the rate of increase of its radius when its surface Area is $500mm^2$. **2**

(d)



(i) Find GI. **1**

(ii) Find the size of ∠EGB, using the Cosine Rule. **2**

QUESTION 4

(a) **1**

(i) Complete the table for $y = \cos^2 x$.

x	o	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y					

(ii) Sketch the curve for $y = \cos^2 x$ for $0 \le x \le \pi$. **1**

(iii) Shaded in the region enclosed by $y = \cos^2 x$ for $o \le x \le \pi$ and the line $y = 1$. **1**

(iv) Find the exact area of this shaded region. **3**

(v) The area **below** the curve $y = \cos^2 x$ and above the x axis is rotated about the x axis from $x = o$ to $x = \pi$. Estimate this volume using the Trapezoidal Rule with 4 strips. **3**

(b) (i) If $(x + 1).\,Q(x) = x^3 + 2x^2 - 1$, find Q(x). **1**

(ii) Sketch the graphs of $y = x^2$ and $y = \frac{1}{x+2}$ on the same set of axes showing clearly the x coordinates for their point(s) of intersection. **3**

(iii) Hence or otherwise solve $\frac{1}{x+2} > x^2$. **2**

QUESTION 5

(a) A particle is moving along the x axis, its velocity V (m/s) is given by $V^2 = 16 + 4x - 2x^2$ where x is the position of the particle in metres.

(i) Show that particle is in Simple Harmonic Motion. **2**

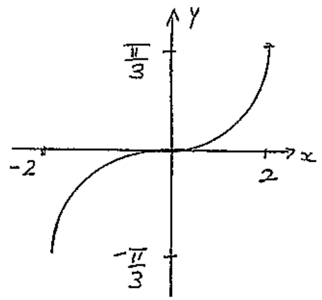
(ii) Find the amplitude and period of the motion. **3**

(iii) Find the maximum speed of the particle **1**

QUESTION 7 (Continued)

(iii) What is the probability that she wins? **2**

(b)



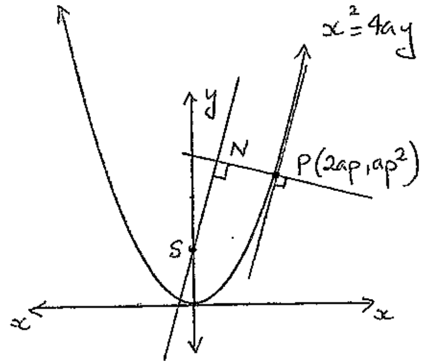
(i) The equation of this graph is in the form $y = a \sin^{-1} bx$, find the values of "a" and "b". **2**

(ii) Show the gradient of the tangent at $x = 0$ is $\frac{1}{3}$. **2**

(iii) For what range of values of "C" will $\frac{x}{3} = a \sin^{-1} bx - c$ have solutions? **3**

QUESTION 5 (Continued)

(b)



P is an arbitrary point on the parabola $x^2 = 4ay$ and PN is perpendicular to NS, where S is the focus.

(i) If PN has equation $x + py = 2ap + ap^3$, find the equation of SN. **2**

(ii) Show that N has coordinates $(ap, ap^2 + a)$. **2**

(iii) Find the locus of N as P moves on the parabola. **2**

QUESTION 6

(a) The rate at which a body's temperature (T) rises is proportional to the difference between its temperature and the surrounding medium (C).

i.e. $\frac{dT}{dt} = k(T - C)$.

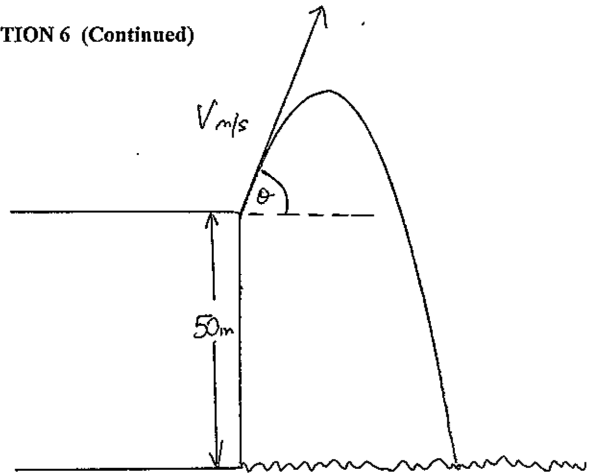
(i) Prove $T = C + A\,e^{kt}$ satisfies the above differential equation. **1**

(ii) If a metal bar at 25^0C is placed in an oven of 300^0C and its temperature rises to 100^0C after 30 minutes, find the value of A and k. **2**

(iii) What will its temperature be after a <u>further</u> 40 minutes? **2**

QUESTION 6 (Continued) **Marks**

(b)



A projectile is fired from a 50 metre cliff into the sea at a velocity V metres per second with an angle of projection θ.

Let $g = 10 m/s^2$

(i) Derive the equations of motion for the horizontal and vertical components of the motion of the projectile.. **2**

(ii) If the time of flight is 5 seconds and the range of the projectile is 100 metres, find the angle of projection and the velocity of the projectile. **3**

(iii) Find the velocity of the projectile at the point of impact with the water.

QUESTION 7

(a) Rebecca invents a game with 2 dice which have their faces coloured the same way. i.e. 3 red, 2 white and 1 black face.

She rolls both dice and wins if she throws 2 red faces and loses if she rolls 1 or more black face on the uppermost face of the dice.

If she gets neither, she rolls again and the game stops when she wins or loses.

(i) Show that the probability of winning in 1 throw is $\frac{1}{4}$ and the probability of losing in 1 throw is $\frac{11}{36}$. **2**

(ii) What is the probability that she wins in the first or 2nd or 3rd throw. **2**

14

1 a) $y = x \rightarrow m_1 = 1$

$y = \sqrt{3}x \rightarrow m_2 = \sqrt{3}$ 1

$\tan\theta = \left|\dfrac{\sqrt{3}-1}{1+\sqrt{3}}\right|$ 1

$\theta = 15^\circ$ 1 (3)

b) $(1,4)$ $(6,-12)$ Internal $(3, -\frac{12}{5})$ 1 mark

$-2:3$

$\left(\dfrac{-2(6)+3(1)}{-2+3}, \dfrac{-2(-12)+3(4)}{-2+3}\right)$

$= (-9, 36)$ (2)

c) $\cos 3x + 2\cos x = 0$

$4\cos^3 x - 3\cos x + 2\cos x = 0$

$4\cos^3 x - \cos x = 0$ } 1

$\cos x(4\cos^2 x - 1) = 0$

$\therefore \cos x = 0 \quad \cos x = \pm\frac{1}{2}$ 1

$x = \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}$ 1 (3)

d) $y = \tan^2 x$

$\frac{dy}{dx} = 2\tan x \sec^2 x$ 1

at $x = \frac{\pi}{4}$ $\quad y' = 2\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}$

$= 2.1.2$

$= 4$ 1

Solutions Total [92]

$\therefore$ Tangent $y - 1 = 4(x - \frac{\pi}{4})$ } 1

$y = 4x - \pi + 1$ (3)

e) $\left(\frac{2}{x} + x^3\right)^{20}$

General term $^{20}C_k\, 2^{20-k} x^{k-20} (x^3)^k$ 1

$= {}^{20}C_k\, 2^{20-k}\, x^{4k-20}$

Now $4k - 20 = 8$

$4k = 28$

$k = 7$ 1

$\therefore$ Coefficient is $^{20}C_7\, 2^{13}$ 1 (3)

Question 2 [11]

a) $f(x) = \sin x + \frac{x}{2} - 1$

$f(0.6) = \sin(0.6) + 0.3 - 1$

$= -0.135\ldots$ 1

$f'(x) = \cos x + \frac{1}{2}$

$f'(0.6) = 1.325\ldots$ 1

$\therefore a_1 = 0.6 - \dfrac{(-0.135\ldots)}{(1.325\ldots)}$

$a_1 = 0.70$ 1 (3)

b) $\displaystyle\int_0^2 \frac{x}{\sqrt{9-x^2}}\,dx$

when $x = 2 \quad u = 5$ } 1

$x = 0 \quad u = 9$

$u = 9 - x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\therefore \displaystyle\int_9^5 \frac{x}{\sqrt{u}} \cdot \frac{du}{-2x}$ } 1

$= -\frac{1}{2}\displaystyle\int_9^5 u^{-\frac{1}{2}}\,du$

$= -\frac{1}{2}\left[2\sqrt{u}\right]_9^5$ 1

$= -\frac{1}{2}\left[2\sqrt{5} - 2\sqrt{9}\right]$

$= -\frac{1}{2}(2\sqrt{5} - 6)$

$= 3 - \sqrt{5}$ 1 (4)

Prove $5^n + 2(11)^n$ is $\div$ by 3.

Step 1. Prove true for $n = 1$

$5 + 22 = 27$ which is $\div$ by 3.

Step 2 Assume true for $n = k$

1 { $\dfrac{5^k + 2(11)^k}{3} = M$ where M is an integer.

$\therefore 5^k = 3M - 2(11)^k$

Step 3 Prove true for $n = k+1$.

$5^{k+1} + 2(11)^{k+1}$

1 { $= 5(5^k) + 2(11)^k \cdot 11^1$

$= 5(3M - 2(11)^k) + 22(11)^k$

$= 15M - 10(11)^k + 22(11)^k$

$= 15M + 12(11)^k$

1 $= 3(5M + 4(11)^k)$

which is $\div$ by 3.

Step 4 Prove true for $n = 1$ & assumed true for $n = k$ & proven true for $n = k+1$,

1 $\therefore$ true for $n = 1, n = 2, \ldots$ & for all n by Mathematical Induction.

(4)

Question 3. [12]

a) $\displaystyle\int \frac{dx}{9+4x^2} = \int \frac{dx}{4(\frac{9}{4} + x^2)}$

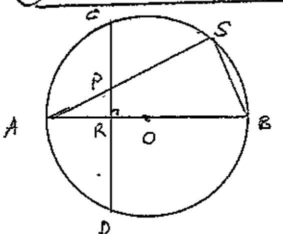
$= \frac{1}{4}\displaystyle\int \frac{dx}{(\frac{3}{2})^2 + x^2}$ } 1

$= \frac{1}{4}\left(\frac{1}{\frac{3}{2}}\right)\tan^{-1}\frac{x}{(\frac{3}{2})} + c$

1 { $= \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$

(2)

b)



$\angle PRB = 90^\circ$ (CD $\perp$ AB)

$\angle ASB = 90^\circ$ (Angle in a semi-circle) 1

$\therefore \angle PRB + \angle ASB = 180^\circ$

Hence $\angle SPR + \angle RBS = 180^\circ$ 1 (Angle Sum of Quad).

Hence PRSB is a cyclic quad (Opposite angles supplementary).

$AP \times AS = AR \cdot AB$

let $BR = x$

$\therefore 5 \times 13 = 4 \times (x + 4)$ 1

$\therefore 65 = 4x + 16$ (4)

$4x = 49 \Rightarrow x = \frac{49}{4}$ 1

c) $\frac{dV}{dt} = 10 \quad V = \frac{4}{3}\pi r^3 \therefore \frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$10 = 4\pi r^2 \cdot \frac{dr}{dt}$ 1 but $4\pi r^2 = 500$

$\therefore 10 = 500 \cdot \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{1}{50}$ 1 (2)

d) (i) $GI = 10$ 1 (Pythagoras)

$EB = \sqrt{136}$ (")

$BG = \sqrt{80}$ (") 1 for either

$\therefore EF = \sqrt{72}$ (") EB, EG, or

$\therefore EG = \sqrt{136}$ (")

let $\angle EGB = x$

$\cos x = \dfrac{(\sqrt{136})^2 + (\sqrt{80})^2 - (\sqrt{136})^2}{2.\sqrt{136}.\sqrt{80}}$ 1

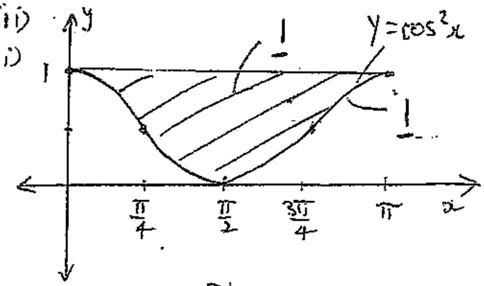
$x = 67°27'$ 1 (4)

4 a) $y = \cos^2 x$ [15]

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1

1

(ii)
(iii)



(iv) $\text{Area} = \int_0^{\pi} 1-\cos^2 x$

$= \int_0^{\pi} \sin^2 x$

$= \int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)$ 1

$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi}$ 1

$= \left(\frac{\pi}{2} - \frac{\sin 2\pi}{4}\right) - (0-0)$

$= \frac{\pi}{2}$ 1

(v) $V = \pi \int y^2 \, dx.$

$= \pi \int_0^{\pi} \cos^4 x \, dx$ 1

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1

1

$V = \pi\left[\frac{\frac{\pi}{4}}{2}\left(1+1+2\left(\frac{1}{4}+0+\frac{1}{4}\right)\right)\right]$

$= \pi\left[\frac{\pi}{8}(3)\right]$

$= \frac{3\pi^2}{8}$ 1 (3·70) (10)

b) (i)

$$\begin{array}{r} x^2 + x - 1 \\ x+1 \overline{) x^3+2x^2+0x-1} \\ x^3+x^2 \\ \hline x^2+0x \\ x^2+x \\ \hline -x-1 \\ -x-1 \\ \hline \end{array}$$

$\therefore (x+1)(x^2+x-1) = x^3+2x^2-1$ 1

(ii) pt of intersection is when

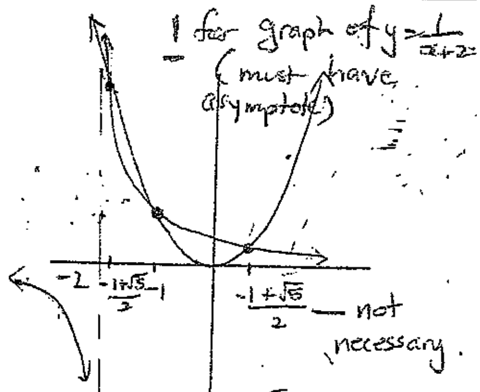
$\frac{1}{x+2} = x^2$

ie $x^3+2x^2-1=0$ from i)

$= (x+1)(x^2+x-1)=0$ 1

$x=-1$ 1 $x = \frac{-1 \pm \sqrt{1+4}}{2}$

$= \frac{-1 \pm \sqrt{5}}{2}.$ 1



From graph.

(iii) $\frac{1}{x+2} > x^2$ when (6)

$-2 < x < \frac{-1-\sqrt{5}}{2}$ and 1

when $-1 < x < \frac{-1+\sqrt{5}}{2}$

5b) (i) $x+py = 2ap+ap^3$ 13

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$py = -x+2ap+ap^3$

$y = -\frac{x}{p} + 2a + ap^2$

$m = -\frac{1}{p}$ 1 $\perp m = p.$

$\therefore$ Eq'n of SN Focus (0, a)

$y = px + a.$ 1

(ii) $x+py = 2ap+ap^3$

$y = px+a$

$\therefore x + p(px+a) = 2ap+ap^3$ 1

$x+p^2x+ap = 2ap+ap^3$

$x(1+p^2) = ap+ap^3$

$x(1+p^2) = ap(1+p^2)$

$\therefore x = ap$ 1

$y = ap^2+a$

$\therefore N(ap, ap^2+a)$

(iv) $x=ap \Rightarrow p = \frac{x}{a}$ 1

$y = ap^2+a$

$= a\left(\frac{x}{a}\right)^2+a$

$y = \frac{x^2}{a} + a$ 1 (6)

$x^2 - ay + a^2 = 0.$

d) (i) $v^2 = 16+4x-2x^2$

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$\frac{1}{2}v^2 = 8+2x-x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2-2x$

$= -2(x-1)$ 1

which is in the form

$\ddot{x} = -n^2(x-b)$ ∴ 1

S.H.

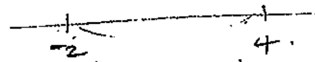
(ii) Endpoints when $v=0$

$\therefore 0 = 16+4x-2x^2$

$x^2-2x-8=0$

$(x-4)(x+2)=0$ 1

$x=-2, 4.$



Amplitude = 3 1

Period $= \frac{2\pi}{n}$ $n^2=2$

$n=\sqrt{2}$

$\therefore = \frac{2\pi}{\sqrt{2}}$ or $\sqrt{2}\pi.$ 1

(iii) Max speed when $\ddot{x}=0$

ie $-2(x-1)=0$

ie at $x=1$ 1

$v^2 = 16+4-2$

$v = \pm\sqrt{18}$

$\therefore$ Max speed $= 3\sqrt{2}$ 1 (7)

Question 6 14

a) $\frac{dT}{dt} = k(T-C)$

$T = C+Ae^{kt} \Rightarrow Ae^{kt} = T-C$

$\frac{dT}{dt} = Ake^{kt}$

$= kAe^{kt}$

$= k(T-C)$ 1

$\therefore T = C+Ae^{kt}$ is a solution to $\frac{dT}{dt} = k(T-C)$

(ii) when $t=0$ $T=25$ $C=300$

$\therefore 25 = 300 + Ae^0$

$\therefore A = -275.$ 1

$\therefore T = 300-275e^{kt}$

when $t=30$ $T=100$

$\therefore 100 = 300-275e^{30k}$

$275e^{30k} = 200$ 1

$30k = \ln\left(\frac{200}{275}\right)$

$k = \frac{\ln\left(\frac{200}{275}\right)}{30}$

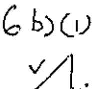
$= -0.0106\ldots$ 1

(iii) Find T when $t=70$ 1

$T = 300-275e^{70(-0.0106)}$

$T = 169·2°.$ 1 (6)

6 b) (i). $\ddot{x}=0$



$\dot{x} = c_1$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + c_2$

when $t=0$ $x=0$ $\therefore c_2=0$

$\therefore x = Vt\cos\theta$ 1

6 b(i) $\ddot{y} = -10$

$\dot{y} = -10t + c_3 \quad c_3 = V\sin\theta$

$\dot{y} = -10t + V\sin\theta$

$y = -5t^2 + Vt\sin\theta + c_4$

when $t = 0$ $y = 50 \Rightarrow c_4 = 50$

$\therefore y = -5t^2 + Vt\sin\theta + 50$ 1

(ii) when $t = 5$ $x = 100$ and $y = 0$

$\therefore 100 = 5V\cos\theta$

$V\cos\theta = 20$ -- ① 1

$0 = -125 + 5V\sin\theta + 50$

$\therefore 75 = 5V\sin\theta$

$15 = V\sin\theta$ --- ② 1

② ÷ ① $\frac{3}{4} = \tan\theta$

$\theta = 36°52'$ 1

$\therefore V\cos 36°52' = 20$

$V = \frac{20}{\cos 36°52'}$

$= 25$ m/s. 1

(iii) At impact $V = \sqrt{(\dot{x})^2 + (\dot{y})^2}$

when $t = 5$ $V = 25$

$\dot{x} = 25 \cos 36°52'$

$= 20$ m/s.

1 $\dot{y} = -10(5) + 25 \sin 36°52'$

$= 25$ m/s.

$\therefore V = \sqrt{(20)^2 + (35)^2}$

[12] $= \sqrt{125}$ m/s (40.3) 1 ⑧

7a)

(i) $P(\text{Winning}) = \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}$ 1

$P(\text{Losing}) = P(B) + P(\text{other, Black})$

$= \frac{1}{6} + \frac{5}{6} \times \frac{1}{6}$

$= \frac{11}{36}$ 1

(ii) $P(\text{Win first go}) = \frac{1}{4}$

$P(\text{Win 2nd Go}) = P(\text{Draw then win})$

1 ← $= \frac{4}{9} \times \frac{1}{4}$

$P(\text{Win 3rd Go}) = P(\text{Draw, Draw, Win})$

$= \frac{4}{9} \times \frac{4}{9} \times \frac{1}{4}$

$= \left(\frac{4}{9}\right)^2 \times \frac{1}{4}$

$\therefore P(\text{Win 1st, 2nd, 3rd..}) =$

$\frac{1}{4} + \frac{1}{4} \times \frac{4}{9} + \frac{1}{4} \times \left(\frac{4}{9}\right)^2$ 1

(iii) $P(\text{Wins}) =$

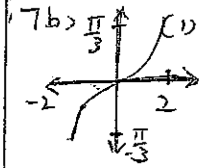
$\frac{1}{4}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 + \cdots\right)$ 1

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{4}\left(\frac{1}{1 - \frac{4}{9}}\right)$

$= \frac{1}{4} \times \frac{9}{5} = \frac{9}{20}$ 1 ⑥

7b) (i)



$y = a \sin^{-1} bx$

range $-\frac{\pi}{3} \Leftrightarrow \frac{\pi}{3}$

Now $y = \sin^{-1} x$

range $-\frac{\pi}{2} \Leftrightarrow \frac{\pi}{2}$

$\frac{\pi}{3}$ is $\frac{2}{3}$'s $\frac{\pi}{2}$ $\therefore a = \frac{2}{3}$ 1

$\frac{y}{a} = \sin^{-1} bx$

$x = \frac{1}{b}\sin\frac{y}{a}$

range is $-2 \Leftrightarrow 2$ (twice that of $y = \sin^{-1} x$)

$\therefore \frac{1}{b} = 2$

$b = \frac{1}{2}$ 1

$a = \frac{2}{3} \quad b = \frac{1}{2}$

(ii) $y = \frac{2}{3}\sin^{-1}\frac{x}{2}$

$y' = \frac{2}{3} \cdot \frac{\frac{1}{2}}{\sqrt{1 - \left(\frac{x}{2}\right)^2}}$

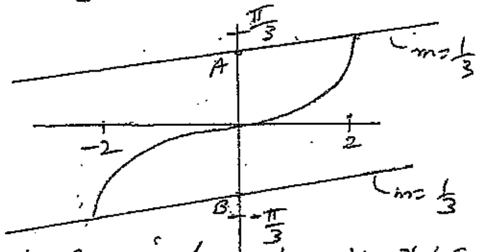
$= \frac{2}{3} \cdot \frac{\frac{1}{2}}{\frac{\sqrt{4 - x^2}}{\sqrt{4}}}$ 1

$= \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{\sqrt{4}}{\sqrt{4 - x^2}}$

$= \frac{2}{3\sqrt{4 - x^2}}$

When $x = 0$ $y' = \frac{1}{3}$ 1

(iii) $\frac{x}{3} + c = \frac{2}{3}\sin^{-1}\frac{x}{2}$



$\therefore$ Require to solve $y = \frac{x}{3} + c$ & $y = \frac{2}{3}\sin^{-1}\frac{x}{2}$ simultaneously. C will lie between A and B.

1 mark for recognising need to solve $y = \frac{x}{3} + c$ & $y = \frac{2}{3}\sin^{-1}\frac{x}{2}$ simultaneously

⑥

At A

$y = \frac{x}{3} + A$ passes through $(2, \frac{\pi}{3})$.

$\therefore \frac{\pi}{3} = \frac{2}{3} + A$

$\therefore A = \frac{\pi}{3} - \frac{2}{3}$

At B $y = \frac{x}{3} + B$ passes through $(-2, -\frac{\pi}{3})$

$-\frac{\pi}{3} = -\frac{2}{3} + B$

$\therefore B = \frac{2 - \pi}{3}$

$\therefore \frac{2 - \pi}{3} \leq c \leq \frac{\pi - 2}{3}$ 1